Theory and Applications of Categories, Vol. 33, No. 29, 2018, pp. 867–897.

CROSSED PRODUCTS OF CROSSED MODULES OF HOPF MONOIDS

J.N. ALONSO ÁLVAREZ, J.M. FERNÁNDEZ VILABOA, AND R. GONZÁLEZ RODRÍGUEZ

Abstract. In this paper we consider a crossed product of two crossed modules of Hopf monoids in a strict symmetric monoidal category $\mathcal{C}$ and give necessary and sufficient conditions to get a new crossed module of Hopf monoids in $\mathcal{C}$. Moreover we introduce the notion of projection of crossed modules of Hopf monoids and show that with additional hypotheses, any of these projections defines a new crossed module of Hopf monoids and allows us to construct an example of this kind of crossed product. Finally, we develop the explicit computations of a crossed product associated to a projection of crossed modules in groups.

1. Introduction

Motivated by the semidirect product construction in the theory of groups and taking into account that the group algebra is a Hopf algebra, Molnar [Molnar, 1997] introduced, in a category of vector spaces over a field k, the smash product $B\#H$ of a Hopf algebra B by a cocommutative Hopf algebra H and gave sufficient conditions to assure that this product is compatible with the tensor product coalgebra structure. This work was initially extended by Radford [Radford, 1985], who found necessary and sufficient conditions for the smash product algebra structure and the smash coproduct coalgebra structure to afford a Hopf algebra structure and characterized such objects via bialgebra projections. Later, Majid [Majid, 1994] interpreted this result in the modern context of braided categories. Also, but in a different direction, Majid [Majid, 1995] gave a different version by considering two-cocycles. More precisely, he established conditions to assure that a product modified with a two-cocycle be compatible with the tensor product coalgebra structure. This result was extended to a braided category in [Alonso Álvarez, Fernández Vilaboa, González Rodríguez, 2001].

On the other hand, crossed modules of groups were initially defined by Whitehead [Whitehead, 1949] as models for (homotopy) 2-types. Recall that the triple (A, H, ∂) is a

The authors were supported by Ministerio de Economía y Competitividad (Spain), grant MTM2016-79661-P (AEI/FEDER, UE, support included) (Homología, homotopía e invariantes categóricos en grupos y álgebras no asociativas)

Received by the editors 2017-05-16 and, in final form, 2018-08-26.

Transmitted by Richard Blute. Published on 2018-08-30.

2010 Mathematics Subject Classification: 18D10, 16W30, 16S40.

Key words and phrases: Hopf monoid, crossed module, entwining structure.

crossed module if, for the group morphism $\partial : A \to H$ and the action $\varphi_A(h,a) = {}^{h}a$, the following identities

(i) $\partial({}^{h}a) = h\partial(a)h^{-1}$,

(ii) ${}^{\partial(a)}a' = aa'a^{-1}$ (Peiffer identity).

hold. An immediate generalization of this notion is obtained if we consider group algebras in place of groups and it is easy to extend the notion to the more general context of Hopf algebras. Unfortunately, there is no agreement as to the definition of a crossed module of Hopf algebras. The most general notion was given by Frégier and Wageman in [Frégier, Wagemann, 2011]. If $\otimes$ denotes the tensor product in $k-\mathbf{Vect}$, they consider two Hopf algebras A and H, an action $\varphi_A : H\otimes A \to A$, such that (A, φ_A) is a left H-module algebra (coalgebra), and a Hopf algebra morphism $\partial : A \to H$ satisfying

$$\partial \circ \varphi_A = ad_H \circ (id_H \otimes \partial), \tag{1}$$

and the Peiffer identity

$$\varphi_A \circ (\partial \otimes id_A) = ad_A, \tag{2}$$

where ad_H, ad_A denote the adjoint action for H and A respectively. The notion given by Majid [Majid, 2012] (see also [Faria, 2016]) assumes (1), (2), the condition of morphism of H-modules for the antipode of A, and the equality

$$(id_H \otimes \varphi_A) \circ (\delta_H \otimes id_A) = (id_H \otimes \varphi_A) \circ ((c_{H,H} \circ \delta_H) \otimes id_A) \tag{3}$$

where δ_H is the coproduct of H and $c_{H,H}$ the twist in $k-\mathbf{Vect}$. On the other hand, in a monoidal setting, Fernández Vilaboa and López López [Fernández Vilaboa, López López, 2007] assume also that H is cocommutative and impose constraints coming from compatibility with cocommutativity.

In this work, we will consider the monoidal version of the Frégier and Wageman notion with the additional equality (3). As Majid pointed in [Majid, 2012], the main reason for assuming this condition is that (3) is a necessary condition to ensure that a crossed product of Hopf monoids be compatible with the tensor coproduct. But we also want to emphasize that, as we prove in this paper, (3) allows us to ensure that any Hopf monoid H induces a crossed module of Hopf monoids (H, H, id_H), in a similar way to what happens in the group setting. In any case, we do not make any assumption on cocommutativity, nor do we ask that the antipode be a morphism of left H-modules. As a consequence, our definition is more general that the ones given in [Fernández Vilaboa, López López, 2007] and [Majid, 2012]. Finally, we want to point out that, although our definition of crossed module of Hopf monoids is equivalent to the one drawn up in terms of module monoid and comonoid structures, we have written it in terms of entwining structures in order to illustrate the possibility of working with crossed modules through the usual entwining techniques.

An outline of the paper is as follows: after this introduction, in section 2 we give the necessary and sufficient conditions to ensure that a crossed product of two crossed modules of Hopf monoids be a crossed module of Hopf monoids. As an application, in section 3 we introduce the notion of projection of crossed modules of Hopf monoids and show that with some additional hypotheses, any of these projections defines a new crossed module of Hopf monoids and allows us to construct an example of this kind of crossed product. Finally, we develop the explicit computations for a crossed product of two crossed modules of Hopf monoids in the case where the Hopf monoids are groups.

An immediate application of the results presented in this paper is the possibility of building new crossed modules of Hopf monoids through crossed products. But we also want to point out that, even though in this paper we work in a monoidal symmetric setting, the definition of crossed module of Hopf monoids and the main results about this objects can be formulated and obtained with similar proofs in a braided context using the corresponding definition of cocommutativity class in a braided category.

2. Crossed product of crossed modules

Throughout this paper $\mathcal{C}$ denotes a strict symmetric monoidal category with tensor product $\otimes$, unit object K and natural isomorphism of symmetry c. We also assume that in $\mathcal{C}$ every idempotent morphism splits, i.e., for any morphism $q : M \to M$ such that $q \circ q = q$ there exists an object Z, called the image of q, and morphisms $i : Z \to M$, $p : M \to Z$ such that $q = i \circ p$ and $p \circ i = id_Z$ where id_Z denotes the identity morphism for Z. The morphisms p and i will be called a factorization of q. Note that Z, p and i are unique up to isomorphism. The categories satisfying this property constitute a broad class that includes, among others, categories with epi-monic decomposition for morphisms and categories with (co)equalizers (see [Borceux, 1994] for details). For each object M in $\mathcal{C}$, we denote the identity morphism by $id_M : M \to M$ and, for simplicity of notation, given objects M, N and P in $\mathcal{C}$ and a morphism $f : M \to N$, we write $P \otimes f$ for $id_P \otimes f$ and $f \otimes P$ for $f \otimes id_P$. There is no loss of generality in assuming the strictness of $\mathcal{C}$ because it is well known that we can construct a strict monoidal category $\mathcal{C}^{st}$ which is tensor equivalent to $\mathcal{C}$ (see [Kassel, 1995] for the details). As a consequence, the results proved in this paper hold for every non-strict symmetric monoidal category (for example $k-\mathbf{Vect}$).

A monoid in $\mathcal{C}$ is a triple $A = (A, \eta_A, \mu_A)$ where A is an object in $\mathcal{C}$ and $\eta_A : K \to A$ (unit), $\mu_A : A \otimes A \to A$ (product) are morphisms in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = id_A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (A \otimes \mu_A) = \mu_A \circ (\mu_A \otimes A)$. Given two monoids $A = (A, \eta_A, \mu_A)$ and $B = (B, \eta_B, \mu_B)$, $f : A \to B$ is a monoid morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$, $f \circ \eta_A = \eta_B$. Also, if A, B are monoids in $\mathcal{C}$, the object $A \otimes B$ is a monoid in $\mathcal{C}$ where $\eta_{A\otimes B} = \eta_A \otimes \eta_B$ and $\mu_{A\otimes B} = (\mu_A \otimes \mu_B) \circ (A \otimes c_{B,A} \otimes B)$.

A comonoid in $\mathcal{C}$ is a triple $D = (D, \varepsilon_D, \delta_D)$ where D is an object in $\mathcal{C}$ and $\varepsilon_D : D \to K$ (counit), $\delta_D : D \to D \otimes D$ (coproduct) are morphisms in $\mathcal{C}$ such that $(\varepsilon_D \otimes D) \circ \delta_D = id_D = (D \otimes \varepsilon_D) \circ \delta_D$, $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$. If $D = (D, \varepsilon_D, \delta_D)$ and $E = (E, \varepsilon_E, \delta_E)$ are comonoids, $f : D \to E$ is a comonoid morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$, $\varepsilon_E \circ f = \varepsilon_D$. When D, E are comonoids in $\mathcal{C}$, $D \otimes E$ is a comonoid in $\mathcal{C}$ where $\varepsilon_{D\otimes E} = \varepsilon_D \otimes \varepsilon_E$ and

$\delta_{D\otimes E} = (D \otimes c_{D,E} \otimes E) \circ (\delta_D \otimes \delta_E)$.

If A is a monoid, B is a comonoid and $f : B \to A$, $g : B \to A$ are morphisms, we define the convolution product by $f * g = \mu_A \circ (f \otimes g) \circ \delta_B$.

A bimonoid H is a monoid (H, η_H, μ_H) and a comonoid $(H, \varepsilon_H, \delta_H)$ such that η_H and μ_H are morphisms of comonoids (equivalently, ε_H and δ_H are morphisms of monoids), i.e., $\delta_H \circ \eta_H = \eta_H \circ \eta_H$ and $\delta_H \circ \mu_H = \mu_{H\otimes H} \circ (\delta_H \otimes \delta_H)$. If moreover there exists a morphism $\lambda_H : H \to H$ (called the antipode of H) such that $id_H * \lambda_H = \lambda_H * id_H = \varepsilon_H \otimes \eta_H$, we will say that H is a Hopf monoid. Moreover, if H and G are Hopf monoids, $f : H \to G$ is a morphism of Hopf monoids if it is a monoid and comonoid morphism. It is not difficult to see that in this case $\lambda_G \circ f = f \circ \lambda_H$. Note that a Hopf monoid in $k - \mathbf{Vect}$ is precisely a classical Hopf algebra.

2.1. Definition. Let H be a Hopf monoid. An object A in $\mathcal{C}$ is said to be a *left H-module* if there is a morphism $\phi_A : H \otimes A \to A$ in $\mathcal{C}$ satisfying that $\phi_A \circ (\eta_H \otimes A) = id_A$ and $\phi_A \circ (H \otimes \phi_A) = \phi_A \circ (\mu_H \otimes A)$. Given two left H-modules (A, ϕ_A) and (B, ϕ_B), $f : A \to B$ is a *morphism of left H-modules* if $\phi_B \circ (H \otimes f) = f \circ \phi_A$.

We will say that a left H-module (A, ϕ_A) is in the cocommutativity class of H if $c_{H,A}$ is a morphism of left H-modules, considering $\phi_{H\otimes A} = (\mu_H \otimes \phi_A) \circ (H \otimes c_{H,H} \otimes A) \circ (\delta_H \otimes H \otimes A)$ and $\phi_{A\otimes H} = (\phi_A \otimes \mu_H) \circ (H \otimes c_{H,A} \otimes H) \circ (\delta_H \otimes A \otimes H)$. By Proposition 2.4 of [Alonso Álvarez, Fernández Vilaboa, González Rodríguez, 2001] this is equivalent to the condition

$$(\phi_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) = c_{H,A} \circ (H \otimes \phi_A) \circ (\delta_H \otimes A) \tag{4}$$

holds. Finally, if A is a monoid and η_A and μ_A are left H-module morphisms, i.e., $\phi_A \circ (H \otimes \eta_A) = \varepsilon_H \otimes \eta_A$ and $\phi_A \circ (H \otimes \mu_A) = \mu_A \circ (\phi_A \otimes \phi_A) \circ (H \otimes c_{H,A} \otimes A) \circ (\delta_H \otimes A \otimes A)$, we will say that A is a *left H-module monoid*. If A is a comonoid and ε_A and δ_A are left H-module morphisms, that is, $\varepsilon_A \circ \phi_A = \varepsilon_H \otimes \varepsilon_A$ and $\delta_A \circ \phi_A = (\phi_A \otimes \phi_A) \circ \delta_{H\otimes A}$, A is said to be a *left H-module comonoid*.

2.2. Examples.

(a) Let H be a Hopf monoid. Then H is a left H-module comonoid via μ_H.

(b) If H is a Hopf monoid, A a monoid and $f : H \to A$ a monoid morphism, we can define the adjoint action of H on A associated to f (see [Molnar, 1997]) as

$$ad_{f,A} = \mu_A \circ (\mu_A \otimes A) \circ (f \otimes A \otimes (f \circ \lambda_H)) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A).$$

Then A is a left H-module monoid via $ad_{f,A}$. In particular, if $A = H$ and $f = id_H$ the action defined above (called the adjoint action of H) is the following:

$$ad_{id_H,H} = \mu_H \circ (\mu_H \otimes \lambda_H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H).$$

In what follows we will denote this action by ad_H.

It is obvious that if H is cocommutative (i.e., $\delta_H = c_{H,H} \circ \delta_H$), any left H-module is in the cocommutativity class of H. Therefore the following result improves Proposition 2.5 (a) in [Molnar, 1997] (note that in the 'if' part we only need that A be a bimonoid).

2.3. PROPOSITION. *Let H and A be Hopf monoids, and let $f : H \to A$ be a bimonoid morphism. The following assertions are equivalent.*

(i) $(ad_{f,A} \otimes (f \circ \lambda_H)) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) = c_{A,A} \circ ((f \circ \lambda_H) \otimes ad_{f,A}) \circ (\delta_H \otimes A)$.

(ii) A is a left H-module comonoid via $ad_{f,A}$.

