

THE GENUINE OPERADIC NERVE

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ABSTRACT. We construct a generalization of the operadic nerve, providing a translation between the equivariant simplicially enriched operadic world to the parametrized ∞ -categorical perspective. This naturally factors through genuine equivariant operads, a model for “equivariant operads with norms up to homotopy”. We introduce the notion of an op-fibration of genuine equivariant operads, extending Grothendieck op-fibrations, and characterize fibrant operads as the image of genuine equivariant symmetric monoidal categories. Moreover, we show that under the operadic nerve, this image is sent to G -symmetric monoidal G - ∞ -categories. Finally, we produce a functor comparing the notion of algebra over an operad in each of these two contexts.

1. Introduction

Operads have proven to be a valuable tool since they were introduced by Boardman-Vogt [BV73] and May [May72]. In stable homotopy theory, Boardman-Vogt and May introduced a class of simplicial operads¹ called E_∞ -operads which encode homotopy coherent multiplication in spaces and spectra. Further, Boardman-Vogt and May showed that spaces equipped with such structures represented infinite loop spaces. Moreover, the homotopy theory of simplicial operads is designed so that any cofibrant replacement of the commutative operad is E_∞ , capturing the notion that E_∞ -algebras are the “correct” homotopical replacements of strict topological monoids.

However, while simplicial operads can encode these homotopical structures, they themselves remain fairly rigid objects. To obtain further homotopical flexibility, an alternative framework has been pioneered by Lurie to work in the language of ∞ -categories, an extension of category theory defined by Boardman-Vogt [BV73] and refined by Joyal [Joy02], where notions are only ever well-defined up to coherent homotopy. Lurie [Lur17] constructs the theory of ∞ -operads, a certain class of ∞ -category equipped with a map to the category of finite pointed sets \mathbf{F}_* .

While these two theories aim to model the same homotopy theory, the equivalence between them was not known for some time, and is not direct. Work of Cisinski-Moerdijk-

Received by the editors 2019-05-01 and, in final form, 2019-09-06.

Transmitted by Julie Bergner. Published on 2019-09-11.

2010 Mathematics Subject Classification: 55P91,55P48,19D23,18D30.

Key words and phrases: infinity operads, equivariant operads, symmetric monoidal categories.

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¹In general, we write “operad” and \mathbf{Op} to refer to the category of *colored* operads, which includes the classical single-colored notion as well as the more general notion (which often goes by the name of “multicategory”). Additionally, we write “simplicial operad” and \mathbf{sOp} to mean the category of (possibly many colored) *operads enriched in simplicial sets*, as opposed to the more general notion of a simplicial object in (colored) operads. Details will be given in Definition 2.2.1.

Weiss [CM13a, CM13b, CM11, MW09], Chu-Haugsgeng-Heuts [CHH18], and Barwick [Bar18] produces a zig-zag of Quillen equivalences between simplicial operads and ∞ -operads.

On the other hand, there is a fairly natural construction between these models. For any simplicial operad $\mathcal{O} \in \mathbf{sOp}$, May-Thomason [MT78] produce an associated simplicial category \mathcal{O}^\otimes , living over the category \mathbf{F}_* of pointed finite sets, called the *category of operators*, and moreover show that the theory of algebras over \mathcal{O} and \mathcal{O}^\otimes coincide. The homotopy coherent nerve of \mathcal{O}^\otimes is denoted $N^\otimes(\mathcal{O})$, and is called the *operadic nerve* by Lurie [Lur17, 2.1.1.22]. This construction has several nice properties. First, in [Lur17, Prop. 2.1.1.27], Lurie showed that this sends a levelwise fibrant simplicial operad (where all mapping spaces are Kan) to an ∞ -operad (and in fact, Lurie’s definition of an ∞ -operad is truly a generalization of these categories of operators). Second, it is expected to be an equivalence of homotopy theories, and has already been shown to be one when restricted to non-unital operads by [HHM16]. Third, the operadic nerve preserves symmetric monoidal categories. That is, there are canonical faithful inclusions of simplicial symmetric monoidal categories into simplicial operads and symmetric monoidal ∞ -categories into ∞ -operads, and the operadic nerve sends one subcategory to the other [Lur17, Prop. 4.1.7.10].

In this paper, we generalize the narrative of the operadic nerve to the equivariant setting, incorporating actions of a finite group G . However, the appropriate source and target of the new map are not simply the categories of G -objects of source and target of the original operadic nerve. Instead, sophisticated categories have been built to capture the more intricate homotopy theory of equivariant operads. This additional complexity comes from the observation, first by Constenoble-Waner [CW91] and explored systematically by Blumberg-Hill [BH15], that there are several possible notions of “equivariant homotopy coherent multiplication”. The distinctions come from whether or not they encode *norm maps*; as a distinguished and archetypal example, in G -spectra these are G -equivariant maps of the form

$$G \cdot_H N^A X \simeq G \cdot_H \bigwedge_{a \in A} i_H^* X \rightarrow X$$

for A some finite H -set with $H \leq G$, X a G -spectrum, and $N^A X$ the indexed smash product, with an H -action that both permutes the indices (via the action on A) and acts on each X . So-called *naïve E_∞ -operads*, E_∞ -operads with a trivial G -action, only encode norm maps for A a trivial H -set, while *genuine G - E_∞ -operads* encode all such maps. As desired, there is a homotopy theory of simplicial G -operads which distinguishes these classes of operads, constructed by the author and his collaborator Luís Pereira [BPc], and independently by Gutierrez-White [GW18].

With respect to the operadic nerve, this has the following consequence. The new source is the category \mathbf{sOp}_G of *simplicial genuine² equivariant operads*, also introduced in [BPc] by the author and Pereira. This is a generalization of simplicial G -operads,

²The *genuine* adjective here has two (related) etymologies. First, the image $i_* \mathcal{O} \in \mathbf{sOp}_G$ of any genuine G - E_∞ -operad $\mathcal{O} \in \mathbf{sOp}^G$ is contractible at every level, while this fails when starting with a naïve E_∞ -operad, and \mathbf{sOp}_G was designed precisely to see this distinction. Second, the adjective “genuine” has been used regularly to describe homotopy theories of equivariant objects which see all possible fixed point

which embed fully-faithfully as part of a right Quillen equivalence, where objects $\mathcal{P} \in \mathbf{sOp}_G$ have evaluations at all finite H -sets A . Morally, the operations in $\mathcal{P}(A)$ precisely encode A -norm maps, while the composition law details their interactions.

For the target, Barwick-Dotto-Glasman-Nardin-Shah [BDG⁺] have produced a beautiful theory of parametrized ∞ -categories and parametrized homotopy theory. In particular, when the base is the *orbit category* \mathbf{O}_G of finite transitive G -sets, they recover a coherent description of equivariant homotopy theory. Encoding algebraic structures here are \mathbf{O}_G - ∞ -operads $\mathcal{O}^\otimes \in \mathbf{Op}_{\infty,G}$, a certain class of \mathbf{O}_G - ∞ -categories equipped with a map to the category \mathbf{F}_*^G of all finite pointed H -sets for all $H \leq G$ (cf. Definition 3.1.1).

Given a simplicial genuine equivariant operad $\mathcal{P} \in \mathbf{sOp}_G$, we construct an analogue of the operadic nerve $N^\otimes(\mathcal{P})$ dubbed the *genuine operadic nerve*, also built as the homotopy coherent nerve of a (genuine) category of operators construction. The main results of this paper are the following extensions of [Lur17, Prop. 2.1.1.27 and 4.1.7.10], providing a (1-categorical) translation between these two theories of homotopical equivariant operads.

First, we prove the following in Section 4.2.

1.0.1. THEOREM. *The genuine operadic nerve is a faithful functor*

$$N^\otimes(-) : \mathbf{sOp}_{G,f} \longrightarrow \mathbf{Op}_{\infty,G}$$

*from the category of level fibrant genuine equivariant operads to the (1)-category of \mathbf{O}_G - ∞ -operads, which recovers the original operadic nerve in the case where $G = *$.*

Additionally, similarly to the inclusion $\mathbf{sSymMon} \hookrightarrow \mathbf{sOp}$ of (simplicial) symmetric monoidal categories into (colored, simplicial) operads, there are analogous notions of “symmetric monoidal category” inside \mathbf{sOp}_G and $\mathbf{Op}_{\infty,G}$, namely the (simplicial) $E\Sigma_G$ -algebras of the author and Pereira [BPd] and G -symmetric monoidal G - ∞ -categories of Barwick et al. [Nar17]. We prove the following in Section 5.4, which says that the genuine operadic nerve preserves these notions of symmetric monoidal categories and the corresponding notions of symmetric monoidal functors.

1.0.2. THEOREM. *The functor $N^\otimes : \mathbf{sOp}_G \rightarrow \mathbf{Op}_{\infty,G}$ restricts to a functor*

$$N^\otimes : \mathbf{sSymMon}_{G,f}^q \longrightarrow \mathbf{SymMon}_{\infty,G}.$$

A major step in the proof of Theorem 1.0.2 is the identification of the image of $E\Sigma_G$ -algebras inside \mathbf{sOp}_G as those genuine operads which are “fibered” over the terminal operad, in a sense which generalizes Grothendieck op-fibrations of categories, made precise in Section 5.

Essentially, we prove the following; a precise statement can be found at Theorem 5.3.1.

information, namely the genuine/fine homotopy theory of G -spaces and the (fully) genuine homotopy theory of G -spectra. Expanding on this, both named theories can be realized as presheaf categories, and in particular their homotopy groups inherit extra structure. Similarly, results in [BPc] prove that the homotopy groups of simplicial G -operads are naturally genuine G -operads of sets, and we should think of genuine equivariant operads as appropriate analogues to presheaves in this algebraic setting.

1.0.3. THEOREM. *The image of simplicial $E\Sigma_G$ -algebras in genuine equivariant operads are those operads such that the canonical map to the terminal operad is a genuine operadic fibration.*

All together, using the notation to be defined in the paper, these results amalgamate into the following commuting diagram of well-defined faithful functors.

$$\begin{array}{ccccccc}
 \text{sPerm}_{G,f} & \xrightarrow[\cong]{\mathcal{P}(-)} & \text{Fib}^f(\text{Comm}) & \xrightarrow{(-)^\otimes} & \text{Fib}_{\text{Segal}}^f(\underline{\mathbb{F}}_*^G) & \xrightarrow{N} & \text{SymMon}_{\infty,G} \\
 \downarrow \simeq & & \downarrow \simeq & & \downarrow & & \parallel \\
 \text{sSymMon}_{G,f}^q & \xrightarrow[\cong]{\mathcal{P}(-)} & \text{Fib}^q(\text{Comm}) & \xrightarrow{(-)^\otimes} & \text{Fib}_{\text{Segal}}^q(\underline{\mathbb{F}}_*^G) & \xrightarrow{N} & \text{SymMon}_{\infty,G} \\
 & & \downarrow & & \downarrow & & \parallel \\
 & & \text{Fib}(\text{Comm}) & \xrightarrow{(-)^\otimes} & \text{Fib}_{\text{Segal}}(\underline{\mathbb{F}}_*^G) & \xrightarrow{N} & \text{SymMon}_{\infty,G} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{Op}_G & \xrightarrow{(-)^\otimes} & \text{CatOp}_G & \xrightarrow{N} & \text{Op}_{\infty,G}
 \end{array} \tag{1.1}$$

We end by showing that this framework preserves algebras over operads, if we make small additional assumptions on \mathcal{V} and our operads \mathcal{O} . In Section 6.1, we build a model $\underline{\mathcal{V}}_{\infty,G}^\square$ for the G -symmetric monoidal G - ∞ -category of strict G -objects in *globally fibrant* symmetric monoidal simplicial categories \mathcal{V} (Definition 6.1.2), and prove the following for algebras over *graph fibrant* operads \mathcal{O} (Definition 6.0.2).