As a consequence, if λ_H is an isomorphism we have that H is a left H-module comonoid via ad_H if and only if (H, ad_H) is in the cocommutativity class of H.

PROOF. (i) $\Rightarrow$ (ii). Trivially, $\varepsilon_A \circ ad_{f,A} = \varepsilon_H \otimes \varepsilon_A$. Moreover

$$\begin{aligned}
&\delta_A \circ ad_{f,A} \\
&= \mu_{A\otimes A} \circ (\mu_{A\otimes A} \otimes (((f \circ \lambda_H) \otimes (f \circ \lambda_H)) \circ c_{H,H} \circ \delta_H)) \circ (((f \otimes f) \circ \delta_H) \otimes \delta_A \otimes H) \\
&\quad \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) \text{ (A is a bimonoid, f is a bimonoid morphism, λ_H is anticomultiplicative)} \\
&= (\mu_A \otimes A) \circ (A \otimes c_{A,A}) \circ (A \otimes ((ad_{f,A} \otimes (f \circ \lambda_H)) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A))) \\
&\quad \circ ((\mu_A \circ (f \otimes A)) \otimes H \otimes A) \circ \delta_{H\otimes A} \text{ (naturality)} \\
&= (\mu_A \otimes A) \circ (A \otimes c_{A,A}) \circ (A \otimes ((A \otimes (f \circ \lambda_H)) \circ c_{H,A} \circ (H \otimes ad_{f,A}) \circ (\delta_H \otimes A))) \\
&\quad \circ ((\mu_A \circ (f \otimes A)) \otimes H \otimes A) \circ \delta_{H\otimes A} \text{ ((i))} \\
&= (ad_{f,A} \otimes ad_{f,A}) \circ \delta_{H\otimes A} \text{ (naturality)},
\end{aligned}$$

and then $ad_{f,A}$ is a comonoid morphism.

(ii) $\Rightarrow$ (i). Assume that $\delta_A \circ ad_{f,A} = (ad_{f,A} \otimes ad_{f,A}) \circ \delta_{H\otimes A}$. First of all we will show that the equality

$$\begin{aligned}
&((\mu_A \circ (A \otimes (f \circ \lambda_H))) \otimes ad_{f,A}) \circ (A \otimes (c_{H,H} \circ \delta_H) \otimes A) \circ (c_{H,A} \otimes A) \circ (H \otimes \delta_A) \\
&= ((\mu_A \circ (A \otimes (f \circ \lambda_H))) \otimes ad_{f,A}) \circ (A \otimes \delta_H \otimes A) \circ (c_{H,A} \otimes A) \circ (H \otimes \delta_A) \qquad (5)
\end{aligned}$$

holds. Indeed,

$$\begin{aligned}
&((\mu_A \circ (A \otimes (f \circ \lambda_H))) \otimes ad_{f,A}) \circ (A \otimes (c_{H,H} \circ \delta_H) \otimes A) \circ (c_{H,A} \otimes A) \circ (H \otimes \delta_A) \\
&= \mu_{A\otimes A} \circ (A \otimes A \otimes A \otimes (\mu_A \circ c_{A,A})) \circ (A \otimes f \otimes (((f \circ \lambda_H) \otimes (f \circ \lambda_H)) \circ c_{H,H} \circ \delta_H) \otimes A) \\
&\quad \circ (A \otimes \delta_H \otimes A) \circ (c_{H,A} \otimes A) \circ (H \otimes \delta_A) \text{ (naturality)} \\
&= (A \otimes \mu_A) \circ (c_{A,A} \otimes A) \circ (f \otimes (\mu_{A\otimes A} \circ (\delta_A \otimes (\delta_A \circ f \circ \lambda_H)))) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A) \\
&\qquad \text{(λ_H is anticomultiplicative, f is a comonoid morphism)} \\
&= \mu_{A\otimes A} \circ (A \otimes \mu_A \otimes \delta_A) \circ (c_{A,A} \otimes A \otimes (f \circ \lambda_H)) \circ (f \otimes \delta_A \otimes H) \circ (H \otimes c_{H,A}) \circ (\delta_H \otimes A)
\end{aligned}$$

(naturality)

$= \mu_{A\otimes A}\circ(\mu_{A\otimes A}\otimes(\delta_A\circ f\circ\lambda_H))\circ((f\circ(\mu_H\circ(\lambda_H\otimes H)\circ\delta_H))\otimes f\otimes\delta_A\otimes H)\circ(\delta_H\otimes c_{H,A})$
$\circ(\delta_H\otimes A)$ (f is a monoid morphism, H is a Hopf monoid)

$= (\mu_A\otimes A)\circ(A\otimes\mu_{A\otimes A})\circ(A\otimes\mu_{A\otimes A}\otimes\delta_A)\circ((f\circ\lambda_H)\otimes((f\otimes f)\circ\delta_H)\otimes\delta_A\otimes(f\circ\lambda_H))$
$\circ(H\otimes H\otimes c_{H,A})\circ(H\otimes\delta_H\otimes A)\circ(\delta_H\otimes A)$ (f is a bimonoid morphism)

$= (\mu_A\otimes A)\circ(A\otimes(\delta_A\circ ad_{f,A}))\circ((((f\circ\lambda_H)\otimes H)\circ\delta_H)\otimes A)$ (A is a bimonoid)

$= (\mu_A\otimes A)\circ(f\otimes ad_{f,A}\otimes ad_{f,A})\circ(\lambda_H\otimes\delta_{H\otimes A})\circ(\delta_H\otimes A)$ ($ad_{f,A}$ is a comonoid morphism)

$= (\mu_A\otimes A)\circ((\mu_A\circ(f\otimes A))\otimes(f\circ\lambda_H)\circ ad_{f,A})\circ((\mu_H\circ(\lambda_H\otimes H)\circ\delta_H)\otimes A\otimes\delta_H\otimes A)\circ\delta_{H\otimes A}$

(f is a bimonoid morphism, A is a bimonoid)

$= ((\mu_A\circ(A\otimes(f\circ\lambda_H)))\otimes ad_{f,A})\circ(A\otimes\delta_H\otimes A)\circ(c_{H,A}\otimes A)\circ(H\otimes\delta_A)$ (f is a bimonoid morphism, A is a Hopf monoid).

Then

$c_{A,A}\circ(\mu_A\otimes A)\circ(A\otimes(((\mu_A\circ(A\otimes(f\circ\lambda_H)))\otimes ad_{f,A})\circ(A\otimes(c_{H,H}\circ\delta_H)\otimes A)\circ(c_{H,A}\otimes A)$
$\circ(H\otimes\delta_A)))\circ(c_{H,A}\otimes A)\circ(H\otimes((\lambda_A\otimes A)\circ\delta_A))$

$= c_{A,A}\circ(\mu_A\otimes A)\circ((\mu_A\circ(\lambda_A\otimes A)\circ\delta_A)\otimes(f\circ\lambda_H)\otimes ad_{f,A})\circ(A\otimes(c_{H,H}\circ\delta_H)\otimes A)$
$\circ(c_{H,A}\otimes A)\circ(H\otimes\delta_A)$ (naturality)

$= (ad_{f,A}\otimes(f\circ\lambda_H))\circ(H\otimes c_{H,A})\circ(\delta_H\otimes A)$ (A is a Hopf monoid),

and on the other hand,

$c_{A,A}\circ(\mu_A\otimes A)\circ(A\otimes(((\mu_A\circ(A\otimes(f\circ\lambda_H)))\otimes ad_{f,A})\circ(A\otimes\delta_H\otimes A)\circ(c_{H,A}\otimes A)\circ(H\otimes\delta_A)))$
$\circ(c_{H,A}\otimes A)\circ(H\otimes((\lambda_A\otimes A)\circ\delta_A))$

$= c_{A,A}\circ(\mu_A\otimes A)\circ((\mu_A\circ(\lambda_A\otimes A)\circ\delta_A)\otimes(f\circ\lambda_H)\otimes ad_{f,A})\circ(A\otimes\delta_H\otimes A)\circ(c_{H,A}\otimes A)$
$\circ(H\otimes\delta_A)$(naturality)

$= c_{A,A}\circ((f\circ\lambda_H)\otimes ad_{f,A})\circ(\delta_H\otimes A)$ (A is a Hopf monoid),

and the proof is complete.

■

Now we recall the notion of entwining structure introduced in [Brzeziński, Majid, 1998].

2.4. DEFINITION. A *left-left entwining structure* on $\mathcal{C}$ consists of a triple $(A, D, \psi_{A,D})$, where A is a monoid, D a comonoid, and $\psi_{A,D} : A \otimes D \to D \otimes A$ is a morphism satisfying the relations

(a1) $\psi_{A,D} \circ (\eta_A \otimes D) = D \otimes \eta_A$,

(a2) $(D \otimes \mu_A) \circ (\psi_{A,D} \otimes A) \circ (A \otimes \psi_{A,D}) = \psi_{A,D} \circ (\mu_A \otimes D)$,

(a3) $(\delta_D \otimes A) \circ \psi_{A,D} = (D \otimes \psi_{A,D}) \circ (\psi_{A,D} \otimes D) \circ (A \otimes \delta_D)$,

(a4) $(\varepsilon_D \otimes A) \circ \psi_{A,D} = A \otimes \varepsilon_D$.

If we only have the conditions (a1) and (a2) we will say that $(A, D, \psi_{A,D})$ is a *left-left semi-entwining structure*. In a similar way, we can define the notions of right-right, right-left and left-right (semi)entwining structure. In particular, $(A, D, \psi_{D,A} : D \otimes A \to A \otimes D)$ will be a *right-right semi-entwining structure* if conditions

(b1) $\psi_{D,A} \circ (D \otimes \eta_A) = \eta_A \otimes D$,

(b2) $(\mu_A \otimes D) \circ (A \otimes \psi_{D,A}) \circ (\psi_{D,A} \otimes A) = \psi_{D,A} \circ (D \otimes \mu_A)$,

hold.

We will say that a left-left entwining structure $(A, D, \psi_{A,D})$ is in the cocommutativity class of D if condition

$$(D \otimes \psi_{A,D}) \circ (c_{A,D} \otimes D) \circ (A \otimes \delta_D) = (D \otimes c_{A,D}) \circ (\psi_{A,D} \otimes D) \circ (A \otimes \delta_D) \tag{6}$$

holds. In a similar way, a right-right entwining structure $(A, D, \psi_{D,A})$ is in the cocommutativity class of D if it satisfies that

$$(\psi_{D,A} \otimes D) \circ (D \otimes c_{D,A}) \circ (\delta_D \otimes A) = (c_{D,A} \otimes D) \circ (D \otimes \psi_{D,A}) \circ (\delta_D \otimes A). \tag{7}$$

2.5. EXAMPLE. Let H be a Hopf monoid. Then

$$(H, H, \psi_{H,H} = (\mu_H \otimes H) \circ (H \otimes c_{H,H}) \circ (\delta_H \otimes H))$$

is a left-left entwining structure on $\mathcal{C}$.

2.6. LEMMA. *Let X and Y be monoids and comonoids and let $\psi_{Y,X} : Y \otimes X \to X \otimes Y$ be a morphism such that (a4) of Definition 2.4 holds. The following assertions are equivalent.*

(i) $\delta_{X\otimes Y} \circ \psi_{Y,X} = (\psi_{Y,X} \otimes \psi_{Y,X}) \circ \delta_{Y\otimes X}$.

(ii) $\psi_{Y,X}$ *is in the cocommutativity class of Y, i.e.,*

$$(\psi_{Y,X} \otimes Y) \circ (Y \otimes c_{Y,X}) \circ (\delta_Y \otimes X) = (c_{Y,X} \otimes Y) \circ (Y \otimes \psi_{Y,X}) \circ (\delta_Y \otimes X), \tag{8}$$

and satisfies (a3) of Definition 2.4 and the equality

$$(X \otimes \delta_Y) \circ \psi_{Y,X} = (\psi_{Y,X} \otimes Y) \circ (Y \otimes c_{Y,X}) \circ (\delta_Y \otimes X). \tag{9}$$

PROOF. (i) ⇒ (ii) We get (9) by composing in (i) on the left with $X\otimes Y\otimes \varepsilon_X\otimes Y$ and using (a4) of Definition 2.4. Note that as a consequence we have that

$$(X\otimes \varepsilon_Y\otimes Y)\circ(\psi_{Y,X}\otimes Y)\circ(Y\otimes c_{Y,X})\circ(\delta_Y\otimes X)=\psi_{Y,X}. \tag{10}$$

Moreover,

$$\begin{aligned}
&(\delta_X\otimes Y)\circ\psi_{Y,X}\\
&=(X\otimes \varepsilon_Y\otimes X\otimes Y)\circ\delta_{X\otimes Y}\circ\psi_{Y,X} \text{ (X is a comonoid)}\\
&=(X\otimes \varepsilon_Y\otimes X\otimes Y)\circ(\psi_{Y,X}\otimes\psi_{Y,X})\circ\delta_{Y\otimes X} \text{ ((i))}\\
&=(X\otimes\psi_{Y,X})\circ(\psi_{Y,X}\otimes X)\circ(Y\otimes\delta_X) \text{ ((10))},
\end{aligned}$$

and we get (a3). Finally,

$$\begin{aligned}
&(X\otimes\delta_Y)\circ\psi_{Y,X}\\
&=(\varepsilon_X\otimes c_{Y,X}\otimes Y)\circ\delta_{X\otimes Y}\circ\psi_{Y,X} \text{ (X is a comonoid)}\\
&=(\varepsilon_X\otimes c_{Y,X}\otimes Y)\circ(\psi_{Y,X}\otimes\psi_{Y,X})\circ\delta_{Y\otimes X} \text{ ((i))}\\
&=(c_{Y,X}\otimes Y)\circ(Y\otimes\psi_{Y,X})\circ(\delta_Y\otimes X) \text{ ((a4) of Definition 2.4)}.
\end{aligned}$$

(ii) ⇒ (i)

$$\begin{aligned}
&\delta_{X\otimes Y}\circ\psi_{Y,X}\\
&=(X\otimes c_{X,Y}\otimes Y)\circ(\delta_X\otimes Y\otimes Y)\circ(c_{Y,X}\otimes Y)\circ(Y\otimes\psi_{Y,X})\circ(\delta_Y\otimes X) \text{ ((9), (8))}\\
&=(c_{Y,X}\otimes X\otimes Y)\circ(Y\otimes((\delta_X\otimes Y)\circ\psi_{Y,X}))\circ(\delta_Y\otimes X) \text{ (naturality)}\\
&=(c_{Y,X}\otimes X\otimes Y)\circ(Y\otimes((X\otimes\psi_{Y,X})\circ(\psi_{Y,X}\otimes X)\circ(Y\otimes\delta_X)))\circ(\delta_Y\otimes X) \text{ ((a3) of Definition 2.4)}\\
&=(\psi_{Y,X}\otimes\psi_{Y,X})\circ\delta_{Y\otimes X} \text{ ((8))}.
\end{aligned}$$

■

In the following Proposition we establish the connection between entwining and module monoid and comonoid structures.