1.0.4. THEOREM. *For any graph fibrant equivariant simplicial operad $\mathcal{O} \in \text{sOp}^G$ and globally fibrant symmetric monoidal simplicial category (\mathcal{V}, \square) , there exists a functor of ∞ -categories*

$$N\text{Alg}_{\mathcal{O}}(\mathcal{V}^G) \longrightarrow \text{Alg}_{N^\otimes(\mathcal{O})}(\underline{\mathcal{V}}_{\infty,G}^\square)$$

between associated categories of algebras.

1.0.5. REMARK. We note that these results are not yet homotopical. However, as in the non-equivariant case, we expect that N^\otimes is an equivalence of homotopy theories.

1.0.6. REMARK. The author’s joint work with Luís Pereira provides another model of equivariant higher algebra, generalizing the dendroidal sets perspective of Moerdijk, Weiss, Cisinski, and Heuts [CHH18, CM13a, CM13b, CM11, Heu, MW09] which has seen much success. A homotopical analogue of Theorem 1.0.1 in this context is the precisely the culmination of the papers [Per18, BPa, BPe, BPb], that the homotopy coherent dendroidal nerve between equivariant simplicial operads and equivariant dendroidal sets is a right Quillen equivalence.

1.0.7. REMARK. These structures — $E\Sigma_G$ -algebras, genuine equivariant operads, G -operads, and G -symmetric monoidal G - ∞ -categories — have corresponding notions for any (weak) indexing system \mathcal{F} in the sense of [BH15], [BH18], [Per18], [BPc], i.e. replacing all instances of the category $\underline{\mathbb{F}}_*^G$ of all finite pointed G -sets with the category $\underline{\mathbb{F}}_*^\mathcal{F}$ of those finite pointed G -sets generated by \mathcal{F} . We expect the results to extend to these settings.

1.1. ORGANIZATION. The paper is planned as follows.

We begin by recalling the relevant parts of the non-equivariant story in Section 2. In Section 3, we discuss equivariant generalizations of the key players from Section 2, namely the category of finite pointed G -sets, the category of (colored) simplicial genuine equivariant operads $\mathcal{P} \in \mathbf{sOp}_G$ from [BPc], and the \mathbf{Op}_G - ∞ -operads from [BDG⁺, Nar17].

In Section 4, we introduce our main constructions, the *genuine equivariant category of operators* \mathcal{P}^\otimes and the *genuine operadic nerve* $N^\otimes(\mathcal{P})$ associated to \mathcal{P} , and prove Theorem 1.0.1. In Section 5 we define and analyze fibrations in \mathbf{sOp}_G . Section 5.1 defines genuine operadic op-fibrations, Section 5.2 recalls $E\Sigma_G$ -algebras from [BPd] with Proposition 5.2.10 giving the faithful inclusion of simplicial $E\Sigma_G$ -algebras into \mathbf{sOp}_G , and Section 5.3 discusses how the different varieties of $E\Sigma_G$ -algebras can be identified with specific classes of fibrations in \mathbf{Op}_G , culminating in the proof of Theorem 1.0.3. Theorem 1.0.2 is finally proved in Section 5.4.

Lastly, in Section 6, we give several examples of G - ∞ -operads and G -symmetric monoidal G - ∞ -categories coming from the genuine operadic nerve, introduce categories of algebras, and prove Theorem 1.0.4.

1.2. ACKNOWLEDGMENTS. This paper owes much to the ongoing collaborations with Luís Pereira; the author would like to thank him for his help and useful discussions throughout their work together. In particular, the definitions of genuine equivariant operads (single- and many-colored) and $E\Sigma_G$ -algebras are joint with him.

The author would like to thank Asaf Horev for asking whether there was a conversion between G -operads and G - ∞ -operads, leading to this work, for helpful conversations, and for sharing notes of work in progress with Inbar Klang and Foling Zou, which in particular influenced the exposition leading up to Definition 6.1.3. The author would also like to thank Kate Ponto, Bert Guillou, and Nat Stapleton for their comments and suggestions. Finally, the author would like to thank the referee for their careful comments.

2. Motivation and background

We begin by recalling the story in the non-equivariant setting (e.g. [Lur17]), which will provide the guiding outline of the necessary results (as well as the style of proof) for many parts of this article.

2.1. CATEGORICAL FIBRATIONS. First, we recall and establish our terminology for the various different notions of “fibrations” of categories, as generalizations of these ideas appear throughout this paper in many forms.

2.1.1. DEFINITION. *Given a functor $p : \mathcal{C} \rightarrow \mathcal{B}$, an arrow $f : c \rightarrow c'$ in \mathcal{C} is called p -cocartesian if for any $b \in \mathcal{B}$, preimage $c'' \in p^{-1}(b)$, and arrows $g : p(c') \rightarrow b$ in \mathcal{B} and $h : c \rightarrow c''$ in \mathcal{C}*

such that $p(h) = gp(f)$, there exists a unique lift $\bar{g} : c' \rightarrow c''$ such that $p(\bar{g}) = g$ and $\bar{g}f = h$.

$$\begin{array}{ccc}
 p(c) & \xrightarrow{p(f)} & p(c') \\
 \searrow p(h) & & \swarrow \forall g \\
 & & b
 \end{array}
 \qquad
 \begin{array}{ccc}
 c & \xrightarrow{f} & c' \\
 \searrow \forall h & & \swarrow \exists! \bar{g} \\
 & & c''
 \end{array}
 \tag{2.1}$$

We say p is a Grothendieck op-fibration if for every arrow in \mathcal{B} and lift of the domain to \mathcal{C} , there exists some p -cocartesian lift. We say p is additionally q -split if we have a chosen system of cocartesian lifts which are natural in the arrows of \mathcal{B} . Finally, p is additionally fully split if this chosen system is closed under composition.

Given two q -split (resp. fully split) op-fibrations p and p' , a functor $F : \mathcal{C} \rightarrow \mathcal{C}'$ is called a map of (split) op-fibrations if F is a functor over \mathcal{B} and sends (chosen) cocartesian arrows to (chosen) cocartesian arrows. We denote the 1-categories of simplicial fully-split and q -split op-fibrations over \mathcal{B} with maps of op-fibrations by $\text{Fib}^f(\mathcal{B}) \subseteq \text{Fib}^q(\mathcal{B}) \subseteq \text{sCat} \downarrow \mathcal{B}$.

Dually, an arrow f is p -cartesian if f is p -cocartesian in \mathcal{C}^{op} , and p is a Grothendieck fibration if p^{op} is a Grothendieck op-fibration. Explicitly, $f : c' \rightarrow c$ is p -cartesian if for every $g : b \rightarrow p(c')$ and $h : c'' \rightarrow c$ with $p(c'') = b$ and $p(h) = p(f)g$, there exists a unique lift $\bar{g} : c'' \rightarrow c'$ of g such that $f\bar{g} = h$.

The Grothendieck construction provides an equivalent characterization.

2.1.2. THEOREM. *The category of functors $\text{Fun}(\mathcal{B}^{op}, \text{Cat})$ is isomorphic to the categories of fully-split Grothendieck fibrations over \mathcal{B} (dually, fully-split Grothendieck op-fibrations over \mathcal{B}^{op}) with maps of split (op)-fibrations.*

More generally, the 2-category of pseudofunctors $\mathcal{B}^{op} \rightarrow \text{Cat}$ is strictly 2-equivalent to the 2-category of fibrations over \mathcal{B} (dually, op-fibrations over \mathcal{B}^{op}) with maps of (op)-fibrations and natural transformations.

More details on these notions can be found in e.g. [Gra66].

We can repackage Definition 2.1.1 as follows:

2.1.3. LEMMA. *Given a functor $p : \mathcal{C} \rightarrow \mathcal{B}$, An arrow $f : c \rightarrow c'$ is p -cocartesian if the diagram*

$$\begin{array}{ccc}
 \mathcal{C}(c', c'') & \xrightarrow{f^*} & \mathcal{C}(c, c'') \\
 F \downarrow & & \downarrow F \\
 \mathcal{B}(Fc', Fc'') & \xrightarrow{Ff^*} & \mathcal{B}(Fc, Fc'')
 \end{array}
 \tag{2.2}$$

is a strict pullback.

2.1.4. REMARK. Beardsley-Wong [BW] show that we can extend these definitions and discussions to the category sCat of categories enriched in simplicial sets (or actually any bicomplete closed symmetric monoidal category \mathcal{V}): Given a functor $p : \mathcal{C} \rightarrow \mathcal{D}$ of simplicially enriched categories, an arrow $f \in \mathcal{C}(c, c')_0$ is p -cocartesian if (2.2) is a pullback in sSet .

We define *p-cartesian* arrows, (*q-split, fully split*) *Grothendieck (op)-fibrations*, and *maps* of (*q-split, fully split*) (*op*)-fibrations exactly as in Definition 2.1.1.

A main result [BW, Thm. 5.9] is precisely the analogue of Theorem 2.1.2 to this context, so in particular we can freely move between fully split fibrations over \mathcal{B} and functors $\mathcal{B}^{op} \rightarrow \mathbf{sCat}$.

2.2. COLORED SIMPLICIAL OPERADS. We begin with our original object of study, a colored simplicial operad.

2.2.1. DEFINITION. *Given any set \mathfrak{C} , a \mathfrak{C} -signature is a sequence $(x_1, \dots, x_n; x)$ of length $n + 1$ of elements in \mathfrak{C} ; we call the first n objects the source of the signature, and the last one the target.*

A colored simplicial operad ³ $\mathcal{O} \in \mathbf{sOp}$ consists of

- a set $\mathfrak{C} = \mathfrak{C}_{\mathcal{O}}$ of colors (or objects);
- for each \mathfrak{C} -signature $\underline{C} = (x_1, \dots, x_n; x)$ of colors of length $n + 1$, a simplicial set $\mathcal{O}(\underline{C}) \in \mathbf{sSet}$ of operations of arity n ;
- for all collections of \mathfrak{C} -signatures of the form $\vec{C} = (x_1, \dots, x_n; x)$, $\vec{D}_i = (x_1^i, \dots, x_{m_i}^i; x_i)$ for $1 \leq i \leq n$, a composition law

$$\mathcal{O}(\vec{C}) \times \prod_i \mathcal{O}(\vec{D}_i) \rightarrow \mathcal{O}(\vec{C} \circ (\vec{D}_i))$$

where

$$\vec{C} \circ (\vec{D}_i) = (x_1^1, \dots, x_{m_1}^1, x_1^2, \dots, \dots, x_{m_n}^n; x);$$

- a unit operation $1_x \in \mathcal{O}(x; x)$ for all colors $x \in \mathfrak{C}$; and
- for each $\sigma \in \Sigma_n$ and sequences C of length n , an action map

$$\mathcal{O}(\vec{C}) \xrightarrow{\sigma} \mathcal{O}(\sigma^* \vec{C}) = \mathcal{O}(x_{\sigma^{-1}1}, \dots, x_{\sigma^{-1}n}; x);$$

such that the actions of Σ_n are unital and associative, composition is unital and associative, and composition commutes with the action of Σ_n .

A map of operads $F : \mathcal{O} \rightarrow \mathcal{P}$ is given by a map of sets $F_0 : \mathfrak{C}_{\mathcal{O}} \rightarrow \mathfrak{C}_{\mathcal{P}}$, and maps $F(\vec{C}) : \mathcal{O}(\vec{C}) \rightarrow \mathcal{P}(F_0^* \vec{C})$ for all $\mathfrak{C}_{\mathcal{O}}$ -signatures \vec{C} which are compatible with all of the above structure.

See Section 6 for some examples of (equivariant) simplicial operads.

We note that there is a natural path-component functor π_0 , and a forgetful functor j^* which only remembers the operations of arity exactly 1.

$$\pi_0 : \mathbf{sOp} \longrightarrow \mathbf{Op}, \quad j^* : \mathbf{sOp} \rightarrow \mathbf{sCat}.$$

Cisinski-Moerdijk have shown that \mathbf{sOp} has a model structure given by the following:

³These have also been called *multicategories enriched in simplicial sets*.

2.2.2. DEFINITION. [CM13b] A map $F : \mathcal{O} \rightarrow \mathcal{P}$ is called a

- weak equivalence if $F(\vec{C})$ is a weak equivalence in \mathbf{sSet} for all \mathfrak{C} -signatures \vec{C} , and $\pi_{0j^*}F$ is an equivalence of categories.
- fibration if $F(\vec{C})$ is a fibration in \mathbf{sSet} for all \mathfrak{C} -signatures \vec{C} , and $\pi_{0j^*}F$ is an isofibration of categories; i.e. F can lift isomorphisms.

Let $\mathbf{Comm} \in \mathbf{sOp}$ denote the terminal operad, with a single color and $\mathbf{Comm}(-) = *$. We say $\mathcal{O} \in \mathbf{sOp}$ is fibrant if the unique map $\mathcal{O} \rightarrow \mathbf{Comm}$ is a fibration; that is, \mathcal{O} is locally fibrant, with each $\mathcal{O}(\vec{C})$ a Kan complex.

2.3. INFINITY OPERADS. The original operadic nerve construction provided a translation between this world of homotopical algebra with the theory of ∞ -categories. We introduce this second setting now; a more thorough discussion can be found at the original source [Lur17, §2].

2.3.1. DEFINITION. We outline some basic concepts in ∞ -category theory we will need: pointed finite sets, cocartesian arrows, and finally ∞ -operads.

- Let \mathbf{F} denote a fixed category of finite ordered sets and unordered set maps, such that the subcategory with ordered maps is skeletal; i.e., if ever two sets in \mathbf{F} are order isomorphic, they are in fact equal. In particular, we may choose \mathbf{F} to be the category with objects $\underline{n} = \{1, 2, \dots, n\}$ for all $n \geq 1$ with unordered maps.

Let \mathbf{F}_s and Σ denote the subcategories of \mathbf{F} consisting of only surjective maps and bijections, respectively.

These models are chosen so that all of the above have canonical choices for all small limits and colimits using lexicographical ordering. In particular, \mathbf{F} is bipermutative with respect to cartesian product and disjoint union.

- Let \mathbf{F}_* denote the category of pointed finite sets $A_+ = \{0\} \sqcup A$ with $A \in \mathbf{F}$, and pointed maps.
- A map $f : A_+ \rightarrow B_+$ in \mathbf{F}_* is called inert if f is surjective and f is injective away from the basepoint, i.e. for all $0 \neq b \in B$, $|f^{-1}(b)| \leq 1$.
- A map $f : A_+ \rightarrow B_+$ is called a projection map if $B = \{1\}$ and $f(a)$ is not the basepoint of B_+ for exactly one $r \in A$; in this case, we denote f by π_r .
- Given a map of simplicial sets $p : X \rightarrow Y$, we say that a 1-simplex $\xi \in X$ is p -cocartesian if for any diagram of the form below with $0 \leq k < n$ and $n \geq 2$, there exists

a lift as denoted.

$$\begin{array}{ccc}
 \Delta^{0,1} & \xrightarrow{\xi} & X \\
 \downarrow & & \downarrow p \\
 \Lambda^0[n] & \longrightarrow & X \\
 \downarrow & \nearrow \exists & \downarrow \\
 \Delta[n] & \longrightarrow & Y
 \end{array} \tag{2.3}$$

The map p is a cocartesian fibration if p is an inner fibration and satisfies the analogue the definition of Grothendieck op-fibration: for all objects $x \in X$ and arrows $f: y \rightarrow p(x)$ in Y_1 , there exists a p -cocartesian $\widehat{f} \in X_1$ such that $p(\widehat{f}) = f$.

It is clear that if an arrow f in some category \mathcal{C} is p -cocartesian for $p: \mathcal{C} \rightarrow \mathcal{B}$, then $f \in N'\mathcal{C}_1$ is $N'(p)$ -cocartesian, and similarly that if p is a Grothendieck op-fibration, then $N'(p)$ is a cocartesian fibration, where $N': \text{Cat} \rightarrow \text{sSet}$ is the nerve.

- Given a map of ∞ -categories $\mathcal{C} \rightarrow \mathcal{D}$ and a 0-simplex $d \in \mathcal{D}$, denote by $\mathcal{C}_{\langle d \rangle}$ the pullback below in sSet .

$$\begin{array}{ccc}
 \mathcal{C}_{\langle d \rangle} & \longrightarrow & * \\
 \downarrow & & \downarrow d \\
 \mathcal{C} & \longrightarrow & \mathcal{D}
 \end{array} \tag{2.4}$$

2.3.2. DEFINITION. An ∞ -operad is an ∞ -category \mathcal{O}^\otimes equipped with a map of simplicial sets $p: \mathcal{O}^\otimes \rightarrow N(\mathbf{F}_*)$ such that the following three conditions hold:

- (i) For all inert maps $f \in \mathbf{F}_*(A, B)$, and for all objects $x \in \mathcal{O}_{\langle A \rangle}^\otimes$, there exists a p -cocartesian morphism $\bar{f}: x \rightarrow x'$ lifting f . In particular, f (and specified choices) induces a functor $f!: \mathcal{O}_{\langle A \rangle}^\otimes \rightarrow \mathcal{O}_{\langle B \rangle}^\otimes$.

- (ii) For all maps $f \in \mathbf{F}_*(A, B)$, objects $x \in \mathcal{O}_{\langle A \rangle}^\otimes$ and $y \in \mathcal{O}_{\langle B \rangle}^\otimes$, and choices of p -cocartesian lifts $y \rightarrow y_b$ for each projection $\pi_b: B_+ \rightarrow \{b\}_+$ from B onto one of its elements b , the induced Segal map

$$\text{Map}_{\mathcal{O}^\otimes}^f(x, y) \longrightarrow \prod_{b \in B} \text{Map}_{\mathcal{O}^\otimes}^{\pi_b f}(x, y_b) \tag{2.5}$$

is a weak equivalence, where $\text{Map}_{\mathcal{O}^\otimes}^f(-, -) \hookrightarrow \text{Map}_{\mathcal{O}^\otimes}(-, -)$ is the fiber over f .

- (iii) For all objects A_+ and all choices of functors $(\pi_a)_!$, the induced Segal map

$$\mathcal{O}_{\langle A \rangle}^\otimes \longrightarrow \prod_{a \in A} \mathcal{O}_{\langle a \rangle}^\otimes \tag{2.6}$$

is an equivalence of ∞ -categories.

An arrow of ∞ -operads is a map of simplicial sets $F: (\mathcal{O}, p) \rightarrow (\mathcal{P}, q)$ over $N(\mathbf{F}_*)$ which sends p -cocartesian maps to q -cocartesian maps.

2.4. OPERADIC NERVE. To complete our motivation, we prove that there is a faithful functor $N^\otimes : \mathbf{sOp}_f \rightarrow \mathbf{Op}_\infty$, the operadic nerve. The first stage of this map is the construction of the category of operators associated to a simplicial operad.

2.4.1. DEFINITION. [Lur17, 2.1.1.22] *Given $\mathcal{O} \in \mathbf{sOp}$, we define the category of operators associated to \mathcal{O} , denoted \mathcal{O}^\otimes , as the following simplicial category. The set of objects is the set of all tuples $(A, (x_a)_{a \in A})$ with $A_+ \in \mathbf{F}_*$ and (x_a) a tuple of colors of \mathcal{O} indexed by A . Given objects $(A, (x_a))$ and $(B, (y_b))$, define the mapping space by*

$$\mathcal{O}^\otimes((A, (x_a)), (B, (y_b))) = \coprod_{f:A_+ \rightarrow B_+} \prod_{b \in B} \mathcal{O}((x_a)_{a \in f^{-1}(b)}; y_b). \tag{2.7}$$

Composition is as expected: given composable arrows $A_+ \xrightarrow{f} B_+ \xrightarrow{g} C_+$ and operations

$$\psi_c \in \mathcal{O}(g^{-1}c), \quad \xi_b \in \mathcal{O}(f^{-1}b)$$

for all $c \in C$ and $b \in B$, define

$$(\psi_c)_{c \in C} \circ (\xi_b)_{b \in B} = (\psi_c \circ (\xi_b)_{b \in g^{-1}(c)})_{c \in C}. \tag{2.8}$$

This construction is functorial and faithful.

2.4.2. DEFINITION. *Given $\mathcal{O} \in \mathbf{sOp}$, the operadic nerve $N^\otimes(\mathcal{O})$ is the simplicial set $N(\mathcal{O}^\otimes)$, where $N : \mathbf{sCat} \rightarrow \mathbf{sSet}$ is the homotopy coherent nerve. Since \mathcal{O}^\otimes has a canonical map to \mathbf{F}_* , $N^\otimes(\mathcal{O})$ has a canonical map to the nerve of \mathbf{F}_* .*

2.4.3. PROPOSITION. [Lur17, Prop. 2.1.1.27] *If \mathcal{O} is a fibrant simplicial operad, then $N^\otimes(\mathcal{O})$ is an ∞ -operad.*

We record the following easy lemma.

2.4.4. LEMMA. *If $D \in \mathbf{sSet}$ is discrete, then a square with final object D is a pullback iff it is a homotopy pullback iff each induced square with final object $\{d\}$ is a pullback.*

Lastly, we show this construction lifts to a functor. We could not find a statement or proof of this result in the literature; we record it here for completeness and later reference.

2.4.5. PROPOSITION. *The assignment $\mathcal{O} \mapsto N^\otimes(\mathcal{O})$ is part of a functor $\mathbf{sOp}_f \rightarrow \mathbf{Op}_\infty$ from fibrant simplicial operads to ∞ -operads.*

PROOF. For any functor $F : \mathcal{O} \rightarrow \mathcal{P}$ in \mathbf{sOp} , $N^\otimes(F)$ is a map over $N(\mathbf{F}_*)$ for the natural maps $N^\otimes(\mathcal{O}) \xrightarrow{p} N(\mathbf{F}_*)$ and $N^\otimes(\mathcal{P}) \xrightarrow{q} N(\mathbf{F}_*)$. It thus suffices to check that for any functor $F : \mathcal{O} \rightarrow \mathcal{P}$, $N^\otimes(F)$ sends p -cocartesian maps to q -cocartesian maps.

By [Lur17, Remark 2.1.2.9], it suffices to check this on p -cocartesian morphisms living over the projection maps $\pi_r : A_+ \rightarrow 1_+$, with $\pi_r(a) = 1$ if $r = a$ and the basepoint 0 of 1_+ otherwise.