2.7. PROPOSITION. *Let X and Y be bimonoids. The following assertions are equivalent.*

- *(i) There is a morphism $\psi_{Y,X}: Y\otimes X\to X\otimes Y$ such that $(Y,X,\psi_{Y,X})$ is a left-left entwining structure and a right-right semi-entwining structure satisfying (9).*
- *(ii) There is a morphism $\phi_X: Y\otimes X\to X$ such that (X,ϕ_X) is a left Y-module monoid and comonoid.*

Moreover, $\psi_{Y,X}$ is in the cocommutativity class of Y iff so is (X,ϕ_X).

PROOF. (i) ⇒ (ii) Define $\phi_X=(X\otimes\varepsilon_Y)\circ\psi_{Y,X}$.

(ii) ⇒ (i) Define $\psi_{Y,X}=(\phi_X\otimes Y)\circ(Y\otimes c_{Y,X})\circ(\delta_Y\otimes X)$.

■

Now we introduce our definition of crossed module of Hopf monoids. We have drafted this notion in terms of entwining structures in order to illustrate the possibility of working in crossed modules through the usual entwining techniques. Note that, by Proposition 2.7, the definition is equivalent to the one drawn up in terms of module monoid and comonoid structures. To make the relationship clear, we also write the classical version at the end of our definition. In what follows we will use entwining or module structures indifferently.

2.8. Definition. Let $\beta : X \to Y$ be a morphism of Hopf monoids and let $\psi_{Y,X} : Y \otimes X \to X \otimes Y$ be a morphism. We will say that $\mathbf{X_Y} = (X, Y, \beta)$ is a *crossed module of Hopf monoids* if

(c1) $\psi_{Y,X}$ is a left-left entwining structure and a right-right semi-entwining structure satisfying (8) and (9).

(c2) $(\beta \otimes \varepsilon_Y) \circ \psi_{Y,X} = ad_Y \circ (Y \otimes \beta)$.

(c3) $(X \otimes \varepsilon_Y) \circ \psi_{Y,X} \circ (\beta \otimes X) = ad_X$.

A morphism between two crossed modules of Hopf monoids $\mathbf{X_Y} = (X, Y, \beta)$ and $\mathbf{T_G} = (T, G, \partial)$ is a pair of Hopf monoid morphisms $u : X \to T$ and $v : Y \to G$ such that $v \circ \beta = \partial \circ u$ and $(u \otimes \varepsilon_Y) \circ \psi_{Y,X} = (T \otimes \varepsilon_G) \circ \psi_{G,T} \circ (v \otimes u)$.

Equivalently, $\mathbf{X_Y} = (X, Y, \beta)$ is a crossed module of Hopf monoids if there is a morphism $\phi_X : Y \otimes X \to X$ such that

(d1) (X, ϕ_X) is a left Y-module monoid and comonoid satisfying (4).

(d2) $\beta \circ \phi_X = ad_Y \circ (Y \otimes \beta)$.

(d3) (Peiffer identity) $\phi_X \circ (\beta \otimes X) = ad_X$,

and a morphism between two crossed modules of Hopf monoids $\mathbf{X_Y}$ and $\mathbf{T_G}$ is a pair of Hopf monoid morphisms $u : X \to T$ and $v : Y \to G$ such that $v \circ \beta = \partial \circ u$ and $u \circ \phi_X = \phi_T \circ (v \otimes u)$.

2.9. Example. Let H be a Hopf monoid such that the antipode is an isomorphism. By Proposition 2.3, (H, H, id_H) is a crossed module of Hopf monoids if and only if the adjoint action ad_H is in the cocommutativity class of H.

Note that our definition of crossed module of Hopf monoids is slightly different from the definition of crossed module of Hopf algebras given by Frégier and Wagemann [Frégier, Wagemann, 2011] because of condition (4). The reason for assuming this condition is to ensure that, as in the group setting, any Hopf monoid H induces a crossed module of Hopf monoids $\mathbf{H_H} = (H, H, id_H)$. In any case, we do not make any assumption on cocommutativity, nor do we ask that the antipode be a morphism of left H-modules. As a consequence, our definition is more general that the ones given in [Fernández Vilaboa, López López, 2007] and [Majid, 2012].

Let Y be a Hopf monoid and let (X, ϕ_X) be a left Y-module monoid. Then the smash product of X by Y defined as

$$X\#Y = (X \otimes Y, \eta_{X\#Y} = \eta_{X\otimes Y}, \mu_{X\#Y} = (\mu_X \otimes \mu_Y) \circ (X \otimes \psi_{Y,X} \otimes Y))$$

in [Sweedler, 1969], is a monoid, where $\psi_{Y,X} = (\phi_X \otimes Y) \circ (Y \otimes c_{Y,X}) \circ (\delta_Y \otimes X)$.

2.10. PROPOSITION. *Let X and Y be Hopf monoids and let $\psi_{Y,X}$ be a left-left entwining structure and a right-right semi-entwining structure satisfying (8) and (9). Then the tensor product comonoid structure is compatible with the smash product monoid structure, making*

$$X \bowtie Y = (X \otimes Y, \eta_{X\bowtie Y} = \eta_{X\#Y}, \mu_{X\bowtie Y} = \mu_{X\#Y}, \varepsilon_{X\bowtie Y} = \varepsilon_{X\otimes Y}, \delta_{X\bowtie Y} = \delta_{X\otimes Y})$$

a Hopf monoid with antipode $\lambda_{X\bowtie Y} = \psi_{Y,X} \circ (\lambda_Y \otimes \lambda_X) \circ c_{X,Y}$.

PROOF. It is a particular case of Proposition 3.8 of [Alonso Álvarez, Fernández Vilaboa, González Rodríguez, 2001]. Note that in this case the cocycle is trivial and then we do not need that the antipode of Y be invertible. ■

The main goal of this section is to construct the crossed product of two crossed modules of Hopf monoids. In order to do so, in what follows we consider two crossed modules of Hopf monoids $\mathbf{X_Y} = (X, Y, \beta)$ and $\mathbf{T_G} = (T, G, \partial)$ and denote the corresponding entwining structures by $\psi_{Y,X}$ and $\psi_{G,T}$, respectively. Moreover, let $t : Y \otimes T \to X$ be a morphism and assume that $\psi_{G,X} : G \otimes X \to X \otimes G$, $\psi_{T,X} : T \otimes X \to X \otimes T$ and $\psi_{G,Y} : G \otimes Y \to Y \otimes G$ are three left-left entwining structures and right-right semi-entwining structures such that (8), (9) and the Yang-Baxter condition

$$(\psi_{Y,X} \otimes G) \circ (Y \otimes \psi_{G,X}) \circ (\psi_{G,Y} \otimes X) = (X \otimes \psi_{G,Y}) \circ (\psi_{G,X} \otimes Y) \circ (G \otimes \psi_{Y,X}) \quad (11)$$

holds. Now define the morphism

$$\phi_{X\bowtie T} : Y \bowtie G \otimes X \bowtie T \to X \bowtie T$$

as

$$\phi_{X\bowtie T} = (\mu_X \otimes T) \circ (X \otimes t \otimes T) \circ (X \otimes Y \otimes \delta_T \otimes \varepsilon_G) \circ (\psi_{Y,X} \otimes \psi_{G,T}) \circ (Y \otimes \psi_{G,X} \otimes T).$$

In the following technical lemmas we will give the necessary and sufficient conditions to get that $X \bowtie T$ is a left $Y \bowtie G$-module monoid and comonoid with action $\phi_{X\bowtie T}$.

2.11. LEMMA. *The following assertions are equivalent.*

(i) $(X \bowtie T, \phi_{X\bowtie T})$ *is a left $Y \bowtie G$-module.*

(ii) The equalities

$$t \circ (\eta_Y \otimes T) = \varepsilon_T \otimes \eta_X, \quad (12)$$

$$(t \otimes \varepsilon_G) \circ (Y \otimes \psi_{G,T}) \circ (\psi_{G,Y} \otimes T) = (X \otimes \varepsilon_G) \circ \psi_{G,X} \circ (G \otimes t), \quad (13)$$

and

$$t \circ (\mu_Y \otimes T) = \mu_X \circ (X \otimes t) \circ (\psi_{Y,X} \otimes T) \circ (Y \otimes t \otimes T) \circ (Y \otimes Y \otimes \delta_T) \quad (14)$$

hold.

PROOF. (i) ⇒ (ii). Assume that $(X \bowtie T, \phi_{X\bowtie T})$ is a left $Y \bowtie G$-module. Then

$$\varepsilon_T \otimes \eta_X$$

$$= (X \otimes \varepsilon_T) \circ \phi_{X\bowtie T} \circ (\eta_Y \otimes \eta_G \otimes \eta_X \otimes T) \text{ ($X \bowtie T$ is a left $Y \bowtie G$-module)}$$

$$= t \circ (\eta_Y \otimes T) \text{ ((a1) of Definition 2.4)},$$

and we get (12). Moreover,

$$(t \otimes \varepsilon_G) \circ (Y \otimes \psi_{G,T}) \circ (\psi_{G,Y} \otimes T)$$

$$= (X \otimes \varepsilon_T) \circ \phi_{X\bowtie T} \circ (\mu_{Y\bowtie G} \otimes X \otimes T) \circ (\eta_Y \otimes G \otimes Y \otimes \eta_G \otimes \eta_X \otimes T)$$

(T is a comonoid, G and Y are monoids, (b1) of Definition 2.4)

$$= (X \otimes \varepsilon_T) \circ \phi_{X\bowtie T} \circ (Y \otimes G \otimes \phi_{X\bowtie T}) \circ (\eta_Y \otimes G \otimes Y \otimes \eta_G \otimes \eta_X \otimes T)$$

($X \bowtie T$ is a left $Y \bowtie G$-module)

$$= (X \otimes \varepsilon_G) \circ \psi_{G,X} \circ (G \otimes t)$$

(T is a comonoid, G is a Hopf monoid, X is a monoid, (a1), (a4) and (b1) of Definition 2.4, (12)),

and we obtain (13).

Finally, in a similar way we get (14). Indeed,

$$t \circ (\mu_Y \otimes T)$$

$$= (X \otimes \varepsilon_T) \circ \phi_{X\bowtie T} \circ (\mu_{Y\bowtie G} \otimes X \otimes T) \circ (Y \otimes \eta_G \otimes Y \otimes \eta_G \otimes \eta_X \otimes T)$$

$$= (X \otimes \varepsilon_T) \circ \phi_{X\bowtie T} \circ (Y \otimes G \otimes \phi_{X\bowtie T}) \circ (Y \otimes \eta_G \otimes Y \otimes \eta_G \otimes \eta_X \otimes T)$$

$$= \mu_X \circ (X \otimes t) \circ (\psi_{Y,X} \otimes T) \circ (Y \otimes t \otimes T) \circ (Y \otimes Y \otimes \delta_T).$$

(ii) ⇒ (i). By (a1) and (12), and using that G and X are Hopf monoids, it is easy to see that

$$\phi_{X\bowtie T} \circ (\eta_{Y\otimes G} \otimes X \otimes T) = id_{X\bowtie T}.$$

Moreover, by (9), (10) and (13) we have that

$$(t \otimes G) \circ (Y \otimes \psi_{G,T}) \circ (\psi_{G,Y} \otimes T) = \psi_{G,X} \circ (G \otimes t), \tag{15}$$

and then

$$\phi_{X\bowtie T} \circ (Y \otimes G \otimes \phi_{X\bowtie T})$$

$$= (\mu_X \otimes T) \circ (X \otimes t \otimes T) \circ (\mu_X \otimes Y \otimes \delta_T \otimes \varepsilon_G) \circ (X \otimes \psi_{Y,X} \otimes \psi_{G,T})$$

$$\circ (\psi_{Y,X} \otimes (\psi_{G,X} \circ (G \otimes t)) \otimes T) \circ (Y \otimes \psi_{G,X} \otimes Y \otimes \delta_T \otimes \varepsilon_G) \circ (Y \otimes G \otimes \psi_{Y,X} \otimes \psi_{G,T})$$

$\circ(Y\otimes G\otimes Y\otimes \psi_{G,X}\otimes T)$ ((b2) of Definition 2.4)

$= (\mu_X\otimes T)\circ(X\otimes t\otimes T)\circ(\mu_X\otimes Y\otimes \delta_T\otimes \varepsilon_G)\circ(X\otimes \psi_{Y,X}\otimes \psi_{G,T})$

$\circ(\psi_{Y,X}\otimes((t\otimes G)\circ(Y\otimes \psi_{G,T})\circ(\psi_{G,Y}\otimes T))\otimes T)\circ(Y\otimes \psi_{G,X}\otimes Y\otimes \delta_T\otimes \varepsilon_G)$

$\circ(Y\otimes G\otimes \psi_{Y,X}\otimes \psi_{G,T})\circ(Y\otimes G\otimes Y\otimes \psi_{G,X}\otimes T)$ ((15))

$= (\mu_X\otimes T)\circ(X\otimes(\mu_X\circ(X\otimes t)\circ(\psi_{Y,X}\otimes T)\circ(Y\otimes t\otimes T)\circ(Y\otimes Y\otimes \delta_T))\otimes T)$

$\circ(X\otimes Y\otimes Y\otimes \delta_T\otimes \varepsilon_G)\circ(\psi_{Y,X}\otimes Y\otimes \psi_{G,T}\otimes \varepsilon_G)\circ(Y\otimes((X\otimes \psi_{G,Y})\circ(\psi_{G,X}\otimes Y)$

$\circ(G\otimes \psi_{Y,X}))\otimes \psi_{G,T})\circ(Y\otimes G\otimes Y\otimes \psi_{G,X}\otimes T)$ ((a3) of Definition 2.4)

$= (\mu_X\otimes T)\circ(X\otimes(t\circ(\mu_Y\otimes T))\otimes T)\circ(X\otimes Y\otimes Y\otimes \delta_T\otimes \varepsilon_G)\circ(\psi_{Y,X}\otimes Y\otimes \psi_{G,T}\otimes \varepsilon_G)$