To that end, consider the rectangle below, where $\sigma_s : 1_+ \rightarrow B_+$ sends 1 to $s \in B$, $\alpha_{r,s} : A_+ \rightarrow B_+$ is $\sigma_s \pi_r$ (so sends r to s and everything else to the basepoint), and $\varphi \in \mathcal{O}_{\pi_r}^\otimes((x_a)_{a \in A}; x) = \mathcal{O}(x_r; x)$ is some p -cocartesian map over π_r .

$$\begin{array}{ccc}
 \mathcal{O}_{\sigma_s}^\otimes(x; (y_b)_{b \in B}) & \xrightarrow{\varphi^*} & \mathcal{O}_{\alpha_{r,s}}^\otimes((x_a)_{a \in A}, (y_b)_{b \in B}) \\
 \downarrow F & & \downarrow F \\
 \mathcal{P}_{\sigma_s}^\otimes(F(x); (F(y_b))_{b \in B}) & \xrightarrow{N^\otimes(\varphi)^*} & \mathcal{P}_{\alpha_{r,s}}^\otimes((F(x_a))_{a \in A}, (F(y_b))_{b \in B}) \\
 \downarrow & & \downarrow \\
 \{\sigma_s\} & \xrightarrow{(\pi_r)^*} & \{\alpha_{r,s}\}
 \end{array} \tag{2.9}$$

But we understand this rectangle explicitly:

$$\begin{aligned}
 \mathcal{O}_{\sigma_s}^\otimes(x; (y_b)_{b \in B}) &= \prod_{b \in B \setminus s} \mathcal{O}(\emptyset; y_b) \times \mathcal{O}(x; y_s) \\
 \mathcal{O}_{\alpha_{r,s}}^\otimes((x_a)_{a \in A}; (y_b)_{b \in B}) &= \prod_{b \in B \setminus s} \mathcal{O}(\emptyset; y_b) \times \mathcal{O}(x_r; y_s)
 \end{aligned}$$

and similarly for \mathcal{P}^\otimes , and φ^* is just pre-composition by the actual operation $\varphi \in \mathcal{O}(x_r; x)$. Since by [Lur09, Prop. 2.4.1.10] and Lemma 2.4.4 the big rectangle is a pullback, φ (and hence φ^*) must be an isomorphism. Thus $F(\varphi)$ (and hence $N^\otimes(\varphi)^*$) is an isomorphism, and hence the lower rectangle is a pullback. The reverse directions of Lemma 2.4.4 and [Lur09, Prop. 2.4.1.10] complete the proof. ■

2.4.6. REMARK. In fact, this gives a functor $N^\otimes(-) : \mathbf{sOp} \rightarrow \mathbf{PreOp}_\infty$ from all simplicial operads to the (1)-category of ∞ -preoperads (see [Lur17, §2.1.4]): the above functoriality was independent from the fibrancy of \mathcal{O} and \mathcal{P} , and the construction $N^\otimes(-)$ always gives a preoperad by *loc cit*.

3. Equivariant preliminaries

For the rest of the paper, we fix a finite group G . We will now generalize the definitions found in Sections 2.2 and 2.3 to an equivariant context.

3.1. FINITE POINTED G -SETS. Analogously to how ∞ -operads live over the category \mathbf{F}_* of finite pointed sets, equivariant ∞ -operads live over a category of finite pointed G -sets. However, to provide for a complete “genuine” equivariant picture, our category of finite G -sets also needs to contain all finite H -sets as well for every $H \leq G$, as in Definition 3.1.3 below.

3.1.1. DEFINITION. Let \mathbf{F}^G denote the category of G -objects in \mathbf{F} , i.e. a fixed full subcategory of all ordered finite G -sets with unordered actions of G and unordered G -maps, such that the wide⁴ subcategory with ordered G -maps is skeletal. In particular, following [GM],

⁴A subcategory is called *wide* if it contains all objects of the original category.

we may choose F^G to be the category with objects (\underline{n}, α) with $\alpha : G \rightarrow \Sigma_n$ a homomorphism. As in Definition 2.3.1, F^G has a natural bipermutative structure with respect to product and disjoint union.

Let O_G denote the full subcategory of F^G of the transitive G -sets. A \mathcal{B} -coefficient system is a functor $O_G^{op} \rightarrow \mathcal{B}$ for some category \mathcal{B} .

Particularly simple coefficient systems are given by the system of fixed points of a G -object $X \in \mathcal{B}^G$. Define the i_*X to be the coefficient system

$$i_*X(G/H) = X^H := \lim(H \rightarrow G \xrightarrow{X} \mathcal{B}).$$

If \mathcal{B} is closed symmetric monoidal with unit I , then $X^H \simeq \mathcal{B}^G(G/H \cdot I, X)$.

3.1.2. CONVENTION. To avoid confusion following Definition 2.1.1, we will specify which type of Grothendieck fibration we mean by name and by identifying the base. Our single exception will be for coefficient systems of *sets*, where we will just write “coefficient system” to mean either the presheaf functor or the associated *cartesian* fibration.

Now, replacing the role of finite pointed sets from §2.3 will not just be finite pointed G -sets F_*^G , but a whole coefficient system.

3.1.3. DEFINITION. [BDG⁺, Nar17] Let $\underline{F}_*^G \rightarrow O_G^{op}$ denote the Grothendieck *op*-fibration associated to the functor below

$$O_G^{op} \longrightarrow \text{Cat}, \quad R \longmapsto F_*^G \downarrow_+ R_+,$$

where $F_*^G \downarrow_+ R_+$ denotes the full subcategory spanned by arrows for the form $(A \rightarrow R)_+$.

We unpack this definition as follows. Objects are G -maps $A \rightarrow R$ with $A \in F^G$ and $R \in O_G$, and an arrow $f : (A \rightarrow R) \rightarrow (B \rightarrow S)$ is given by a triple (q, A_f, \bar{f}) of a G -map $q : S \rightarrow R$, an inclusion $A_f \hookrightarrow q^*A$ over S , and a G -map $\bar{f} : A_f \rightarrow B$ over S .

$$\begin{array}{ccccccc}
 A & \longleftarrow & q^*A & \longleftrightarrow & A_f & \xrightarrow{\bar{f}} & B \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 R & \xleftarrow{q} & S & \xlongequal{\quad} & S & \xlongequal{\quad} & S
 \end{array} \tag{3.1}$$

Composition is given by pullbacks (see (3.3) below), and we have an obvious map $\underline{F}_*^G \rightarrow O_G$ sending $(A \rightarrow R)$ to R .

3.1.4. REMARK. This description above indeed recovers \underline{F}_*^G , the amalgamation of the categories of pointed finite H -sets for all $H \leq G$ given by Definition 3.1.3. When $R = G/H$, an arrow $A \rightarrow G/H$ is equivalent to data of an H -set by considering the fiber A_H over $eH \in G/H$. Moreover, for $S = G/K$ with $K \leq H$ and $q : G/K \rightarrow G/H$ the canonical quotient map, the pullback $q^*A \rightarrow G/K$ represents the restriction of A_H to a K -set, as the fiber over eK is precisely $i_K^*A_H$. Finally, a pointed map of K -sets $A_{K,+} \rightarrow B_{K,+}$ is the same as a partially defined map of K -sets $A_K \leftarrow (A_f)_K \rightarrow B_K$, where the orbits of $A_K \setminus (A_f)_K$ are sent to the basepoint of $B_{K,+}$. Thus, we should think of (3.1) as representing a pointed equivariant map from a pointed H -set to a pointed K -set.

3.1.5. DEFINITION. For any object $(A \rightarrow R)$, we define the set of orbits to be A/G , where we remember that each element $U \in A/G$ comes with a specified inclusion $U \hookrightarrow A$ over R .

3.1.6. NOTATION. As a general convention, we will use A, B, C to denote arbitrary G -sets, U, V, W to denote their orbits, and R, S, T to denote transitive G -sets acting as bases of the objects in $\underline{\mathbf{F}}_*^G$.

Following Definition 2.3.1, we name several classes of maps in $\underline{\mathbf{F}}_*^G$.

3.1.7. DEFINITION. A map $f = (q, A_f, \bar{f}) : A \rightarrow B$ in $\underline{\mathbf{F}}_*^G$ is called

- inert if \bar{f} is an isomorphism.
- a projection map if $A_f = B = U$ for some $U \in A/G$, and both q and \bar{f} are identities; in this case, we denote f by π_U .
- a quotient map if $A_f = q^*A$ and \bar{f} is an isomorphism.
- an orbit map if $A = R$, $A_f = q^*A$, and \bar{f} is the identity.

3.1.8. REMARK. Remark 3.1.4 provides the equivalence of the above notions of *inert* and *projection* with those from Definition 2.3.1: a map is *inert* (resp. a *projection*) if the map of pointed G -sets over S is surjective and additionally injective away from the basepoint (resp. $B = U$ for some orbit $U \in A/G$, $\bar{f}(V)$ is not the basepoint for all $V \neq U$ in A/G , and $\bar{f}|_U$ is the identity).

Let $\underline{\mathbf{F}}_{*,in}^G, \underline{\Sigma}^G \subseteq \underline{\mathbf{F}}_*^G$ denote the wide subcategories with inert maps and quotient maps respectively.

3.1.9. REMARK. We note that $\underline{\Sigma}^G$ is actually a subcategory of $\underline{\mathbf{F}}^G$ *unpointed* finite G -sets (and in fact is the maximal G -subgroupoid over \mathbf{O}_G). Moreover, in the case $G = *$, $\underline{\Sigma}^G$ is just the *symmetric category* $\Sigma = \sqcup \Sigma_n$, the disjoint union of all symmetric groups. Keeping with this terminology, we call $\underline{\Sigma}^G$ the *G -symmetric category*.

We end this subsection with a technical look at composition in $\underline{\mathbf{F}}_*^G$. Specifically, given arrows

$$(q, A_f, \bar{f}) : (A \rightarrow R) \rightarrow (B \rightarrow S), \quad (p, B_g, \bar{g}) : (B \rightarrow S) \rightarrow (C \rightarrow T), \quad (3.2)$$

define A_{gf} to be the pullback of A_f and B_g over B . These pieces fit together in the following commutative diagram, where in particular the denoted squares are pullbacks

and we define the composite $g \circ f$ to be the outer rectangle.

$$\begin{array}{ccccccccc}
 A & \longleftarrow & p^*q^*A & \longleftarrow & p^*A_f & \xlongequal{\quad} & p^*A_f & \dashrightarrow & A_{gf} & \xrightarrow{\bar{g}f} & C \\
 \parallel & & \downarrow & & \downarrow & & \parallel & & \parallel & & \parallel \\
 A & \longleftarrow & q^*A & \longleftarrow & A_f & \xleftarrow{p} & p^*A_f & \dashrightarrow & A_{gf} & & \\
 \downarrow & & \downarrow & & \bar{f} \downarrow & \lrcorner & \downarrow p^*\bar{f} & \lrcorner & \downarrow & & \downarrow \\
 R & \xleftarrow{q} & S & \xlongequal{\quad} & S & \xleftarrow{p} & T & \xlongequal{\quad} & T & \xlongequal{\quad} & T
 \end{array} \tag{3.3}$$

We may identify the inverse image of orbits in C under the composite $\bar{g}f$.

3.1.10. NOTATION. Given $q : S \rightarrow R$ in \mathbf{O}_G and $(A \rightarrow R) \in \mathbf{F}_*^G$, for all $\bar{U} \in q^*A/G$ we write $q_{\bar{U}} : \bar{U} \rightarrow q(\bar{U})$ for the induced map in \mathbf{O}_G .

3.1.11. LEMMA. For arrows f and g as in (3.2), and any $W \in C/G$, we have

$$\bar{g}f^{-1}(W) = \coprod_{\bar{V} \in \bar{g}^{-1}(W)/G} p_{\bar{V}}^*(\bar{f}^{-1}(p(\bar{V}))). \tag{3.4}$$

PROOF. First, we note that for any $V \in B/G$, we have that

$$\coprod_{\bar{V} \in p^{-1}(V)/G} p_{\bar{V}}^*(\bar{f}^{-1}(V)) = (p^*\bar{f})^{-1}(p^{-1}(V)). \tag{3.5}$$

Indeed, the triple of inclusions $(\bar{V} \hookrightarrow p^*B, V \hookrightarrow B, \bar{f}^{-1}(V) \hookrightarrow A_f)$ induces an inclusion of pullbacks $p_{\bar{V}}^*(\bar{f}^{-1}(V)) \hookrightarrow p^*A_f$, whence we conclude $(p^*\bar{f})^{-1}(\bar{V}) = p_{\bar{V}}^*(\bar{f}^{-1}(V))$. Second, we in fact have the more general statement that for any G -closed subset $S \subseteq (p^*V)/G$,

$$\coprod_{\bar{V} \in S} p_{\bar{V}}^*(\bar{f}^{-1}(V)) = (p^*\bar{f})^{-1}(S). \tag{3.6}$$

Finally, (3.4) follows since we have the identifications below for all $W \in C/G$ by (3.3).

$$\begin{aligned}
 \bar{g}f^{-1}(W) &= (p^*\bar{f})^{-1}(\bar{g}^{-1}(W)) \\
 \coprod_{\bar{V} \in \bar{g}^{-1}(W)/G} p_{\bar{V}}^*(\bar{f}^{-1}(p(\bar{V}))) &= \coprod_{V \in B/G} \left(\coprod_{\bar{V} \in ((p^{-1}(V)) \cap (\bar{g}^{-1}(W)))/G} p_{\bar{V}}^*(\bar{f}^{-1}(p(\bar{V}))) \right)
 \end{aligned}$$

■

3.2. EQUIVARIANT OPERADS. In this section, we introduce two of the major players of this paper, colored genuine equivariant operads and \mathbf{O}_G - ∞ -operads.

COLORED GENUINE EQUIVARIANT OPERADS. We begin with the former. As noted in the introduction, the category $\mathbf{sOp}_G^{\{*\}}$ of single-colored genuine equivariant operads was introduced in [BPc] as a projective model category Quillen equivalent to the category $\mathbf{sOp}_{\{*\}}^G$ of single-colored simplicial equivariant operads, i.e. operads in simplicial G -sets (or equivalently G -objects in $\mathbf{sOp}_{\{*\}}$). Objects $\mathcal{P} \in \mathbf{sOp}_G$ have evaluations for all finite H -sets for all $H \leq G$ — in fact, have an underlying functor $\underline{\Sigma}^G \rightarrow \mathbf{sSet}$ — and have composition laws which respect the orbit structures of the various participating G -sets. Morally, these play the same role coefficient system of spaces played in the Elmendorf-Piacenza Theorem [Elm83, Pia91] showing the Quillen equivalence $\mathbf{Top}^G \simeq_Q \mathbf{Top}^{\mathbf{O}_G^{op}}$.

Below, we give a description of a generalization of this structure which allows for multiple objects/colors⁵. When working with many-colored equivariant simplicial operads, the associated set of colors is in fact a G -set, with action inherited by the G -action on the operad itself (see e.g. [BPe]). However, in the *genuine* setting, we are instead allowed to have a non-trivial *coefficient system* of colors, agreeing with our moral intuition.

The main ingredient in this many-colored generalization is a replacement of $\underline{\Sigma}^G \rightarrow \mathbf{O}_G^{op}$ with a many-colored variant $\underline{\Sigma}_{\mathcal{C}}^G$. First, we recall the following categories of *tuples*.

3.2.1. DEFINITION. For any category \mathcal{C} , let $\mathbf{F} \wr \mathcal{C} \rightarrow \mathbf{F}$ denote the (split) Grothendieck fibration associated to the functor

$$\mathbf{F}^{op} \longrightarrow \mathbf{Cat}, \quad A \mapsto \mathcal{C}^{\times A}.$$

Explicitly, objects are tuples $(A, (X_a))$ of elements in \mathcal{C} , and arrows $(A, (X_a)) \rightarrow (B, (Y_b))$ are maps $\alpha : A \rightarrow B$ in \mathbf{F} and arrows $f_a : X_a \rightarrow Y_{\alpha(a)}$ in \mathcal{C} .

We write $\mathbf{F}_s \wr \mathcal{C}$ (resp. $\Sigma \wr \mathcal{C}$) for the analogous definition replacing \mathbf{F} with the wide subcategory \mathbf{F}_s of surjective maps (resp. Σ of isomorphisms).

3.2.2. DEFINITION. Define the edge orbit and leaf orbit functors⁶

$$\mathbf{E}_G : \underline{\Sigma}^{G,op} \longrightarrow \mathbf{F}_s \wr \mathbf{O}_G, \quad \mathbf{L}_G : \underline{\Sigma}^{G,op} \longrightarrow \mathbf{F}_s \wr \mathbf{O}_G, \tag{3.7}$$

by letting $\mathbf{E}_G(A \rightarrow R)$ be the tuple of all orbits $(A/G, (U)) \sqcup (*, R)$, and $\mathbf{L}_G(A \rightarrow R) = (A/G, (U))$ the tuple of orbits of the source.

⁵It would not be wrong to call these structures *genuine equivariant multicategories*. However, as with our earlier conventions, we follow Lurie, Berger, Cisinski, and Moerdijk and write “operad” to refer to both the classical single-colored notion as well as the more general many-colored variety.

⁶This terminology comes from recognizing $\underline{\Sigma}^G$ as the category of *G-corollas* as in [BPc, Per18].

3.2.3. DEFINITION. Fix a coefficient system of sets \mathfrak{C} . The \mathfrak{C} -colored G -symmetric category, denoted $\underline{\Sigma}_{\mathfrak{C}}^{G,op}$, is the pullback below.

$$\begin{array}{ccc} \underline{\Sigma}_{\mathfrak{C}}^{G,op} & \longrightarrow & \mathbf{F}_s \wr \mathfrak{C} \\ \downarrow & & \downarrow \\ \underline{\Sigma}^{G,op} & \xrightarrow{\mathbf{E}_G} & \mathbf{F}_s \wr \mathbf{O}_G \end{array} \tag{3.8}$$

Objects are called \mathfrak{C} -signatures, and are written

$$\vec{C} = (A \rightarrow R, ((x_U); x_R)) = \left(\begin{array}{c} A \\ \downarrow \\ R \end{array}, ((x_U); x_R) \right) \tag{3.9}$$

with $(A \rightarrow R)$ in $\underline{\Sigma}^G$, $U \in A/G$, and $x_U \in \mathfrak{C}_U$ and $x_R \in \mathfrak{C}_R$. We call $(A \rightarrow R)$ the arity of the signature, and will sometimes denote the arity of \vec{C} by C .

Arrows in the opposite⁷ category $\underline{\Sigma}_{\mathfrak{C}}^G$

$$f: \left(\begin{array}{c} A \\ \downarrow \\ R \end{array}, ((x_U); x_R) \right) \rightarrow \left(\begin{array}{c} B \\ \downarrow \\ S \end{array}, ((y_V); y_S) \right) \tag{3.10}$$

are given by quotient maps $(q, \bar{f}) : (A \rightarrow R) \rightarrow (B \rightarrow S)$ in $\underline{\mathbf{F}}_*^G$ (with $q : S \rightarrow R$ and $\bar{f} : q^*A \xrightarrow{\cong} B$ as in Defn. 3.1.7) such that

$$q_{\bar{U}}^* x_{q(\bar{U})} = y_{\bar{f}(\bar{U})} \quad \text{and} \quad q_S^* x_R = y_S \tag{3.11}$$

for all $\bar{U} \in q^*A/G$ (where we note $q(S) = R$).

A \mathfrak{C} -colored G -symmetric sequence is a functor $\underline{\Sigma}_{\mathfrak{C}}^G \rightarrow \mathbf{sSet}$.

A \mathfrak{C} -colored genuine equivariant operad consists of a \mathfrak{C} -colored G -symmetric sequence, equipped with a “composition law” for all appropriately-compatible signatures.

3.2.4. DEFINITION. Given some $\vec{C} = (A \rightarrow R, ((x_U); x_R)) \in \underline{\Sigma}_{\mathfrak{C}}^G$, a compatible collection is a collection of objects $\vec{D}_U = (B_U \rightarrow U, ((x_{U,V}); x_U))$, one for each $U \in A/G$. The composite of the compatible collection is another object in $\underline{\Sigma}_{\mathfrak{C}}^G$, denoted $\vec{C} \circ (\vec{D}_U)$, defined to be

$$\left(\begin{array}{c} B \\ \downarrow \\ R \end{array}, ((x_{U,V}); x_R) \right), \quad \text{with} \quad B = \coprod_{U \in A/G} B_U.$$

3.2.5. DEFINITION. [cf. [BPc, Eq. (1.11)]] A colored genuine equivariant operad \mathcal{P} is given by the following data:

- A coefficient system $\mathfrak{C} = \mathfrak{C}(\mathcal{P})$ of colors;
- A \mathfrak{C} -colored G -symmetric sequence $\mathcal{P} : \underline{\Sigma}_{\mathfrak{C}}^G \rightarrow \mathbf{sSet}$;

⁷This convention is further discussed in Warning 3.2.6.

- For all compatible collections $\vec{C}, (\vec{D}_U)$ as in Definition 3.2.4, a composition structure map

$$\mu : \mathcal{P}\left(\begin{smallmatrix} A \\ \downarrow \\ R \end{smallmatrix}, ((x_U); x_R)\right) \times \prod_{U \in A/G} \mathcal{P}\left(\begin{smallmatrix} B_U \\ \downarrow \\ U \end{smallmatrix}, ((x_{U,V}), x_U)\right) \longrightarrow \mathcal{P}\left(\begin{smallmatrix} B \\ \downarrow \\ R \end{smallmatrix}, ((x_{U,V}), x_R)\right)$$

where $(B \rightarrow R, ((x_{U,V}); x_R))$ is the composite of the compatible collection.

These composition structures are natural in $\underline{\Sigma}_{\mathfrak{C}}^G$, associative, and unital. Spelling out naturality, we have that for any compatible collection as in Definition 3.2.4 with composite $\vec{E} = \vec{C} \circ (\vec{D}_U)$, and arrows $f = (q, \bar{f}) : (A \rightarrow R) \rightarrow (C \rightarrow S)$ in $\underline{\Sigma}_{\mathfrak{C}}^G$, the diagram

$$\begin{array}{ccc} \mathcal{P}\left(\begin{smallmatrix} A \\ \downarrow \\ R \end{smallmatrix}, ((x_U); (x_R))\right) \times \prod_{U \in A/G} \mathcal{P}\left(\begin{smallmatrix} B_U \\ \downarrow \\ U \end{smallmatrix}, ((x_{U,V}); x_U)\right) & \xrightarrow{\mu} & \mathcal{P}(\vec{E}) \\ \downarrow (f, \Delta_q) & & \downarrow f \\ \mathcal{P}\left(\begin{smallmatrix} C \\ \downarrow \\ S \end{smallmatrix}, ((q_W^* x_{q(W)}); q_S^* x_R)\right) \times \prod_{W \in C/G} \mathcal{P}\left(\begin{smallmatrix} q_W^* B_{q(W)} \\ \downarrow \\ W \end{smallmatrix}, ((q_W^* x_{q(W),V}); q_W^* x_{q(W)})\right) & \xrightarrow{\mu} & \mathcal{P}(\vec{E}') \end{array} \tag{3.12}$$

commutes, where $q_W : W \rightarrow \bar{f}^{-1}(W) \rightarrow q(\bar{f}^{-1}(W))$ is the induced map on orbits, Δ_q is the “q-twisted diagonal”

$$\prod_U \mathcal{P}\left(\begin{smallmatrix} B_U \\ \downarrow \\ U \end{smallmatrix}\right) \xrightarrow{\Delta} \prod_W \mathcal{P}\left(\begin{smallmatrix} B_{q(W)} \\ \downarrow \\ q(W) \end{smallmatrix}\right) \xrightarrow{\Pi q_W} \prod_W \mathcal{P}\left(\begin{smallmatrix} q_W^* B_{q(W)} \\ \downarrow \\ W \end{smallmatrix}\right),$$

and $\vec{E}' = q^* \vec{C} \circ (q_W^* \vec{D}_U)$ is the composite of the compatible collection written in the bottom row of (3.12), which is naturally isomorphic to $q^* \vec{E}$.