$\circ(Y\otimes((\psi_{Y,X}\otimes G)\circ(Y\otimes \psi_{G,X})\circ(\psi_{G,Y}\otimes X))\otimes \psi_{G,T})\circ(Y\otimes G\otimes Y\otimes \psi_{G,X}\otimes T)$

((14), (11))

$= (\mu_X\otimes T)\circ(X\otimes t\otimes T)\circ(X\otimes Y\otimes \delta_T\otimes(\varepsilon_G\circ\mu_G))\circ(\psi_{Y,X}\otimes \psi_{G,T}\otimes G)\circ(\mu_Y\otimes \psi_{G,X}\otimes \psi_{G,T})$

$\circ(Y\otimes \psi_{G,Y}\otimes \psi_{G,X}\otimes T)$ ((b2) of Definition 2.4, G is a Hopf monoid)

$= \phi_{X\bowtie T}\circ(\mu_{Y\bowtie G}\otimes X\otimes T)$ ((b2) of Definition 2.4),

and the proof is complete. ■

2.12. Lemma. *The following assertions are equivalent.*

(i) $\phi_{X\bowtie T}$ *is a monoid morphism.*

(ii) The equalities

$$t\circ(Y\otimes \eta_T) = \varepsilon_Y\otimes \eta_X, \tag{16}$$

$$t\circ(Y\otimes \mu_T) = \tag{17}$$

$$(\mu_X\otimes \varepsilon_T)\circ(t\otimes \psi_{T,X})\circ(Y\otimes \delta_T\otimes X)\circ(Y\otimes T\otimes t)\circ(Y\otimes c_{Y,T}\otimes T)\circ(\delta_Y\otimes T\otimes T)$$

and

$$\mu_X\circ(X\otimes t)\circ(\psi_{Y,X}\otimes T)\circ(Y\otimes \psi_{T,X}) \tag{18}$$

$$= (\mu_X\otimes \varepsilon_T)\circ(t\otimes \psi_{T,X}\otimes \varepsilon_Y)\circ(Y\otimes \delta_T\otimes \psi_{Y,X})\circ(Y\otimes c_{Y,T}\otimes X)\circ(\delta_Y\otimes T\otimes X),$$

hold.

PROOF. (i) ⇒ (ii). Assume that $\phi_{X\bowtie T}$ is a monoid morphism. Then

$$\varepsilon_Y \otimes \eta_X = (X \otimes \varepsilon_T) \circ \phi_{X\bowtie T} \circ (Y \otimes \eta_G \otimes \eta_X \otimes \eta_T) = t \circ (Y \otimes \eta_T)$$

and we get (16). Moreover, using that G and T are Hopf monoids, and (a1) and (b1) of Definition 2.4, we have that

$$\begin{aligned}
& t \circ (Y \otimes \mu_T) \\
&= (X \otimes \varepsilon_T) \circ \phi_{X\bowtie T} \circ (Y \otimes G \otimes \mu_{X\bowtie T}) \circ (Y \otimes \eta_G \otimes \eta_X \otimes T \otimes \eta_X \otimes T) \\
&= (X \otimes \varepsilon_T) \circ \mu_{X\bowtie T} \circ (\phi_{X\bowtie T} \otimes \phi_{X\bowtie T}) \circ (Y \otimes G \otimes c_{Y\otimes G, X\otimes T} \otimes X \otimes T) \\
&\quad \circ (\delta_{Y\otimes G} \otimes X \otimes T \otimes X \otimes T) \circ (Y \otimes \eta_G \otimes \eta_X \otimes T \otimes \eta_X \otimes T) \\
&= (\mu_X \otimes \varepsilon_T) \circ (t \otimes \psi_{T,X}) \circ (Y \otimes \delta_T \otimes X) \circ (Y \otimes T \otimes t) \circ (Y \otimes c_{Y,T} \otimes T) \circ (\delta_Y \otimes T \otimes T).
\end{aligned}$$

Finally,

$$\begin{aligned}
& \mu_X \circ (X \otimes t) \circ (\psi_{Y,X} \otimes T) \circ (Y \otimes \psi_{T,X}) \\
&= (X \otimes \varepsilon_T) \circ \phi_{X\bowtie T} \circ (Y \otimes G \otimes \mu_{X\bowtie T}) \circ (Y \otimes \eta_G \otimes \eta_X \otimes T \otimes X \otimes \eta_T) \\
&= (X \otimes \varepsilon_T) \circ \mu_{X\bowtie T} \circ (\phi_{X\bowtie T} \otimes \phi_{X\bowtie T}) \circ (Y \otimes G \otimes c_{Y\otimes G, X\otimes T} \otimes X \otimes T) \\
&\quad \circ (\delta_{Y\otimes G} \otimes X \otimes T \otimes X \otimes T) \circ (Y \otimes \eta_G \otimes \eta_X \otimes T \otimes X \otimes \eta_T) \\
&= (\mu_X \otimes \varepsilon_T) \circ (t \otimes \psi_{T,X} \otimes \varepsilon_Y) \circ (Y \otimes \delta_T \otimes \psi_{Y,X}) \circ (Y \otimes c_{Y,T} \otimes X) \circ (\delta_Y \otimes T \otimes X).
\end{aligned}$$

(ii) ⇒ (i). Using that T and X are Hopf monoids, (a1) and (b1) of Definition 2.4, and (16), it is easy to see that $\phi_{X\bowtie T} \circ (Y \otimes G \otimes \eta_X \otimes \eta_T) = \varepsilon_{Y\otimes G} \otimes \eta_X \otimes \eta_T$. On the other hand, the equality

$$\begin{aligned}
& (\mu_X \otimes T \otimes Y) \circ (t \otimes \psi_{T,X} \otimes Y) \circ (Y \otimes \delta_T \otimes \psi_{Y,X}) \circ (Y \otimes c_{Y,T} \otimes X) \circ (\delta_Y \otimes T \otimes X) \\
&\quad = (\mu_X \otimes T \otimes Y) \circ (X \otimes t \otimes c_{Y,T}) \circ (X \otimes \delta_{Y\otimes T}) \circ (\psi_{Y,X} \otimes T) \circ (Y \otimes \psi_{T,X}) \qquad (19)
\end{aligned}$$

holds. Indeed,

$$\begin{aligned}
& (\mu_X \otimes T \otimes Y) \circ (t \otimes \psi_{T,X} \otimes Y) \circ (Y \otimes \delta_T \otimes \psi_{Y,X}) \circ (Y \otimes c_{Y,T} \otimes X) \circ (\delta_Y \otimes T \otimes X) \\
&= (\mu_X \otimes T \otimes Y) \circ (t \otimes ((X \otimes \varepsilon_T \otimes T) \circ (\psi_{T,X} \otimes T) \circ (T \otimes c_{T,X}) \circ (\delta_T \otimes X)) \otimes Y) \\
&\quad \circ (Y \otimes \delta_T \otimes ((X \otimes \varepsilon_Y \otimes Y) \circ (\psi_{Y,X} \otimes Y) \circ (Y \otimes c_{Y,X}) \circ (\delta_Y \otimes X))) \circ (Y \otimes c_{Y,T} \otimes X) \\
&\quad \circ (\delta_Y \otimes T \otimes X) \ \text{((10))} \\
&= (((\mu_X \otimes \varepsilon_T) \circ (t \otimes \psi_{T,X} \otimes \varepsilon_Y) \circ (Y \otimes \delta_T \otimes \psi_{Y,X}) \circ (Y \otimes c_{Y,T} \otimes X) \circ (\delta_Y \otimes T \otimes X)) \otimes T \otimes Y) \\
&\quad \circ (Y \otimes T \otimes c_{T,X} \otimes Y) \circ (Y \otimes \delta_T \otimes c_{Y,X}) \circ (Y \otimes c_{Y,T} \otimes X) \circ (\delta_Y \otimes T \otimes X) \ \text{(naturality)}
\end{aligned}$$

$$= ((\mu_X \circ (X \otimes t) \circ (\psi_{Y,X} \otimes T) \circ (Y \otimes \psi_{T,X})) \otimes T \otimes Y) \circ (Y \otimes T \otimes c_{T,X} \otimes Y) \circ (Y \otimes \delta_T \otimes c_{Y,X})$$

$$\circ (Y \otimes c_{Y,T} \otimes X) \circ (\delta_Y \otimes T \otimes X) \text{ ((18))}$$

$$= (\mu_X \otimes T \otimes Y) \circ (X \otimes t \otimes c_{Y,T}) \circ (X \otimes \delta_{Y \otimes T}) \circ (\psi_{Y,X} \otimes T) \circ (Y \otimes \psi_{T,X}) \text{ ((9))}.$$

As a consequence,

$$\mu_{X \bowtie T} \circ (\phi_{X \bowtie T} \otimes \phi_{X \bowtie T}) \circ (Y \otimes G \otimes c_{Y \otimes G, X \otimes T} \otimes X \otimes T) \circ (\delta_{Y \otimes G} \otimes X \otimes T \otimes X \otimes T)$$

$$= \mu_{X \bowtie T} \circ ((\mu_X \circ (X \otimes t)) \otimes T \otimes (\mu_X \circ (X \otimes t)) \otimes T \otimes \varepsilon_G) \circ (X \otimes Y \otimes \delta_T \otimes \psi_{Y,X} \otimes \delta_T \otimes G)$$

$$\circ (X \otimes Y \otimes c_{Y,T} \otimes X \otimes \psi_{G,T}) \circ (X \otimes \delta_Y \otimes T \otimes \psi_{G,X} \otimes T) \circ (\psi_{Y,X} \otimes \psi_{G,T} \otimes X \otimes T)$$

$$\circ (Y \otimes \psi_{G,X} \otimes T \otimes X \otimes T) \text{ ((9), naturality)}$$

$$= (\mu_X \otimes T) \circ (X \otimes \mu_{X \bowtie T}) \circ (X \otimes X \otimes T \otimes ((t \otimes T) \circ (Y \otimes \delta_T)) \otimes \varepsilon_G)$$

$$\circ (X \otimes ((\mu_X \otimes T \otimes Y) \circ (t \otimes \psi_{T,X} \otimes Y) \circ (Y \otimes \delta_T \otimes \psi_{Y,X}) \circ (Y \otimes c_{Y,T} \otimes X)$$

$$\circ (\delta_Y \otimes T \otimes X)) \otimes \psi_{G,T}) \circ (X \otimes Y \otimes T \otimes \psi_{G,X} \otimes T) \circ (\psi_{Y,X} \otimes \psi_{G,T} \otimes X \otimes T)$$

$$\circ (Y \otimes \psi_{G,X} \otimes T \otimes X \otimes T) \text{ ((a2) of Definition 2.4)}$$

$$= (\mu_X \otimes T) \circ (X \otimes \mu_{X \bowtie T}) \circ (X \otimes X \otimes T \otimes ((t \otimes T) \circ (Y \otimes \delta_T)) \otimes \varepsilon_G)$$

$$\circ (X \otimes ((\mu_X \otimes T \otimes Y) \circ (X \otimes t \otimes c_{Y,T}) \circ (X \otimes \delta_{Y \otimes T}) \circ (\psi_{Y,X} \otimes T) \circ (Y \otimes \psi_{T,X})) \otimes \psi_{G,T})$$

$$\circ (X \otimes Y \otimes T \otimes \psi_{G,X} \otimes T) \circ (\psi_{Y,X} \otimes \psi_{G,T} \otimes X \otimes T) \circ (Y \otimes \psi_{G,X} \otimes T \otimes X \otimes T) \text{ ((19))}$$

$$= (\mu_X \otimes T) \circ (X \otimes ((\mu_X \otimes \varepsilon_T) \circ (t \otimes \psi_{T,X}) \circ (Y \otimes \delta_T \otimes X) \circ (Y \otimes T \otimes t) \circ (Y \otimes c_{Y,T} \otimes T)$$

$$\circ (\delta_Y \otimes T \otimes T)) \otimes \mu_T) \circ (X \otimes Y \otimes \delta_{T \otimes T} \otimes \varepsilon_G) \circ (\psi_{Y,X} \otimes T \otimes \psi_{G,T})$$

$$\circ (Y \otimes ((X \otimes \psi_{G,T}) \circ (\psi_{G,X} \otimes T) \circ (G \otimes \mu_X \otimes T)) \otimes T) \circ (Y \otimes G \otimes X \otimes \psi_{T,X} \otimes T)$$

((10), (b2) of Definition 2.4, (11))

$$= (\mu_X \otimes T) \circ (X \otimes (t \circ (Y \otimes \mu_T)) \otimes \mu_T) \circ (X \otimes Y \otimes \delta_{T \otimes T} \otimes \varepsilon_G) \circ (\psi_{Y,X} \otimes T \otimes \psi_{G,T})$$

$$\circ (Y \otimes ((X \otimes \psi_{G,T}) \circ (\psi_{G,X} \otimes T) \circ (G \otimes \mu_X \otimes T)) \otimes T) \circ (Y \otimes G \otimes X \otimes \psi_{T,X} \otimes T)$$

(by (17))

$$= \phi_{X \bowtie T} \circ (Y \otimes G \otimes \mu_{X \bowtie T}) \text{ (T is a Hopf monoid, (b2) of Definition 2.4).}$$

■

2.13. LEMMA. *The following assertions are equivalent.*

(i) $\phi_{X\bowtie T}$ *is a comonoid morphism.*

(ii) The map t *is a comonoid morphism and the equality*

$$c_{X,T}\circ(t\otimes T)\circ(Y\otimes\delta_T)=(T\otimes t)\circ(c_{Y,T}\otimes T)\circ(Y\otimes\delta_T) \tag{20}$$

holds.

PROOF. (i) ⇒ (ii). In a similar way to the 'if' part of Lemma 2.11, we get that $\varepsilon_X\circ t=\varepsilon_Y\otimes\varepsilon_T$ by composing $\phi_{X\bowtie T}$ with $\varepsilon_X\otimes\varepsilon_T$ on the left and with $Y\otimes\eta_G\otimes\eta_X\otimes T$ on the right and using that $\phi_{X\bowtie T}$ is a comonoid morphism. On the other hand, by (a1) and (b1) of Definition 2.4 and taking into account that G and T are Hopf monoids,

$$\begin{aligned}&\delta_X\circ t\\ &=(X\otimes\varepsilon_T\otimes X\otimes\varepsilon_T)\circ\delta_{X\otimes T}\circ\phi_{X\bowtie T}\circ(Y\otimes\eta_G\otimes\eta_X\otimes T)\\ &=(X\otimes\varepsilon_T\otimes X\otimes\varepsilon_T)\circ(\phi_{X\bowtie T}\otimes\phi_{X\bowtie T})\circ\delta_{Y\otimes G\otimes X\otimes T}\circ(Y\otimes\eta_G\otimes\eta_X\otimes T)\\ &=(t\otimes t)\circ\delta_{Y\otimes T}.\end{aligned}$$

Finally, using (a1), (b1) of Definition 2.4 and that $\varepsilon_X\circ t=\varepsilon_Y\otimes\varepsilon_T$,

$$\begin{aligned}&c_{X,T}\circ(t\otimes T)\circ(Y\otimes\delta_T)\\ &=(\varepsilon_X\otimes T\otimes X\otimes\varepsilon_T)\circ\delta_{X\otimes T}\circ\phi_{X\bowtie T}\circ(Y\otimes\eta_G\otimes\eta_X\otimes T)\\ &=(\varepsilon_X\otimes T\otimes X\otimes\varepsilon_T)\circ(\phi_{X\bowtie T}\otimes\phi_{X\bowtie T})\circ\delta_{Y\otimes G\otimes X\otimes T}\circ(Y\otimes\eta_G\otimes\eta_X\otimes T)\\ &=(T\otimes t)\circ(c_{Y,T}\otimes T)\circ(Y\otimes\delta_T).\end{aligned}$$

(ii) ⇒ (i). Using that X and T are Hopf monoids, t a comonoid morphism and (a4) of Definition 2.4, it is easy to see that $(\varepsilon_X\otimes\varepsilon_T)\circ\phi_{X\bowtie T}=\varepsilon_Y\otimes\varepsilon_G\otimes\varepsilon_X\otimes\varepsilon_T$. Moreover,

$$\begin{aligned}&\delta_{X\bowtie T}\circ\phi_{X\bowtie T}\\ &=(X\otimes c_{X,T}\otimes T)\circ(\mu_{X\otimes X}\otimes\delta_T)\circ(\delta_X\otimes t\otimes t\otimes T)\circ(X\otimes\delta_{Y\otimes T}\otimes T)\circ(X\otimes Y\otimes\delta_T\otimes\varepsilon_G)\\ &\quad\circ(\psi_{Y,X}\otimes\psi_{G,T})\circ(Y\otimes\psi_{G,X}\otimes T)\ \text{(X is a Hopf monoid, t is a comonoid morphism)}\\ &=(X\otimes c_{X,T}\otimes T)\circ(\mu_{X\otimes X}\otimes T\otimes T)\circ(X\otimes X\otimes t\otimes((t\otimes T)\circ(Y\otimes\delta_T))\otimes T)\\ &\quad\circ(X\otimes c_{Y,X}\otimes c_{Y,T}\otimes\delta_T)\circ(\psi_{Y,X}\otimes\psi_{Y,X}\otimes\delta_T\otimes\varepsilon_G)\circ(\delta_{Y\otimes X}\otimes\psi_{G,T})\circ(Y\otimes\psi_{G,X}\otimes T)\\ &\quad\text{((8), (9))}\\ &=(X\otimes c_{X,T}\otimes T)\circ(\mu_{X\otimes X}\otimes T\otimes T)\circ(X\otimes X\otimes t\otimes(c_{T,X}\circ(T\otimes t)\circ(c_{Y,T}\otimes T)\circ(Y\otimes\delta_T))\otimes T)\end{aligned}$$

$$\circ(X \otimes c_{Y,X} \otimes c_{Y,T} \otimes \delta_T) \circ (\psi_{Y,X} \otimes \psi_{Y,X} \otimes \delta_T \otimes \varepsilon_G) \circ (\delta_{Y\otimes X} \otimes \psi_{G,T}) \circ (Y \otimes \psi_{G,X} \otimes T)$$

((20))

$$= ((\mu_X \circ (X \otimes t)) \otimes T \otimes (\mu_X \circ (X \otimes t)) \otimes T) \circ (\psi_{Y,X} \otimes \delta_T \otimes \psi_{Y,X} \otimes \delta_T)$$

$$\circ(Y \otimes X \otimes ((c_{Y,T} \otimes X) \circ (Y \otimes c_{X,T})) \otimes \varepsilon_G \otimes T \otimes \varepsilon_G) \circ (\delta_{Y\otimes X} \otimes \psi_{G,T} \otimes \psi_{G,T}) \circ (Y \otimes X \otimes \delta_{G\otimes T})$$

$$\circ(Y \otimes \psi_{G,X} \otimes T) \quad \text{((a3) of Definition 2.4, (10)),}$$

$$= (\phi_{X\bowtie T} \otimes \phi_{X\bowtie T}) \circ \delta_{Y\otimes G\otimes X\otimes T} \quad \text{((9), (8)),}$$

and then $\phi_{X\bowtie T}$ is a comonoid morphism.

■

2.14. LEMMA. *If (12) and (16) hold, the following assertions are equivalent.*

(i) $(X \bowtie T, \phi_{X\bowtie T})$ *is in the cocommutativity class of* $Y \otimes G$.

(ii) $\psi_{Y,X}$ *is in the cocommutativity class of* Y, $\psi_{G,T}$ *and* $\psi_{G,X}$ *are in the cocommutativity class of* G *and the equality*

$$(t \otimes Y) \circ (Y \otimes c_{Y,T}) \circ (\delta_Y \otimes T) = c_{Y,X} \circ (Y \otimes t) \circ (\delta_Y \otimes T) \tag{21}$$

holds.

PROOF. (i) ⇒ (ii). Using that T and G are Hopf monoids, (a1) and (b1) of Definition 2.4 and the hypothesis we obtain (21) because

$$(t \otimes Y) \circ (Y \otimes c_{Y,T}) \circ (\delta_Y \otimes T)$$

$$= (X \otimes \varepsilon_T \otimes Y \otimes \varepsilon_G) \circ (\phi_{X\bowtie T} \otimes Y \otimes G) \circ (Y \otimes G \otimes c_{Y\otimes G,X\otimes T}) \circ (\delta_{Y\otimes G} \otimes X \otimes T)$$

$$\circ(Y \otimes \eta_G \otimes \eta_X \otimes T)$$

$$= (X \otimes \varepsilon_T \otimes Y \otimes \varepsilon_G) \circ c_{Y\otimes G,X\otimes T} \circ (Y \otimes G \otimes \phi_{X\bowtie T}) \circ (\delta_{Y\otimes G} \otimes X \otimes T) \circ (Y \otimes \eta_G \otimes \eta_X \otimes T)$$

$$= c_{Y,X} \circ (Y \otimes t) \circ (\delta_Y \otimes T).$$

To finish the 'if' part we will show that $\psi_{Y,X}$ is in the cocommutativity class of Y. The proof for $\psi_{G,T}$ and $\psi_{G,X}$ follows a similar pattern. First of all, consider the cocommutativity class condition for $\phi_{X\bowtie T}$. By composing with $X \otimes \varepsilon_T \otimes Y \otimes \varepsilon_G$ on the left and with $Y \otimes \eta_G \otimes X \otimes \eta_T$ on the right, and using that T and G are Hopf monoids, (a1) of Definition 2.4, (10) and (16), we obtain that $\psi_{Y,X} = (c_{Y,X} \otimes \varepsilon_Y) \circ (Y \otimes \psi_{Y,X}) \circ (\delta_Y \otimes X)$. Then, by coassociativity and (10),

$$(\psi_{Y,X} \otimes Y) \circ (Y \otimes c_{Y,X}) \circ (\delta_Y \otimes X)$$

$$= (((c_{Y,X} \otimes \varepsilon_Y) \circ (Y \otimes \psi_{Y,X}) \circ (\delta_Y \otimes X)) \otimes Y) \circ (Y \otimes c_{Y,X}) \circ (\delta_Y \otimes X)$$

$= (((c_{Y,X} \otimes \varepsilon_Y) \circ (Y \otimes \psi_{Y,X})) \otimes Y) \circ (Y \otimes Y \otimes c_{Y,X}) \circ (Y \otimes \delta_Y \otimes X) \circ (\delta_Y \otimes X)$

$= (c_{Y,X} \otimes Y) \circ (Y \otimes \psi_{Y,X}) \circ (\delta_Y \otimes X).$

(ii) $\Rightarrow$ (i). Indeed,

$c_{Y\otimes G,X\otimes T} \circ (Y \otimes G \otimes \phi_{X\bowtie T}) \circ (\delta_{Y\otimes G} \otimes X \otimes T)$

$= (X\otimes c_{Y,T}\otimes G)\circ(c_{Y,X}\otimes c_{G,T})\circ(Y\otimes(c_{G,X}\circ(G\otimes\mu_X)\circ(c_{X,G}\otimes t)\circ(X\otimes c_{Y,G}\otimes T))\otimes T)$

$\circ(Y\otimes\psi_{Y,X}\otimes G\otimes\delta_T\otimes\varepsilon_G)\circ(\delta_Y\otimes\psi_{G,X}\otimes\psi_{G,T})\circ(Y\otimes((G\otimes c_{G,X})\circ(\delta_G\otimes X))\otimes T)$

($\psi_{G,X}$ is in the cocommutativity class of G)

$= (X \otimes c_{Y,T} \otimes G) \circ (c_{Y,X} \otimes T \otimes G) \circ (Y \otimes (\mu_X \circ (X \otimes t) \circ (\psi_{Y,X} \otimes T)) \otimes T \otimes G)$

$\circ(\delta_Y \otimes X \otimes \delta_T \otimes G)(Y \otimes ((c_{T,X} \otimes G) \circ (T \otimes \psi_{G,X}) \circ (\psi_{G,T} \otimes X) \circ (G \otimes c_{X,T})))$

($\psi_{G,T}$ is in the cocommutativity class of G)

$= ((\mu_X\circ c_{X,X})\otimes c_{Y,T}\otimes G)\circ(X\otimes\psi_{Y,X}\otimes T\otimes G)\circ((c_{Y,X}\circ(Y\otimes t)\circ(\delta_Y\otimes T))\otimes c_{T,X}\otimes G)$

$\circ(Y \otimes \delta_T \otimes \psi_{G,X}) \circ (Y \otimes \psi_{G,T} \otimes X) \circ (Y \otimes G \otimes c_{X,T})$ ($\psi_{Y,X}$ is in the cocommutativity class of X)

$= ((\mu_X\circ c_{X,X})\otimes c_{Y,T}\otimes G)\circ(X\otimes\psi_{Y,X}\otimes T\otimes G)\circ(((t\otimes Y)\circ(Y\otimes c_{Y,T})\circ(\delta_Y\otimes T))\otimes c_{T,X}\otimes G)$

$\circ(Y \otimes \delta_T \otimes \psi_{G,X}) \circ (Y \otimes \psi_{G,T} \otimes X) \circ (Y \otimes G \otimes c_{X,T})$ ((21))

$= ((\mu_X\circ c_{X,X})\otimes T\otimes Y\otimes G)\circ(t\otimes c_{T,X}\otimes Y\otimes G)\circ(Y\otimes\delta_T\otimes\psi_{Y,X}\otimes G)\circ(Y\otimes c_{Y,T}\otimes\psi_{G,X})$

$\circ(\delta_Y \otimes ((T \otimes \varepsilon_G \otimes G) \circ (\psi_{G,T} \otimes G) \circ (G \otimes c_{G,T}) \circ (\delta_G \otimes T)) \otimes X) \circ (Y \otimes G \otimes c_{X,T})$

((10))

$= (((\mu_X \otimes T) \circ (X \otimes t \otimes T) \circ (X \otimes Y \otimes \delta_T)) \otimes Y \otimes G) \circ (X \otimes Y \otimes c_{Y,T} \otimes \varepsilon_G \otimes G)$

$\circ(((\psi_{Y,X} \otimes Y) \circ (Y \otimes c_{Y,X}) \circ (\delta_Y \otimes X)) \otimes ((\psi_{G,T} \otimes G) \circ (G \otimes c_{G,T})))$

$\circ(Y \otimes ((\psi_{G,X} \otimes G) \circ (G \otimes c_{G,X}) \circ (\delta_G \otimes X)) \otimes T)$

($\psi_{G,X}$ and $\psi_{Y,X}$ are in the cocommutativity class of G and Y, respectively)

$= (\phi_{X\bowtie T} \otimes Y \otimes G) \circ (Y \otimes G \otimes c_{Y\otimes G,X\otimes T}) \circ (\delta_{Y\otimes G} \otimes X \otimes T)$ (naturality).

■

2.15. LEMMA. *The following assertions are equivalent.*

(i) $(\beta \otimes \partial) \circ \phi_{X \bowtie T} = ad_{Y \bowtie G} \circ (Y \otimes G \otimes \beta \otimes \partial)$

(ii) The equalities

$$((\beta \circ t) \otimes \partial) \circ (Y \otimes \delta_T) = (\mu_Y \otimes G) \circ (Y \otimes (\psi_{G,Y} \circ c_{Y,G} \circ (\lambda_Y \otimes \partial))) \circ (\delta_Y \otimes T) \quad (22)$$

and

$$(\beta \otimes G) \circ \psi_{G,X} = \psi_{G,Y} \circ (G \otimes \beta) \quad (23)$$

hold.