A functor $F : \mathcal{P} \rightarrow \mathcal{P}'$ of genuine equivariant operads consists of a map of coefficient systems $F_0 : \mathfrak{C}(\mathcal{P}) \rightarrow \mathfrak{C}(\mathcal{P}')$ and maps $F(\underline{C}) : \mathcal{P}(\underline{C}) \rightarrow \mathcal{P}'(F(\underline{C}))$ for all \mathfrak{C} -signatures \underline{C} , compatible with the composition structure maps.

We denote the category of genuine equivariant operads and functors by \mathbf{sOp}_G .

See [BPc] for a monadic definition of the single-colored case and further discussion.

3.2.6. WARNING. We record that some of the notational conventions in the previous definition of genuine equivariant operads are *dual* to those written in [BPc]. This comes out of [Nar17] and the author having chosen the *opposite* convention for which category forms the base of the Grothendieck fibration associated to a functor $\mathbf{Op}_G^{op} \rightarrow \mathbf{Cat}$ (see e.g. Definition 3.1.3).

Specifically, the category Σ_G^{op} from [BPc] is canonically isomorphic (as a cartesian fibration over \mathbf{Op}_G^{op}) to $\underline{\Sigma}^G$.

3.2.7. NOTATION. For $\mathcal{P} \in \mathbf{sOp}_G$ and $(A \rightarrow R) \in \underline{\Sigma}^G$, we let $\mathcal{P}(A \rightarrow R)$ denote

$$\mathcal{P}\left(\begin{matrix} A \\ \downarrow \\ R \end{matrix}\right) := \coprod \mathcal{P}\left(\begin{matrix} A \\ \downarrow \\ R \end{matrix}, ((x_U); x_R)\right)$$

where the disjoint union runs over all possible \mathfrak{C} -signatures in $\underline{\Sigma}^G$ with arity $(A \rightarrow R) \in \underline{\Sigma}^G$.

As in the non-equivariant case, we expect there to be a model structure on colored genuine equivariant operads, following [BPc, BPe]. For this paper, we will just need the following.

3.2.8. DEFINITION. A genuine equivariant operad $\mathcal{P} \in \mathbf{sOp}_G$ is called locally fibrant if $\mathcal{P}(\vec{C})$ is a Kan complex in \mathbf{sSet} for all \mathfrak{C} -signatures $\vec{C} \in \underline{\Sigma}^G$. We denote the full-subcategory spanned by locally fibrant operads by $\mathbf{sOp}_{G,f} \subseteq \mathbf{sOp}_G$.

\mathbf{O}_G - ∞ -OPERADS. For the second player, we follow [Nar17],[BDG⁺] to define \mathbf{O}_G - ∞ -operads as a particular case of parametrized ∞ -operads. Parallel to replacing Σ with $\underline{\Sigma}^G$, we replace $N(\mathbf{F}_*)$ with $N(\underline{\mathbf{F}}_*^G)$. Specifying Definition 2.3.1 to this case, if $p: X \rightarrow N(\underline{\mathbf{F}}_*^G)$ is a fixed map of simplicial sets, we refer to p -cocartesian morphisms in X as G -cocartesian.

3.2.9. DEFINITION. [BDG⁺, Nar17], cf. Defn. 2.3.2 An \mathbf{O}_G - ∞ -operad is an ∞ -category \mathcal{O}^\otimes equipped with a map of simplicial sets $p: \mathcal{O}^\otimes \rightarrow N(\underline{\mathbf{F}}_*^G)$ such that the following three conditions hold:

- (i) For all inert maps $f \in \underline{\mathbf{F}}_*^G(A \rightarrow R, B \rightarrow S)$ and for all objects $x \in \mathcal{O}_{\langle A \rightarrow R \rangle}^\otimes$, there exists a G -cocartesian morphism $\bar{f}: x \rightarrow x'$ lifting f . In particular, f (and specified choices) induces a functor $f_!: \mathcal{O}_{\langle A \rightarrow R \rangle}^\otimes \rightarrow \mathcal{O}_{\langle B \rightarrow S \rangle}^\otimes$.
- (ii) For all maps $f \in \underline{\mathbf{F}}_*^G(A \rightarrow R, B \rightarrow S)$, objects $x \in \mathcal{O}_{\langle A \rightarrow R \rangle}^\otimes$ and $y \in \mathcal{O}_{\langle B \rightarrow S \rangle}^\otimes$, and choices of G -cocartesian lifts $y \rightarrow y_V$ for each projection $\pi_V \in \underline{\mathbf{F}}_*^G(B \rightarrow S, V \rightarrow S)$ of B onto one of its orbits V , the induced Segal map

$$\mathrm{Map}_{\mathcal{O}^\otimes}^f(x, y) \longrightarrow \prod_{V \in B/G} \mathrm{Map}_{\mathcal{O}^\otimes}^{\pi_V f}(x, y_V) \tag{3.13}$$

is a weak equivalence, where $\mathrm{Map}_{\mathcal{O}^\otimes}^f(-, -) \hookrightarrow \mathrm{Map}_{\mathcal{O}^\otimes}(-, -)$ is the fiber over f .

- (iii) For all objects $(A \rightarrow R)$ with set of orbits $\{U \rightarrow R\}_{A/G}$, and all choices of functors $(\pi_U)_!$, the induced Segal map

$$\mathcal{O}_{\langle A \rightarrow R \rangle}^\otimes \longrightarrow \prod_{U \in A/G} \mathcal{O}_{\langle U \rightarrow R \rangle}^\otimes \tag{3.14}$$

is an equivalence of ∞ -categories.

3.2.10. REMARK. We will call an ∞ -category \mathcal{O}^\otimes satisfying (i) a G -inert (cocartesian) fibration, and those satisfying (i) and (iii) to be of Segal type.

4. The genuine operadic nerve

In this section, we extend the non-equivariant construction $N^\otimes(-) : \mathbf{sOp}_f \rightarrow \mathbf{Op}_\infty$ to the genuine equivariant setting, and prove Theorem 1.0.1. As in Section 2, this will be the composition of a “category of operators” construction followed by the homotopy coherent nerve.

4.1. GENUINE CATEGORY OF OPERATORS. We begin by extending Definition 2.4.1 by again applying the philosophy of replacing \mathbf{F}_* with \mathbf{F}_*^G . We first restrict to the case of a single color.

4.1.1. DEFINITION. Let $\mathcal{P} \in \mathbf{sOp}_G$ be a genuine equivariant simplicial operad with a single color. We define the genuine equivariant category of operators associated to \mathcal{P} , denoted \mathcal{P}^\otimes , as follows. The set of objects is precisely $\text{Ob}(\mathbf{F}_*^G)$. Given objects $(A \rightarrow R)$ and $(B \rightarrow S)$ in \mathbf{F}_*^G , define the mapping space

$$\mathcal{P}^\otimes \left(\begin{array}{c} A \quad B \\ \downarrow \quad \downarrow \\ R \quad S \end{array} \right) = \coprod_{f \in \mathbf{F}_*^G(A,B)} \prod_{V \in B/G} \mathcal{P} \left(\begin{array}{c} \bar{f}^{-1}(V) \\ \downarrow \\ V \end{array} \right). \tag{4.1}$$

Given composable arrows

$$(A \rightarrow R) \xrightarrow{(q, \bar{f})} (B \rightarrow S) \xrightarrow{(p, \bar{g})} (C \rightarrow T)$$

and operations

$$\xi_V \in \mathcal{P} \left(\begin{array}{c} \bar{f}^{-1}(V) \\ \downarrow \\ V \end{array} \right), \quad \psi_W \in \mathcal{P} \left(\begin{array}{c} \bar{g}^{-1}(W) \\ \downarrow \\ W \end{array} \right)$$

for all $V \in B/G$ and $W \in C/G$ respectively, the composite is given by

$$(\psi_W)_{W \in C/G} \circ (\xi_V)_{V \in B/G} = (\psi_W \circ (p_{\bar{V}}^* \xi_{p(\bar{V})})_{\bar{V} \in \bar{g}^{-1}(W)/G})_{W \in C/G}. \tag{4.2}$$

Heuristically, we need to pull back the operations ξ_V along $p_{\bar{V}}$ until they line up with the orbits of C , and then compose as in the non-equivariant case (2.8). Explicitly, this is

the composite of the following arrows in \mathbf{sSet} :

$$\begin{aligned}
 \mathcal{P}^\otimes(B, C) \times \mathcal{P}^\otimes(A, B) &= \left(\coprod_{g \in \underline{\mathbb{F}}_*^G(B, C)} \prod_{W \in C/G} \mathcal{P} \left(\begin{array}{c} \bar{g}^{-1}(W) \\ \downarrow \\ \bar{W} \end{array} \right) \right) \times \left(\coprod_{f \in \underline{\mathbb{F}}_*^G(A, B)} \prod_{V \in B/G} \mathcal{P} \left(\begin{array}{c} \bar{f}^{-1}(V) \\ \downarrow \\ \bar{V} \end{array} \right) \right) \\
 &= \coprod_{(g, f)} \left[\prod_{W \in C/G} \mathcal{P} \left(\begin{array}{c} \bar{g}^{-1}(W) \\ \downarrow \\ \bar{W} \end{array} \right) \times \prod_{V \in B/G} \mathcal{P} \left(\begin{array}{c} \bar{f}^{-1}(V) \\ \downarrow \\ \bar{V} \end{array} \right) \right] \\
 &\xrightarrow{\Delta} \coprod_{(g, f)} \left[\prod_{W \in C/G} \left(\mathcal{P} \left(\begin{array}{c} \bar{g}^{-1}(W) \\ \downarrow \\ \bar{W} \end{array} \right) \times \prod_{\bar{V} \in \bar{g}^{-1}(W)/G} \mathcal{P} \left(\begin{array}{c} \bar{f}^{-1}(p(\bar{V})) \\ \downarrow \\ p(\bar{V}) \end{array} \right) \right) \right] \\
 &\xrightarrow{p_{\bar{V}}} \coprod_{(g, f)} \left[\prod_{W \in C/G} \left(\mathcal{P} \left(\begin{array}{c} \bar{g}^{-1}(W) \\ \downarrow \\ \bar{W} \end{array} \right) \times \prod_{\bar{V} \in \bar{g}^{-1}(W)/G} \mathcal{P} \left(\begin{array}{c} p_{\bar{V}}^* \bar{f}^{-1}(p(\bar{V})) \\ \downarrow \\ \bar{V} \end{array} \right) \right) \right] \\
 &\xrightarrow{\mu} \coprod_{(g, f)} \prod_{W \in C/G} \mathcal{P} \left(\begin{array}{c} \overline{gf}^{-1}(W) \\ \downarrow \\ \bar{W} \end{array} \right) \\
 &\hookrightarrow \coprod_{h \in \underline{\mathbb{F}}_*^G(A, C)} \prod_{W \in C/G} \mathcal{P} \left(\begin{array}{c} \bar{h}^{-1}(W) \\ \downarrow \\ \bar{W} \end{array} \right), \tag{4.3}
 \end{aligned}$$

where μ is the genuine operadic composition map, and $p_{\bar{V}}$ denotes the map $\bar{V} \rightarrow p(\bar{V})$ in \mathcal{O}_G as well as the associated cartesian arrow in $\underline{\Sigma}^G$.