PROOF. (i) ⇒ (ii). Using that β is a monoid morphism, G and T Hopf monoids, and (a1) and (b1) of Definition 2.4, we get (22) because

$$\begin{aligned} &((\beta \circ t) \otimes \partial) \circ (Y \otimes \delta_T) \\ &= (\beta \otimes \partial) \circ \phi_{X \bowtie T} \circ (Y \otimes \eta_G \otimes \eta_X \otimes T) \\ &= ad_{Y \bowtie G} \circ (Y \otimes G \otimes \beta \otimes \partial) \circ (Y \otimes \eta_G \otimes \eta_X \otimes T) \\ &= (\mu_Y \otimes G) \circ (Y \otimes (\psi_{G,Y} \circ c_{Y,G} \circ (\lambda_Y \otimes \partial))) \circ (\delta_Y \otimes T), \end{aligned}$$

On the other hand, by composing with $Y \otimes \varepsilon_G$ on the left and with $Y \otimes \eta_T$ on the right we obtain that $\beta \circ t \circ (Y \otimes \eta_T) = \varepsilon_Y \otimes \eta_Y$ and then

$$\begin{aligned} &(\beta \otimes \varepsilon_G) \circ \psi_{G,X} \\ &= (Y \otimes \varepsilon_G) \circ (\beta \otimes \partial) \circ \phi_{X \bowtie T} \circ (\eta_Y \otimes G \otimes X \otimes \eta_T) \\ &= (Y \otimes \varepsilon_G) \circ ad_{Y \bowtie G} \circ (Y \otimes G \otimes \beta \otimes \partial) \circ (\eta_Y \otimes G \otimes X \otimes \eta_T) \\ &= (Y \otimes \varepsilon_G) \circ \psi_{G,Y} \circ (G \otimes \beta). \end{aligned}$$

As a consequence, using (10),

$$\begin{aligned} &(\beta \otimes G) \circ \psi_{G,X} \\ &= (\beta \otimes \varepsilon_G \otimes G) \circ (\psi_{G,X} \otimes G) \circ (G \otimes c_{G,X}) \circ (\delta_G \otimes X) \\ &= (Y \otimes \varepsilon_G \otimes G) \circ (\psi_{G,Y} \otimes G) \circ (G \otimes \beta \otimes G) \circ (G \otimes c_{G,X}) \circ (\delta_G \otimes X) \\ &= \psi_{G,Y} \circ (G \otimes \beta), \end{aligned}$$

and we obtain (23).

(ii) ⇒ (i). First of all, by (10) and (c2) of Definition 2.8 it is easy to see that

$$(\beta \otimes Y) \circ \psi_{Y,X} = (ad_Y \otimes Y) \circ (Y \otimes c_{Y,Y}) \circ (\delta_Y \otimes \beta). \quad (24)$$

Therefore

$$(\beta\otimes\partial)\circ\phi_{X\bowtie T}$$

$$=(\mu_Y\otimes G)\circ(\beta\otimes(((\beta\circ t)\otimes\partial)\circ(Y\otimes\delta_T))\otimes\varepsilon_G)\circ(\psi_{Y,X}\otimes\psi_{G,T})\circ(Y\otimes\psi_{G,X}\otimes T)$$

(β is a monoid morphism)

$$=(\mu_Y\otimes G)\circ(ad_Y\otimes((\mu_Y\otimes G)\circ(Y\otimes(\psi_{G,Y}\circ c_{Y,G}\circ(\lambda_Y\otimes\partial)))\circ(\delta_Y\otimes T))\otimes\varepsilon_G)$$

$$\circ(Y\otimes c_{Y,Y}\otimes\psi_{G,T})\circ(\delta_Y\otimes((\beta\otimes G)\circ\psi_{G,X})\otimes T)$$ ((22), (24))

$$=(\mu_Y\otimes G)\circ((\mu_Y\circ(\mu_Y\otimes(\mu_Y\circ(\lambda_Y\otimes Y)\circ\delta_Y)))\otimes(\psi_{G,Y}\circ c_{Y,G}))\circ(Y\otimes c_{Y,Y}\otimes\lambda_Y\otimes ad_G)$$

$$\circ(\delta_Y\otimes c_{Y,Y}\otimes G\otimes G)\circ(\delta_Y\otimes(\psi_{G,Y}\circ(G\otimes\beta))\otimes\partial)$$ ((23), (c2) of Definition 2.8)

$$=(\mu_Y\otimes G)\circ(Y\otimes(\psi_{G,Y}\circ c_{Y,G}))\circ(\mu_Y\otimes\lambda_Y\otimes\mu_G)\circ(Y\otimes c_{Y,Y}\otimes\mu_G\otimes G)$$

$$\circ(\delta_Y\otimes Y\otimes G\otimes(c_{G,G}\circ(\lambda_G\otimes G)))\circ(Y\otimes((\psi_{G,Y}\otimes G)\circ(G\otimes c_{G,Y})\circ(\delta_G\otimes Y))\otimes G)$$

$$\circ(Y\otimes G\otimes\beta\otimes\partial)$$ (Y is a Hopf monoid, (9))

$$=ad_{Y\bowtie G}\circ(Y\otimes G\otimes\beta\otimes\partial)$$ ((b2) of Definition 2.4).

■

2.16. Lemma. *The following assertions are equivalent.*

(i) $\phi_{X\bowtie T}\circ(\beta\otimes\partial\otimes X\otimes T)=ad_{X\bowtie T}$

(ii) The equalities

$$(t\otimes T)\circ(\beta\otimes\delta_T)=(\mu_X\otimes T)\circ(X\otimes(\psi_{T,X}\circ c_{X,T}\circ(\lambda_X\otimes T)))\circ(\delta_X\otimes T)\quad(25)$$

and

$$\psi_{G,X}\circ(\partial\otimes X)=(X\otimes\partial)\circ\psi_{T,X}\quad(26)$$

hold.

Proof. (i) ⇒ (ii). Using that ∂ is a monoid morphism, G a Hopf monoid, and (a1) and (b1) of Definition 2.4,

$$(t\otimes T)\circ(\beta\otimes\delta_T)$$

$$=\phi_{X\bowtie T}\circ(\beta\otimes\partial\otimes X\otimes T)\circ(X\otimes\eta_T\otimes\eta_X\otimes T)$$

$$=ad_{X\bowtie T}\circ(X\otimes\eta_T\otimes\eta_X\otimes T)$$

$$=(\mu_X\otimes T)\circ(X\otimes(\psi_{T,X}\circ c_{X,T}\circ(\lambda_X\otimes T)))\circ(\delta_X\otimes T),$$

and we get (25). By composing in this equality with $X\otimes\varepsilon_T$ on the left and with $\eta_X\otimes T$ on the right we obtain that $t\circ(\eta_Y\otimes T)=\varepsilon_T\otimes\eta_X$ and then

$$(X \otimes \varepsilon_G) \circ \psi_{G,X} \circ (\partial \otimes X)$$
$$= (X \otimes \varepsilon_T) \circ \phi_{X \bowtie T} \circ (\beta \otimes \partial \otimes X \otimes T) \circ (\eta_X \otimes T \otimes X \otimes \eta_T)$$
$$= (X \otimes \varepsilon_T) \circ ad_{X \bowtie T} \circ (\eta_X \otimes T \otimes X \otimes \eta_T)$$
$$= (X \otimes \varepsilon_T) \circ \psi_{T,X}.$$

As a consequence, using (10) and that ∂ is a comonoid morphism,

$$(X \otimes \partial) \circ \psi_{T,X}$$
$$= (X \otimes \varepsilon_T \otimes \partial) \circ (\psi_{T,X} \otimes T) \circ (T \otimes c_{T,X}) \circ (\delta_T \otimes X)$$
$$= (X \otimes \varepsilon_G \otimes \partial) \circ (\psi_{G,X} \otimes T) \circ (\partial \otimes c_{T,X}) \circ (\delta_T \otimes X)$$
$$= \psi_{G,X} \circ (\partial \otimes X),$$

and we obtain (26).

(ii) $\Rightarrow$ (i). In a similar way to (24) it is easy to see that

$$\psi_{Y,X} \circ (\beta \otimes X) = (ad_X \otimes \beta) \circ (X \otimes c_{X,X}) \circ (\delta_X \otimes X). \tag{27}$$

Therefore

$$\phi_{X \bowtie T} \circ (\beta \otimes \partial \otimes X \otimes T)$$
$$= (\mu_X \otimes T) \circ (X \otimes ((t \otimes T) \circ (\beta \otimes \delta_T))) \circ (((ad_X \otimes X) \circ (X \otimes c_{X,X}) \circ (\delta_X \otimes X)) \otimes ((T \otimes \varepsilon_G)$$
$$\circ \psi_{G,T} \circ (\partial \otimes T)) \circ (X \otimes \psi_{T,X} \otimes T) \quad \text{((26), (27))}$$
$$= (\mu_X \otimes T) \circ (X \otimes ((\mu_X \otimes T) \circ (X \otimes (\psi_{T,X} \circ c_{X,T} \circ (\lambda_X \otimes T))) \circ (\delta_X \otimes T)))$$
$$\circ (((ad_X \otimes X) \circ (X \otimes c_{X,X}) \circ (\delta_X \otimes X)) \otimes ad_T) \circ (X \otimes \psi_{T,X} \otimes T) \quad \text{((c3) of Definition 2.8, (25))}$$
$$= (\mu_X \otimes T) \circ (\mu_X \otimes X \otimes T) \circ (X \otimes (\mu_X \circ (\lambda_X \otimes X) \circ \delta_X)) \otimes (\psi_{T,X} \circ c_{X,T} \circ (\lambda_X \otimes T)))$$
$$\circ (\mu_X \otimes \delta_X \otimes ad_T) \circ (X \otimes c_{X,X} \otimes T \otimes T) \circ (\delta_X \otimes \psi_{T,X} \otimes T) \quad \text{(naturality)}$$
$$= (\mu_X \otimes T) \circ (X \otimes (\psi_{T,X} \circ c_{X,T} \circ (\lambda_X \otimes T))) \circ (((\mu_X \otimes X) \circ (X \otimes c_{X,X})$$
$$\circ (\delta_X \otimes X)) \otimes (\mu_T \circ (\mu_T \otimes \lambda_T) \circ (T \otimes c_{T,T}) \circ (\delta_T \otimes T))) \circ (X \otimes \psi_{T,X} \otimes T) \quad (X \text{ is a Hopf monoid})$$
$$= ad_{X \bowtie T} \quad \text{((b2) of Definition 2.4, (9))},$$

and the proof is complete.

■

As a direct consequence of the Lemma we have the main result of this section that gives the necessary and sufficient conditions to assure that a crossed product of two crossed modules of Hopf monoids is a crossed module of Hopf monoids.

2.17. THEOREM. *Let* $\mathbf{T_G} = (T, G, \partial)$ *and* $\mathbf{X_Y} = (X, Y, \beta)$ *be crossed modules of Hopf monoids and denote the corresponding entwining structures by* $\psi_{G,T} : G \otimes T \to T \otimes G$ *and* $\psi_{Y,X} : Y \otimes X \to X \otimes Y$*, respectively. Let* $t : Y \otimes T \to X$ *be a morphism and assume that* $\psi_{G,X} : G \otimes X \to X \otimes G$*,* $\psi_{T,X} : T \otimes X \to X \otimes T$ *and* $\psi_{G,Y} : G \otimes Y \to Y \otimes G$ *are three left-left entwining structures and right-right semi-entwining structures such that (8), (9) and (11) hold. Then the following assertions are equivalent.*

(i) $\mathbf{X_Y} \bowtie \mathbf{T_G} = (X \bowtie T, Y \bowtie G, \beta \otimes \partial)$ *is a crossed module of Hopf monoids via* $\phi_{X \bowtie T}$*.*

(ii) The morphism t *is a comonoid morphism and the equalities (12), (13), (14), (16), (17), (18), (20), (22), (23), (25) and (26) hold.*

Moreover, in this case, $(X \bowtie T, \phi_{X \bowtie T})$ *is in the cocommutativity class of* $Y \bowtie G$ *if and only if* $\psi_{Y,X}$ *is in the cocommutativity class of* Y*,* $\psi_{G,T}$ *and* $\psi_{G,X}$ *are in the cocommutativity class of* G *and condition (21) holds.*

3. Projections of crossed modules of Hopf monoids

Projections of Hopf algebras were introduced by Radford [Radford, 1985]. In this section we develop an example of crossed product of crossed modules by working with projections of Hopf monoids. First of all we recall the notion and the main properties of a projection of Hopf monoids. The interested reader can see the proofs in [Alonso Álvarez, Fernández Vilaboa, 2000], [Majid, 1994] or [Radford, 1985].

3.1. DEFINITION. A *projection of Hopf monoids* is a quartet (T, B, u, w) where T, B are Hopf monoids, and $u : T \to B$, $w : B \to T$ are Hopf monoid morphisms such that $w \circ u = id_T$.

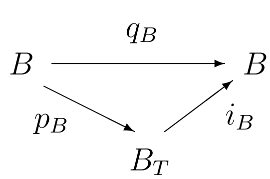
A morphism between projections of Hopf monoids (T, B, u, w) and (G, H, v, y) is a pair (∂, γ), where $\partial : T \to G$, $\gamma : B \to H$ are Hopf monoid morphisms such that

$$v \circ \partial = \gamma \circ u, \quad \partial \circ w = y \circ \gamma. \tag{28}$$

Let (T, B, u, w) be a projection of Hopf monoids. The morphism

$$q_B = \mu_B \circ (B \otimes (u \circ \lambda_T \circ w)) \circ \delta_B$$

is an idempotent and, as a consequence, there exist an epimorphism p_B, a monomorphism i_B, and an object B_T such that the diagram



commutes and $p_B \circ i_B = id_{B_T}$. Also,

$$B_T \xrightarrow{\quad i_B \quad} B \overset{(B\otimes w)\circ\delta_B}{\underset{B\otimes\eta_T}{\rightrightarrows}} B\otimes T$$

is an equalizer diagram and

$$B\otimes T \overset{\mu_B\circ(B\otimes u)}{\underset{B\otimes\varepsilon_T}{\rightrightarrows}} B \xrightarrow{\quad p_B \quad} B_T$$

is a coequalizer diagram.

It is easy to see that i_B (p_B) is a monoid and comonoid morphism, where the monoid and comonoid structures in B_T are $\eta_{B_T} = p_B \circ \eta_B$, $\mu_{B_T} = p_B \circ \mu_B \circ (i_B \otimes i_B)$, and $\varepsilon_{B_T} = \varepsilon_B \circ i_B$, $\delta_{B_T} = (p_B \otimes p_B) \circ \delta_B \circ i_B$, respectively. Also, the equalities

$$\delta_B \circ i_B = (B \otimes q_B) \circ \delta_B \circ i_B, \;\; p_B \circ \mu_B = p_B \circ \mu_B \circ (B \otimes q_B) \tag{29}$$

hold. On the other hand, the morphism $ad_{u,B} \circ (T \otimes i_B)$ factorizes through the equalizer i_B, and the factorization $\phi_{B_T} = p_B \circ \mu_B \circ (u \otimes i_B) : T \otimes B_T \to B_T$ gives a left H-module monoid and comonoid structure for B_T. Moreover, if i_B is a comonoid morphism, B_T is a Hopf monoid with antipode $\lambda_{B_T} = p_B \circ \lambda_B \circ i_B$ and, as a consequence of Proposition 2.3, (B_T, ϕ_{B_T}) is in the cocommutativity class of T. Finally, there is a Hopf monoid isomorphism between $B_T \bowtie T$ and B defined as $\pi_B = \mu_B \circ (i_B \otimes u)$ and with inverse $\pi_B^{-1} = (p_B \otimes w) \circ \delta_B$.

Now we apply the general theory of Hopf monoid projections to study projections between crossed modules of Hopf monoids.

3.2. Definition. Let $\mathbf{T_G} = (T, G, \partial)$ and $\mathbf{B_H} = (B, H, \gamma)$ be crossed modules of Hopf monoids and assume that (T, B, u, w) and (G, H, v, y) are projections of Hopf monoids. We say that

$$(\mathbf{B_H}, \mathbf{T_G}, (u, v), (w, y))$$

is a *projection of crossed modules of Hopf monoids* if (∂, γ) is a morphism between (T, B, u, w) and (G, H, v, y) such that the equalities

$$(u\otimes\varepsilon_G)\circ\psi_{G,T} = (B\otimes\varepsilon_H)\circ\psi_{H,B}\circ(v\otimes u), \;\; (w\otimes\varepsilon_H)\circ\psi_{H,B} = (T\otimes\varepsilon_G)\circ\psi_{G,T}\circ(y\otimes w), \tag{30}$$

hold (equivalently, if ϕ_T and ϕ_B are the left G-module and H-module monoid and comonoid structures for T and B, respectively, and the following equalities hold:

$$u \circ \phi_T = \phi_B \circ (v \otimes u), \;\; w \circ \phi_B = \phi_T \circ (y \otimes w)). \tag{31}$$

3.3. Proposition. *Let* $\mathbf{T_G} = (T, G, \partial)$ *and* $\mathbf{B_H} = (B, H, \gamma)$ *be crossed modules of Hopf monoids and denote the corresponding entwining structures by* $\psi_{G,T} : G \otimes T \to T \otimes G$ *and* $\psi_{Y,X} : Y \otimes X \to X \otimes Y$*, respectively. Let* $(\mathbf{B_H}, \mathbf{T_G}, (u, v), (w, y))$ *be a projection of crossed modules of Hopf monoids such that* i_B *and* i_H *are comonoid morphisms. Then*

$$(B_T, H_G, \sigma = p_H \circ \gamma \circ i_B)$$

is a crossed module of Hopf monoids and the left H_G*-module monoid and comonoid structure for* B_T*, defined as* $\varphi_{B_T} = p_B \circ \phi_B \circ (i_H \otimes i_B)$*, is in the cocommutativity class of* H_G*.*

Proof. Taking into account that p_H, γ and i_B are comonoid morphisms, so is σ. Moreover, the equality

$$q_H \circ \gamma = \gamma \circ q_B \tag{32}$$

holds. Indeed,

$$
\begin{aligned}
& q_H \circ \gamma \\
&= \mu_H \circ (\gamma \otimes (v \circ \lambda_G \circ y \circ \gamma)) \circ \delta_B \text{ (γ is a comonoid morphism)} \\
&= \mu_H \circ (\gamma \otimes (v \circ \lambda_G \circ \partial \circ w)) \circ \delta_B \text{ ((28))} \\
&= \mu_H \circ (\gamma \otimes (\gamma \circ u \circ \lambda_T \circ w)) \circ \delta_B \text{ (∂ is a Hopf monoid morphism, (28))} \\
&= \gamma \circ q_B \text{ (γ is a monoid morphism).}
\end{aligned}
$$

As a consequence it is easy to see that σ is a monoid morphism and then we get that σ is a Hopf monoid morphism. Now consider the morphism $\varphi_{B_T} = p_B \circ \phi_B \circ (i_H \otimes i_B)$. Trivially, $\varphi_{B_T} \circ (\eta_{H_G} \otimes B_T) = id_{B_T}$. Moreover, if ϕ_B is the left H-module monoid and comonoid structure for B, the equality

$$q_B \circ \phi_B \circ (H \otimes i_B) = \phi_B \circ (H \otimes i_B) \tag{33}$$

holds, because

$$
\begin{aligned}
& q_B \circ \phi_B \circ (H \otimes i_B) \\
&= \mu_B \circ (\phi_B \otimes (u \circ \lambda_T \circ w \circ \phi_B)) \circ \delta_{H \otimes B} \circ (H \otimes i_B) \text{ (ϕ_B is a comonoid morphism)} \\
&= \mu_B \circ (\phi_B \otimes (u \circ \lambda_T \circ \phi_T \circ (y \otimes w))) \circ \delta_{H \otimes B} \circ (H \otimes i_B) \text{ ((30))} \\
&= \phi_B \circ (H \otimes i_B) \text{ (properties of i_B, ϕ_B is a comonoid morphism),}
\end{aligned}
$$

and then $\varphi_{B_T} \circ (H_G \otimes \varphi_{B_T}) = \varphi_{B_T} \circ (\mu_{H_G} \otimes B_T)$.

On the other hand, $\varphi_{B_T} \circ (H_G \otimes \eta_{B_T}) = \varepsilon_{H_G} \otimes \eta_{B_T}$ and, using that ϕ_B is a monoid morphism, i_B a comonoid morphism and (33), we get that $\varphi_{B_T} \circ (H_G \otimes \mu_{B_T}) = \mu_{B_T} \circ (\varphi_{B_T} \otimes \varphi_{B_T}) \circ (H_G \otimes c_{H_G, B_T} \otimes B_T) \circ (\delta_{H_G} \otimes B_T \otimes B_T)$ and then (B_T, φ_{B_T}) is a left H_G-module monoid. In a similar way, but using that ϕ_B is a comonoid morphism, it is not difficult to see that $\varepsilon_{B_T} \circ \varphi_{B_T} = \varepsilon_{H_G} \otimes \varepsilon_{B_T}$ and $\delta_{B_T} \circ \varphi_{B_T} = (\varphi_{B_T} \otimes \varphi_{B_T}) \circ \delta_{H_G \otimes B_T}$ and then (B_T, φ_{B_T}) is a left H_G-module comonoid. As far as condition (c2),

$$
\begin{aligned}
& ad_{H_G} \circ (H_G \otimes \sigma) \\
&= p_H \circ \mu_H \circ (H \otimes q_H) \circ ((q_H \circ \mu_H \circ (H \otimes q_H)) \otimes H) \circ (H \otimes (c_{H,H} \circ (\lambda_H \otimes H))) \\
&\quad \circ(((q_H \otimes q_H) \circ \delta_H \circ i_H) \otimes (\gamma \circ i_B)) \text{ (definitions)} \\
&= p_H \circ ad_H \circ (i_H \otimes (\gamma \circ i_B)) \text{ } (i_B \text{ is a monoid and comonoid morphism, (29))} \\
&= p_H \circ \gamma \circ \phi_B \circ (i_H \otimes i_B) \text{ ((c2) of Definition 2.8 for } (B, H, \gamma)) \\
&= \sigma \circ \varphi_{B_T} \text{ ((33))}.
\end{aligned}
$$

In a similar way it is easy to see condition (c3). Finally (B_T, φ_{B_T}) is in the cocommutativity class of H_G by applying that (B, ϕ_B) is in the cocommutativity class of H, and the proof is complete.

■

3.4. THEOREM. *Let* $\mathbf{T_G} = (T, G, \partial)$ *and* $\mathbf{B_H} = (B, H, \gamma)$ *be crossed modules of Hopf monoids and denote the corresponding entwining structures by* $\psi_{G,T} : G \otimes T \to T \otimes G$ *and* $\psi_{Y,X} : Y \otimes X \to X \otimes Y$*, respectively. Let*

$$(\mathbf{B_H}, \mathbf{T_G}, (u, v), (w, y))$$

be a projection of crossed modules of Hopf monoids such that i_B *and* i_H *are comonoid morphisms. Then*

$$(B_T \bowtie T, H_G \bowtie G, \chi = \sigma \otimes \partial)$$

is a crossed module of Hopf monoids and $(B_T, H_G, \gamma') \bowtie \mathbf{T_G} \simeq \mathbf{B_H}$*, where* γ' *is the restriction of* γ *to* B_T*. Moreover, the left* $H_G \bowtie G$*-module monoid and comonoid structure for* $B_T \bowtie T$ *is in the cocommutativity class of* $H_G \bowtie G$*.*

PROOF. By [Radford, 1985], we know that $B_T \bowtie T$ and $H_G \bowtie G$ are Hopf monoids and there is a left T-module monoid and comonoid structure for B_T and a left G-module monoid and comonoid structure for H_G such that (B_T, ϕ_{B_T}) and (H_G, ϕ_{H_G}) are in the cocommutativity class of T and G, respectively. Moreover, by Proposition 3.3 we have that (B_T, H_G, σ) is a crossed module of Hopf monoids and the left H_G-module monoid and comonoid (B_T, φ_{B_T}) is in the cocommutativity class of H_G. Now consider the morphisms

$$\psi_{T,B_T} = (\phi_{B_T} \otimes T) \circ (T \otimes c_{T,B_T}) \circ (\delta_T \otimes B_T),$$

$$\psi_{G,H_G} = (\phi_{H_G} \otimes G) \circ (G \otimes c_{G,H_G}) \circ (\delta_G \otimes H_G)$$

and

$$\psi_{G,B_T} = ((p_B \circ \phi_B \circ (v \otimes i_B)) \otimes G) \circ (G \otimes c_{G,B_T}) \circ (\delta_G \otimes B_T).$$

It is not difficult to see that these morphism are left-left entwining and right-right semi-entwining structures and satisfy (8) and (9). Then, define

$$\phi_{B_T \bowtie T} : H_G \bowtie G \otimes B_T \bowtie T \to B_T \bowtie T$$

as

$$\phi_{B_T\bowtie T}=(\mu_{B_T}\otimes T)\circ(B_T\otimes t\otimes T)\circ(B_T\otimes H_G\otimes\delta_T\otimes\varepsilon_G)\circ(\psi_{H_G,B_T}\otimes\psi_{G,T})\circ(H_G\otimes\psi_{G,B_T}\otimes T),$$

where $t=p_B\circ\phi_B\circ(i_H\otimes u)$. We will see that $\phi_{B_T\bowtie T}=\pi_B^{-1}\circ\phi_B\circ(\pi_H\otimes\pi_B)$. First of all, note that the equalities

$$(\pi_H\otimes G)\circ(H_G\otimes\delta_G)=(H\otimes y)\circ\delta_H\circ\pi_H,\quad(\pi_B\otimes T)\circ(B_T\otimes\delta_T)=(B\otimes w)\circ\delta_B\circ\pi_B\quad(34)$$

hold. Indeed,

$$\begin{aligned}&(\pi_H\otimes G)\circ(H_G\otimes\delta_G)\\&=(\mu_H\otimes G)\circ(i_H\otimes((v\otimes(y\circ v))\circ\delta_G))\\&=\mu_{H\otimes G}\circ(((H\otimes y)\circ\delta_H\circ i_H)\otimes((H\otimes y)\circ\delta_H\circ v))\\&=(H\otimes y)\circ\delta_H\circ\pi_H,\end{aligned}$$

and in a similar way for the second equality. Then

$$\begin{aligned}&\phi_{B_T\bowtie T}\\&=((p_B\circ\mu_B\circ(q_B\otimes q_B)\circ(\phi_B\otimes\phi_B)\circ(i_H\otimes B\otimes i_H\otimes B))\otimes T)\circ(H_G\otimes c_{H_G,B}\otimes u\otimes T)\\&\circ(\delta_{H_G}\otimes(q_B\circ\phi_B\circ(H\otimes i_B))\otimes(\delta_T\circ\phi_T))\circ(H_G\otimes v\otimes c_{G,B_T}\otimes T)\circ(H_G\otimes\delta_G\otimes B_T\otimes T)\end{aligned}$$

(definitions)

$$\begin{aligned}&=((p_B\circ\mu_B\circ(\phi_B\otimes\phi_B))\otimes T)\circ(H\otimes\phi_B\otimes H\otimes(((u\circ\phi_T)\otimes\phi_T)\circ\delta_{G\otimes T}))\\&\circ(H\otimes H\otimes c_{H,B}\otimes G\otimes T)\circ(H\otimes c_{H,H}\otimes c_{G,B}\otimes T)\circ((\delta_H\circ i_H)\otimes((v\otimes G)\circ\delta_G)\otimes i_B\otimes T)\end{aligned}$$

(i_H is a comonoid morphism, T is a left G-module comonoid, (33), (29))

$$\begin{aligned}&=((p_B\circ\mu_B\circ(\phi_B\otimes\phi_B))\otimes T)\circ(H\otimes c_{H,B}\otimes u\otimes\phi_T)\circ(\mu_{H\otimes H}\otimes B\otimes c_{G,T}\otimes T)\\&\circ((\delta_H\circ i_H)\otimes((v\otimes v)\circ\delta_G)\otimes c_{G,B}\otimes\delta_T)\circ(H_G\otimes\delta_G\otimes i_B\otimes T)\end{aligned}$$

(B is a left H-module, (31))

$$\begin{aligned}&=((p_B\circ\mu_B\circ(\phi_B\otimes\phi_B)\circ(H\otimes c_{H,B}\otimes B)\circ((\delta_H\circ\mu_H)\otimes B\otimes u))\otimes\phi_T)\\&\circ(i_H\otimes v\otimes B\otimes c_{G,T}\otimes T)\circ(H_G\otimes G\otimes c_{G,B}\otimes\delta_T)\circ(H_G\otimes\delta_G\otimes i_B\otimes T)\end{aligned}$$

(v is a comonoid morphism, H is a Hopf monoid)

$$=((p_B\circ\phi_B)\otimes\phi_T)\circ(H\otimes c_{G,B}\otimes T)\circ(\pi_H\otimes G\otimes\pi_B\otimes T)\circ(H_G\otimes\delta_G\otimes B_T\otimes\delta_T)$$

(B is a left H-module monoid)

$$= ((p_B \circ \phi_B) \otimes (\phi_T \circ (y \otimes w))) \circ \delta_{H\otimes B} \circ (\pi_H \otimes \pi_B) \ ((34))$$

$$= ((p_B \circ \phi_B) \otimes (w \circ \phi_B)) \circ \delta_{H\otimes B} \circ (\pi_H \otimes \pi_B) \ ((31))$$

$$= \pi_B^{-1} \circ \phi_B \circ (\pi_H \otimes \pi_B) \ (B \text{ is a left } H\text{-module comonoid}).$$

As a consequence, it is easy to see that $(B_T \bowtie T, \phi_{B_T\bowtie T})$ is a left $H_G \bowtie G$-module monoid and comonoid and $\phi_{B_T\bowtie T}$ is in the cocommutativity class of $H_G \bowtie G$ because ϕ_B is in the cocommutativity class of H.