Now, let \mathcal{P} be an arbitrary genuine equivariant simplicial operad, with coefficient system of colors \mathfrak{C} . The genuine equivariant category of operators associated to \mathcal{P} , denoted \mathcal{P}^\otimes , is defined as follows. The set of objects is the set of equivariant tuples⁸

$$\left(\begin{array}{c} A \\ \downarrow \\ R \end{array}, (x_U)_{U \in A/G} \right)$$

with $x_U \in \mathfrak{C}_U$ (compared to \mathfrak{C} -signatures from (3.9), only the “input” orbits are labeled).

Given such tuples $(A \rightarrow R, (x_U))$ and $(B \rightarrow S, (y_V))$, we define the mapping spaces

$$\mathcal{P}^\otimes \left(\left(\begin{array}{c} A \\ \downarrow \\ R \end{array}, (x_U) \right), \left(\begin{array}{c} B \\ \downarrow \\ S \end{array}, (y_V) \right) \right) = \coprod_{f \in \underline{\mathbb{F}}_*^G(A, B)} \prod_{V \in B/G} \mathcal{P} \left(\begin{array}{c} \bar{f}^{-1}(V) \\ \downarrow \\ \bar{V} \end{array}, ((q_{\bar{U}}^* x_{q(\bar{U})})_{\bar{U} \in \bar{f}^{-1}(V)}; y_V) \right). \tag{4.4}$$

Composition is defined analogously as in (4.3), by using the naturality of \mathcal{P} with respect to quotient maps in $\underline{\Sigma}^G$ as in (4.5), for $\bar{V} \in p^*B/G$ and $\bar{U} \in q^*A/G$ (cf. (3.3)).

$$\begin{aligned}
 &\left(\begin{array}{c} \bar{f}^{-1}(p(\bar{V})) \\ \downarrow \\ p(\bar{V}) \end{array} \right) \xrightarrow{f=(p_{\bar{V}}, id)} \left(\begin{array}{c} p_{\bar{V}}^*(\bar{f}^{-1}(p(\bar{V}))) \\ \downarrow \\ \bar{V} \end{array} \right), \tag{4.5} \\
 &\mathcal{P} \left(\begin{array}{c} \bar{f}^{-1}(p(\bar{V})) \\ \downarrow \\ p(\bar{V}) \end{array}, ((q_{\bar{U}}^* x_{q(\bar{U})})_{\bar{U} \in \bar{f}^{-1}(p(\bar{V}))}; y_{p(\bar{V})}) \right) \xrightarrow{p_{\bar{V}}} \mathcal{P} \left(\begin{array}{c} p_{\bar{V}}^*(\bar{f}^{-1}(p(\bar{V}))) \\ \downarrow \\ \bar{V} \end{array}, (p_{\bar{V}}^*(q_{\bar{U}}^* X_{q(\bar{U})}); p_{\bar{V}}^* Y_{p(\bar{V})}) \right),
 \end{aligned}$$

where we observe the following.

$$p_{\bar{V}}^* \left((q_{\bar{U}}^* x_{q(\bar{U})})_{\bar{U} \in \bar{f}^{-1}(p(\bar{V}))} \right) = \left(p_{\bar{U}}^* q_{p(\bar{U})}^* x_{qp(\bar{U})} \right)_{\bar{U} \in p_{\bar{V}}^*(\bar{f}^{-1}(p(\bar{V})))} = \left((qp)_{\bar{U}}^* x_{qp(\bar{U})} \right)_{\bar{U}}$$

⁸Following Definition 5.2.1, this will be the set of objects of the category denoted $\Sigma_G \wr \mathfrak{C}$.

4.1.2. CONVENTION. The results in this section about $\mathcal{P} \in \mathbf{sOp}_G$ will have proofs which only speak to the case where \mathcal{P} has a single color. The methods can be carried through without issue — beyond excessive bookkeeping — into the many-colored setting (following (3.11)).

The following is the main result of this subsection.

4.1.3. PROPOSITION. For $\mathcal{P} \in \mathbf{sOp}_G$, \mathcal{P}^\otimes is a simplicial category, and the construction extends to a functor $(-)^\otimes : \mathbf{sOp}_G \rightarrow \mathbf{sCat}$.

PROOF. It remains to check associativity, unitality, and functoriality.

The identity on an object A is given by the identity map in $\underline{\mathbf{F}}_*^G(A, A)$ along with the identity in each $\mathcal{P}(U \rightarrow U)$, $U \in A/G$, and hence unitality of \mathcal{P}^\otimes follows from the naturality of the unitality of \mathcal{P} with respect to orbit maps.

Associativity will follow from the associativity of cartesian lifts in split Grothendieck fibrations and the associativity of \mathcal{P} . Specifically, given arrows

$$\left(\begin{array}{c} A \\ \downarrow \\ R \end{array} \right) \xrightarrow{f=(q,\bar{f})} \left(\begin{array}{c} B \\ \downarrow \\ S \end{array} \right) \xrightarrow{g=(p,\bar{g})} \left(\begin{array}{c} C \\ \downarrow \\ T \end{array} \right) \xrightarrow{h=(r,\bar{h})} \left(\begin{array}{c} D \\ \downarrow \\ M \end{array} \right),$$

in $\underline{\mathbf{F}}_*^G$, (3.4) implies

$$\bar{h}g^{-1}(Q) = \coprod_{\bar{W} \in \bar{h}^{-1}(Q)/G} r_{\bar{W}}^*(\bar{g}^{-1}(r(\bar{W}))) \tag{4.6}$$

for all $Q \in D/G$, and hence for all $\ddot{V} \in \bar{h}g^{-1}(Q)/G \subseteq r^*B_g/G$, the following triangle commutes.

$$\begin{array}{ccc} \ddot{V} & \xrightarrow{r_{\ddot{V}}} & r(\ddot{V}) \\ & \searrow & \swarrow \\ (pr)_{\ddot{V}} & & p_{p(\ddot{V})} \\ & \searrow & \swarrow \\ & pr(\ddot{V}) & \end{array} \tag{4.7}$$

Thus, for each $Q \in D/G$, we have a factorization of $(rp)_{\ddot{V}} \circ \Delta$ as

$$\begin{array}{ccccc} & & \prod_{V \in B/G} \mathcal{P} \left(\begin{array}{c} \bar{f}^{-1}(V) \\ \downarrow \\ \ddot{V} \end{array} \right) & & \\ & \swarrow \Delta & & \searrow \Delta & \\ \prod_{\substack{\bar{W} \in \bar{h}^{-1}(Q)/G, \\ \bar{V} \in \bar{g}^{-1}(r(\bar{W}))/G}} \mathcal{P} \left(\begin{array}{c} \bar{f}^{-1}(p(\bar{V})) \\ \downarrow \\ p(\bar{V}) \end{array} \right) & \xrightarrow{\Delta} & \prod_{\substack{\bar{W}, \\ \ddot{V} \in r_{\bar{W}}^* \bar{g}^{-1}(r(\bar{W}))/G}} \mathcal{P} \left(\begin{array}{c} \bar{f}^{-1}(pr(\ddot{V})) \\ \downarrow \\ pr(\ddot{V}) \end{array} \right) & & \\ \downarrow p_{\bar{V}} & & \downarrow (rp)_{\ddot{V}} & & \\ \prod_{\bar{W}, \bar{V}} \mathcal{P} \left(\begin{array}{c} p_{\bar{V}}^* \bar{f}^{-1}(p(\bar{V})) \\ \downarrow \\ \bar{V} \end{array} \right) & \xrightarrow{\Delta} & \prod_{\bar{W}, \ddot{V}} \mathcal{P} \left(\begin{array}{c} p_{r(\ddot{V})}^* \bar{f}^{-1}(pr(\ddot{V})) \\ \downarrow \\ pr(\ddot{V}) \end{array} \right) & \xrightarrow{r_{\ddot{V}}} & \prod_{\bar{W}, \ddot{V}} \mathcal{P} \left(\begin{array}{c} (rp)_{\ddot{V}}^* \bar{f}^{-1}(pr(\ddot{V})) \\ \downarrow \\ \ddot{V} \end{array} \right). \end{array} \tag{4.8}$$

Hence, by the naturality of the multiplication μ in our genuine equivariant operad \mathcal{P} with respect to quotient maps, either order of the iterated composition factors through the simplicial set

$$\prod_{Q \in D/G} \left(\mathcal{P} \left(\begin{array}{c} \bar{h}^{-1}(Q) \\ \downarrow \bar{Q} \end{array} \right) \times \prod_{\bar{W} \in \bar{h}^{-1}(Q)/G} \mathcal{P} \left(\begin{array}{c} r_{\bar{W}}^* \bar{g}^{-1}(r(\bar{W})) \\ \downarrow \bar{V} \end{array} \right) \times \prod_{\bar{V} \in \bar{h}g^{-1}(Q)/G} \mathcal{P} \left(\begin{array}{c} (pr)_{\bar{V}}^* \bar{f}^{-1}(pr(\bar{V})) \\ \downarrow \bar{V} \end{array} \right) \right), \quad (4.9)$$

and thus associativity of \mathcal{P}^\otimes follows from associativity of \mathcal{P} .

Lastly, functoriality is immediate, as maps of genuine equivariant operads are natural with respect to maps in \mathbf{O}_G and preserve multiplication. ■

4.1.4. **EXAMPLE.** The terminal genuine equivariant operad $\mathbf{Comm} \in \mathbf{sOp}_G$ has a single color and $\mathbf{Comm}(-) = *$. The associated genuine category of operators \mathbf{Comm}^\otimes is simply all of $\underline{\mathbf{F}}_*^G$, generalizing [Lur17, Example 2.1.1.18].

4.2. **PROOF OF THEOREM 1.0.1.** As indicated previously, we make the following definition.

4.2.1. **DEFINITION.** [cf. Definition 2.4.2] *Given $\mathcal{P} \in \mathbf{sOp}_G$, the genuine operadic nerve of \mathcal{P} , denoted $N^\otimes \mathcal{P}$, is the homotopy coherent nerve of the genuine category of operators*

$$N^\otimes \mathcal{P} = N(\mathcal{P}^\otimes).$$

To prove Theorem 1.0.1, we now need to show that $N^\otimes \mathcal{P}$ is an \mathbf{O}_G - ∞ -operad whenever \mathcal{P} is locally fibrant, and that N^\otimes extends to a functor, sending maps of genuine operads to maps of \mathbf{O}_G - ∞ -operads. We take care of the first requirement now, extending [Lur17, Prop. 2.1.1.26].