On the other hand, and by similar techniques,

$$\pi_H^{-1} \circ \gamma \circ \pi_B$$

$$= (p_H \otimes y) \circ \mu_{H\otimes H} \circ (\delta_H \otimes \delta_H) \circ ((\gamma \circ i_B) \otimes (\gamma \circ u))$$

$$= (p_H \otimes G) \circ \mu_{H\otimes G} \circ (\gamma \otimes (y \circ \gamma) \otimes H \otimes y) \circ ((\delta_B \circ i_B) \otimes (\delta_H \circ v \circ \partial))$$

$$= (p_H \otimes G) \circ \mu_{H\otimes G} \circ (\gamma \otimes \partial \otimes v \otimes (y \circ v)) \circ (((B \otimes w) \circ \delta_B \circ i_B) \otimes (\delta_G \circ \partial)$$

$$= ((p_H \circ \mu_H \circ (H \otimes q_H) \circ (\gamma \otimes v)) \otimes G) \circ (i_B \otimes (\delta_G \circ \partial))$$

$$= ((p_H \circ \mu_H \circ (\gamma \otimes (\eta_H \circ \varepsilon_G))) \otimes G) \circ (i_B \otimes (\delta_G \circ \partial))$$

$$= \sigma \otimes \partial$$

$$= \chi,$$

and then we get that conditions (d2) and (d3) of Definition 2.8 hold. Finally,

$$\pi_H \circ \chi = \pi_H \circ \pi_H^{-1} \circ \gamma \circ \pi_B = \gamma \circ \pi_B$$

and

$$\pi_B \circ \phi_{B_T\bowtie T} = \pi_B \circ \pi_B^{-1} \circ \phi_B \circ (\pi_H \otimes \pi_B) = \phi_B \circ (\pi_H \otimes \pi_B)$$

and then (π_B, π_H) is an isomorphism of crossed modules of Hopf monoids.

■

3.5. Example. The category **Set** of sets is a non-strict monoidal category, where the tensor product is the cartesian product and the unit object is a one element set; the associative and unit constraints are obvious (for simplicity of notation, relying on coherence, we omit explicit mention of them). A group G is an example of a cocommutative Hopf monoid in **Set** where the comultiplication is the duplication, the counit is the obvious one, and the antipode is $\lambda_G(g) = g^{-1}$. A morphism between two groups is a Hopf monoid morphism between the corresponding Hopf monoids in **Set**.

Let T and G be groups with a morphism $\partial : T \to G$ of groups. Let $\varphi_T : G \times T \to T$, $\varphi_T(g,t) = {}^g t$, be an action of G over T, that is, a morphism satisfying the axioms of identity, i.e., $\varphi_T(e_G, t) = t$ where e_G is the unit element of G, and compatibility, i.e., $\varphi_T(gg', t) = \varphi_T(g, \varphi_T(g', t))$. Note that, under these conditions, it is easy to show that (T, φ_T) is a left G-module monoid and comonoid in **Set**. The notion of crossed module of groups was introduced by Whitehead [Whitehead, 1949] in his investigation of the monoidal structure of second relative homotopy groups. Recall that the triple $\mathbf{T_G} = (T, G, \partial)$ is a crossed module if, for the group morphism $\partial : T \to G$ and the action $\varphi_T(g, t) = {}^g t$, the following identities

(i) $\partial({}^g t) = g\partial(t)g^{-1}$ (Precrossed identity),

(ii) ${}^{\partial(t)}t' = tt't^{-1}$ (Peiffer identity).

hold. Then, any crossed module $\mathbf{T_G}$ is an example of a crossed module of Hopf algebras, in the sense of Frégier and Wagemann [Frégier, Wagemann, 2011], in the category of **Set**. Also, it is a crossed module of Hopf monoids in the sense of Definition 2.8, because in this setting the condition (4) holds trivially.

Let $\mathbf{T_G} = (T, G, \partial)$ and $\mathbf{B_H} = (B, H, \gamma)$ be crossed modules. We will say that (u, v) is a morphism of crossed modules between $\mathbf{T_G}$ and $\mathbf{B_H}$ if

(i) $u : T \to B$, $v : G \to H$ are group morphisms,

(ii) $v \circ \partial = \gamma \circ u$,

(ii) For all $g \in G,\ t \in T,\ \ u({}^g t) = {}^{v(g)}u(t)\ \ (u(\varphi_T(g, t)) = \varphi_B(v(g), u(t)))$.

Therefore, $(u, v) : \mathbf{T_G} \to \mathbf{B_H}$ is a morphism of crossed modules of Hopf monoids in **Set**.

As a consequence, in the following, the terms ”crossed module” and ”crossed module of Hopf monoids” are interchangeably used throughout this example.

Let $\mathbf{T_G} = (T, G, \partial)$ and $\mathbf{B_H} = (B, H, \gamma)$ be crossed modules. Let $\Psi_{G,T} : G \times T \to T \times G$, $\Psi_{H,B} : H \times B \to B \times H$

$$\psi_{G,T}(g, t) = ({}^g t, g), \quad \psi_{H,B}(h, b) = ({}^h b, h)$$

be the corresponding entwining structures. The morphisms $\psi_{G,T}$ and $\psi_{H,B}$ trivially satisfy (a4) of Definition 2.4, (8) and (9).

Let $(\mathbf{B_H}, \mathbf{T_G}, (u, v), (w, y))$ be a projection of crossed modules. Then the idempotent morphism $q_B : B \to B$ is defined by

$$q_B(b) = b(u \circ w)(b^{-1})$$

and

$$B_T = ker(w)$$

because, by the general theory of projections, B_T is the equalizer of the morphisms

$$b \to (b, e_T), \quad b \to (b, w(b)).$$

The associated action $\varphi_{B_T} : T \times B_T \to B_T$ is

$$\varphi_{B_T}(t, b) = u(t)bu(t^{-1})$$

and the coaction $\rho_{B_T} : B_T \to T \times B_T$ is trivial, i.e., $\rho_{B_T}(b) = (e_T, b)$. Therefore $i_T : B_T \to B$ is a comonoid morphism.

The cartesian product $B_T \times T$ is a group, denoted by $B_T \bowtie T$, with product

$$\mu_{B_T \bowtie T} = (\mu_{B_T} \times \mu_T) \circ (B_T \times \psi_{T,B_T} \times T),$$

where $\psi_{T,B_T} = (\varphi_{B_T} \times T) \circ (T \times c_{T,B_T}) \circ (\delta_T \times B_T)$.

Then $\psi_{T,B_T}(t, b) = (u(t)bu(t^{-1}), t)$ and, as a consequence,

$$\mu_{B_T \bowtie T}((b, t), (b', t')) = (bu(t)b'u(t^{-1}), tt').$$

For this product, the unit is (e_B, e_T) and the inverse of (b, t) is

$$(b, t)^{-1} = (u(t^{-1})b^{-1}u(t), t^{-1}).$$

Similarly, $q_H : H \to H$ is defined by

$$q_H(h) = h(v \circ y)(h^{-1})$$

and

$$H_G = ker(y).$$

The associated action $\varphi_{H_G} : G \times H_G \to H_G$ is

$$\varphi_{H_G}(g, h) = v(g)hv(g^{-1})$$

and the coaction $\rho_{H_G} : H_G \to G \times H_G$ is also trivial. Therefore $i_G : H_G \to H$ is a comonoid morphism.

The cartesian product $H_G \times G$ is a group, denoted by $H_G \bowtie G$, with product

$$\mu_{H_G \bowtie G} = (\mu_{H_G} \times \mu_G) \circ (H_G \times \psi_{G,H_G} \times G)$$

with

$$\psi_{G,H_G} = (\varphi_{H_G} \times G) \circ (G \times c_{G,H_G}) \circ (\delta_G \times H_G).$$

Then $\psi_{G,H_G}(g, h) = (v(g)hv(g^{-1}), g)$ and

$$\mu_{H_G \bowtie G}((h, g), (h', g')) = (hv(g)h'v(g^{-1}), gg').$$

For this product, the unit is (e_H, e_G) and the inverse of (h, g) is

$$(h, g)^{-1} = (v(g^{-1})h^{-1}v(g), g^{-1}).$$

In these conditions there exist two group isomorphisms

$$\omega_B : B_T \bowtie T \to B, \quad \omega_H : H_G \bowtie G \to H$$

defined by

$$\omega_B(b,t) = bu(t), \quad \omega_H(h,g) = hv(g)$$

and with inverses

$$\omega_B^{-1}(b) = (b(u\circ w)(b^{-1}), w(b)), \quad \omega_H^{-1}(h) = (h(v\circ y)(h^{-1}), y(h)).$$

Moreover, the triple

$$(B_T \bowtie T, H_G \bowtie G, \alpha)$$

is a crossed module where, if γ' is the restriction of γ to B_T,

$$\alpha = \gamma' \times \partial$$

and

$$\varphi_{B_T\bowtie T} = (\mu_{B_T} \times T)\circ(B_T\otimes\Gamma_T\otimes T)\circ(\psi_{H_G,B_T}\otimes(\delta_T\circ\varphi_T))\circ(H_G\otimes\psi_{G,B_T}\otimes T)$$

is the morphism defined in Section 1. In our example we have

$$\psi_{G,B_T} = (m_{B_T}\times G)\circ(G\times c_{G,B_T})\circ(\delta_G\times B_T) : G\times B_T \to B_T\times G,$$

where

$$m_{B_T} : G\times B_T\to B_T, \;\; m_{B_T}(g,b) = {}^{v(g)}b$$

and then,

$$\psi_{G,B_T}(g,b) = ({}^{v(g)}b, g).$$

On the other hand,

$$\psi_{H_G,B_T} = (\phi_{B_T}\times H_G)\circ(H_G\times c_{H_G,B_T})\circ(\delta_{H_G}\times B_T) : H_G\times B_T\to B_T\times H_G$$

where

$$\phi_{B_T} : H_G\times B_T\to B_T, \;\; \phi_{B_T}(h,b) = {}^{h}b$$

and, as a consequence

$$\psi_{H_G,B_T}(h,b) = ({}^{h}b, h).$$

The morphism $\Gamma_T : H_G\times T\to B_T$ is defined by

$$\Gamma_T(h,t) = {}^{h}u(t)u(t^{-1})$$

and then

$$\varphi_{B_T\bowtie T}((h,g),(b,t)) = ({}^{hv(g)}(bu(t))u({}^{g}(t^{-1})), \; {}^{g}t).$$

The pair (ω_B, ω_H) is an isomorphism between the crossed modules $(B_T \bowtie T, H_G \bowtie G, \alpha)$ and $\mathbf{B_H}$ with inverse $(\omega_B^{-1}, \omega_H^{-1})$. Also, (B_T, H_G, γ') is a crossed module with action ϕ_{B_T} and using the general theory we can assert that

$$(B_T, H_G, \gamma') \bowtie \mathbf{T_G} = (B_T \bowtie T, H_G \bowtie G, \alpha) \simeq \mathbf{B_B}.$$

A particular case of this example is the one associated to the projection

$$(\mathbf{B_B}, \mathbf{T_T}, (u, u), (w, w)),$$

where $\mathbf{B_B} = (B, B, id_B)$ and $\mathbf{T_T} = (T, T, id_T)$ with the corresponding adjoint actions.

Then,

$$(B_T, B_T, id_{B_T}) \bowtie \mathbf{T_T} = (B_T \bowtie T, B_T \bowtie T, id_{B_T \bowtie T}) \simeq \mathbf{B_B}$$

and the identity

$$\varphi_{B_T \bowtie T}((b, t), (b', t')) = (b, t)(b', t')(b, t)^{-1}$$

holds.

References

J.N. Alonso Álvarez, J.M. Fernández Vilaboa, Cleft extensions in braided categories, Comm. Algebra 28, 3185-3196, 2000.

J.N. Alonso Álvarez, J.M. Fernández Vilaboa, R. González Rodríguez, On the (co)-commutativity class of a Hopf algebra and crossed products in a braided category, Comm. Algebra 29, 5857-5878, 2001.

F. Borceux, Handbook of categorical algebra I. Basic category theory, Encyclopedia of Mathematics and its applications, 50, Cambridge University Press, Cambridge, 1994.

T. Brzeziński, S. Majid, Coalgebra bundles, Comm. Math. Phys. 191, 467-492, 1998.

J. Faria Martins, Crossed modules of Hopf algebras and of associative algebras and two-dimensional holonomy, J. Geom. Phys. 99, 68-110, 2016.

J.M. Fernández Vilaboa, M.P. López López, Cat^1-Hopf algebras and crossed modules, Comm. Algebra 35, 181-191, 2007.

Y. Frégier, F. Wagemann, On Hopf 2-algebras, Int. Math. Res. Notices 2011, 3471-3501, 2011.

C. Kassel, Quantum groups, Graduate Texts in Mathematics 155, Springer-Verlag, New York, 1995.

S. Majid, Cross products by braided groups and bosonization, J. of Algebra 163, 165-190, 1994.

S. Majid, Foundations of quantum group theory, Cambridge University Press, 1995.

S. Majid, Strict quantum 2-groups, arXiv:1208.6265v1, 2012.

R. K. Molnar, Semi-direct products of Hopf algebras, J. Algebra 47, 29-51, 1997.

D.E. Radford, The structure of Hopf algebras with a projection, J. of Algebra 92, 322-347, 1985.

M. E. Sweedler, Hopf algebras, WA. Benjamin, Inc., New York, 1969.

J.H.C. Whitehead, Combinatorial homotopy II, Bull. Amer. Math. Soc. 55, 453-496, 1949.

Departamento de Matemáticas, Universidad de Vigo,
Campus Universitario Lagoas-Marcosende, E-36280 Vigo, Spain

Departamento de Matemáticas, Universidad de Santiago de Compostela.
E-15771 Santiago de Compostela, Spain

Departamento de Matemática Aplicada II, Universidad de Vigo,
Campus Universitario Lagoas-Marcosende, E-36310 Vigo, Spain

Email: jnalonso@ uvigo.es
josemanuel.fernandez@usc.es
rgon@dma.uvigo.es

This article may be accessed at http://www.tac.mta.ca/tac/