4.2.2. **THEOREM.** *If $\mathcal{P} \in \mathbf{sOp}_G$ is locally fibrant, then $N^\otimes(\mathcal{P})$ is a \mathbf{O}_G - ∞ -operad.*

PROOF. Since \mathcal{P} is locally fibrant, \mathcal{P}^\otimes is fibrant in \mathbf{sCat} (as Kan complexes are closed under products and coproducts), and hence $N^\otimes(\mathcal{P})$ is an ∞ -category. Moreover, the unique arrow $\mathcal{P} \rightarrow \mathbf{Comm}$ induces a map $p : N^\otimes(\mathcal{P}) \rightarrow N(\underline{\mathbf{F}}_*^G)$ which is an inner fibration by [Lur09, Prop. 2.3.1.5].

Now, for all inert maps $f : A \rightarrow B$ in $\underline{\mathbf{F}}_*^G$, we have a canonical map in $\mathcal{P}^\otimes(A, B)$, given by isomorphisms in each component, which we identify with a 1-simplex \widehat{f} in $N^\otimes(\mathcal{P})$ lying over f . By [Lur09, Prop. 2.4.1.10], \widehat{f} is G -cocartesian, and hence (i) is satisfied.

For (ii), we note in particular that for all B and all orbits $V \in B/G$, there exist G -cocartesian $\widehat{f}_V \in \mathcal{P}^\otimes(B, V)$ over the projection $\pi_V : B \rightarrow V$ in $\underline{\mathbf{F}}_*^G(B, V)$. We must show that for all maps $g : A \rightarrow B$ in $\underline{\mathbf{F}}_*^G$, the product of canonical post-composition maps

$$\mathrm{Map}_{\mathcal{P}^\otimes}^g(A, B) \longrightarrow \prod_{V \in B/G} \mathrm{Map}_{\mathcal{P}^\otimes}^{\pi_V g}(A, V) \quad (4.10)$$

is a weak equivalence. In fact, it is clear that this map is an isomorphism.

Finally, we need to show that for all objects $(A \rightarrow R)$ in $\underline{\mathbf{F}}_*^G$, the induced map

$$N^\otimes(\mathcal{P}) \times_{N(\underline{\mathbf{F}}_*^G)} \{A \rightarrow R\} \rightarrow \prod_{U \in A/G} N^\otimes(\mathcal{P}) \times_{N(\underline{\mathbf{F}}_*^G)} \{U \rightarrow R\} \quad (4.11)$$

is an equivalence. However, this is again an isomorphism. First, we note that for any G -orbit U and object $U \rightarrow R$ in \underline{F}_*^G , the simplicial category $\mathcal{P}_{(U \rightarrow R)}^\otimes$ has a single object $(U \rightarrow R)$ with mapping space $\mathcal{P}(U \rightrightarrows U)$. More generally, for any object in \underline{F}_*^G of the form $(A \sqcup B \rightarrow R)$, $\mathcal{P}_{(A \sqcup B \rightarrow R)}^\otimes$ also has a single object $(A \sqcup B \rightarrow R)$, with mapping space

$$\mathcal{P}_{(A \sqcup B \rightarrow R)}^\otimes(A \sqcup B, A \sqcup B) = \prod_{U \in A/G} \mathcal{P}\left(\begin{array}{c} U \\ \downarrow \\ U \end{array}\right) \times \prod_{V \in B/G} \mathcal{P}\left(\begin{array}{c} V \\ \downarrow \\ V \end{array}\right) = \mathcal{P}_{(A \rightarrow R)}^\otimes(A, A) \times \mathcal{P}_{(B \rightarrow R)}^\otimes(B, B). \tag{4.12}$$

The result then follows as the homotopy coherent nerve N preserves pullbacks and products. ■

4.2.3. REMARK. Following Remark 2.4.6, there is a notion of a *pre- \mathbf{O}_G - ∞ -operad* consisting of marked simplicial sets over \underline{F}_*^G marked with inert morphisms. Analogously to the non-equivariant case, the above proof shows that N^\otimes is a functor $\mathbf{sOp}_G \rightarrow \mathbf{PreOp}_{\infty, G}$.

The first main theorem now follows.

PROOF OF THEOREM 1.0.1. It remains to show functoriality. As [Lur17, Remark 2.1.2.9] naturally generalizes in the \mathbf{O}_G - ∞ setting to say that a map preserves all inert maps if and only if it preserves all inert maps over the projection maps $\pi_V : B \rightarrow V$, functoriality follows exactly as in the proof of Proposition 2.4.5. ■

5. Genuine operadic op-fibrations

In this section, we prove Theorem 1.0.2 about a specialization of the functor N^\otimes . Here, we twice extend the work of [Her00] and [Heu] — once each for equivariance and simplicial enrichment — to define genuine operadic op-fibrations in \mathbf{sOp}_G in Section 5.1, a generalization of Grothendieck op-fibrations of categories. Section 5.2 then recalls an appropriate notion of “symmetric monoidal (simplicial) category” in this genuine equivariant context, dubbed (simplicial) $E\Sigma_G$ -algebra, and shows that there is a faithful inclusion generalizing $\mathbf{SymMon} \rightarrow \mathbf{Op}$. The remaining two subsections finish the proofs of Theorems 1.0.2 and 1.0.3, by identifying the image of simplicial $E\Sigma_G$ -algebras in genuine equivariant operads and showing that the genuine operadic nerve sends this notion of symmetric monoidal category to the \mathbf{O}_G - ∞ -categorical version.

5.1. GENUINE OPERADIC OP-FIBRATIONS. In this subsection, we define genuine operadic op-fibrations.

5.1.1. DEFINITION. *Let $F: \mathcal{P} \rightarrow \mathcal{Q}$ be a map in \mathbf{sOp}_G , $C \in \underline{\Sigma}^G$, and $\xi \in \mathcal{P}(A \rightarrow R, ((x_U); x_R))$. The operation ξ called level F -cocartesian if, for every compatible $(B \sqcup R \rightarrow S) \in \underline{\Sigma}^G$, $Y_V \in \mathfrak{C}_V$ for each $V \in B/G$, and $Y_S \in \mathfrak{C}_S$, the commuting diagram below is a strict pullback*

of simplicial sets.

$$\begin{array}{ccc}
 \mathcal{P}\left(\begin{array}{c} B \sqcup R \\ \downarrow \\ S \end{array}, \left(((y_V), x_R); y_S \right) \right) & \xrightarrow{\xi^*} & \mathcal{P}\left(\begin{array}{c} B \sqcup A \\ \downarrow \\ S \end{array}, \left(((y_V), x_U); y_S \right) \right) \\
 \downarrow F & & \downarrow F \\
 \mathcal{Q}\left(\begin{array}{c} B \sqcup R \\ \downarrow \\ S \end{array}, \left(((F(y_V)), F(x_R)); F(y_S) \right) \right) & \xrightarrow{F(\xi)^*} & \mathcal{Q}\left(\begin{array}{c} B \sqcup A \\ \downarrow \\ S \end{array}, \left(((F(y_V)), F(x_U)); F(y_S) \right) \right)
 \end{array} \tag{5.1}$$

The operation ξ is called *F-cocartesian* if it is level *F-cocartesian* and additionally for any $q : (B \rightarrow S) \rightarrow (A \rightarrow R)$ in $\underline{\Sigma}^G$, $q^*\xi$ is level *F-cocartesian*.

If $F : \mathcal{P} \rightarrow \mathbf{Comm}$ is the unique map to the terminal genuine operad, we refer to *F-cocartesian operations* simply as *cocartesian*.

5.1.2. REMARK. We make several remarks.

- We are being slightly cavalier with the ordering of the input colors and source G -sets. However, as it is clear that $\xi \in \mathcal{P}(A \rightarrow R)$ is *F-cocartesian* iff $\sigma \cdot \xi$ is *F-cocartesian* for any isomorphism σ in $\underline{\Sigma}^G$, we will often omit these distinctions.
- We will mostly be restricting to working with *F-cocartesian operations* when $F : \mathcal{P} \rightarrow \mathcal{Q}$ is a *local fibration*, i.e. each

$$\mathcal{P}(\vec{C}) \rightarrow \mathcal{Q}(F\vec{C})$$

is a Kan fibration in \mathbf{sSet} for all $\mathfrak{C}(\mathcal{P})$ -signatures \vec{C} . In particular, any map between locally discrete genuine operads is a local fibration.

If F is a local fibration, then (5.1) is a pullback iff it is a homotopy pullback. As such, in this restricted setting, we can bypass defining the more homotopical notion of an “*F-h-cocartesian*” operation, where we instead require (5.1) to be a homotopy pullback.

- We can repackage our definition of *F-(h)-cocartesian* to be of the form in (2.3) (and [Heu, Defn. 1.3.1]) if we use the combinatorics of the *genuine G-trees* Ω_G and *genuine equivariant dendroidal sets* \mathbf{dSet}_G from [Per18, BPe], and an appropriate enhancement of the homotopy coherent nerve to a functor $N : \mathbf{sOp}_G \rightarrow \mathbf{dSet}_G$: The notion of level *F-h-cocartesian* can be captured by a similar ξ -restricted outer horn lifting condition on the map $N(F)$ (cf. [Lur09, Lemma 2.4.1.10(ii)]).

$$\begin{array}{ccc}
 \Omega_G[C] & & \\
 \downarrow & \searrow \xi & \\
 \Lambda^C[T] & \longrightarrow & N(\mathcal{P}) \\
 \downarrow & \nearrow \exists & \downarrow F \\
 \Omega[T] & \longrightarrow & N(\mathcal{Q})
 \end{array} \tag{5.2}$$

However, this perspective, while meaningful, will not play a large role in the proofs that follow. Thus for the sake of brevity and continuity, we will not elaborate on this description.

We collect several results about F -cocartesian operations.

5.1.3. LEMMA. *Let $F : \mathcal{P} \rightarrow \mathcal{Q}$ be a map between genuine equivariant operads. The following hold:*

(i) *An F -cocartesian operation $\xi \in \mathcal{P}(R \xrightarrow{\bar{}} R, (x; y))$ is an isomorphism iff its image in \mathcal{Q} is an isomorphism.*

(ii) *Sequential composites of F -cocartesian arrows are F -cocartesian: If the operations*

$$\xi \in \mathcal{P}\left(\begin{smallmatrix} A \\ \downarrow \\ R \end{smallmatrix}, \left(\left((x_U), x_{U_0}\right); x_R\right)\right) \quad \text{and} \quad \psi \in \mathcal{P}\left(\begin{smallmatrix} B \\ \downarrow \\ U_0 \end{smallmatrix}, \left(\left(x_V\right); x_{U_0}\right)\right)$$

are F -cocartesian, then so is their composite

$$\psi \circ_{U_0} \xi \in \mathcal{P}\left(\begin{smallmatrix} B \sqcup A \\ \downarrow \\ R \end{smallmatrix}, \left(\left(\left(x_V\right), \left(x_U\right)\right); x_R\right)\right).$$

(iii) *Parallel composites of F -cocartesian arrows have a similar universal property: If the operations*

$$\xi_R \in \mathcal{P}\left(\begin{smallmatrix} A \\ \downarrow \\ R \end{smallmatrix}, \left(\left(x_{R,U}\right); x_R\right)\right)$$

are all F -cocartesian for some collection of objects $(A \rightarrow R) \in \underline{\Sigma}^G$, then for any compatible $(B \sqcup \coprod R \rightarrow S) \in \underline{\Sigma}^G$, $y_V \in \mathfrak{C}(\mathcal{P})_V$ for each $V \in B/G$, and $y_S \in \mathfrak{C}(\mathcal{P})_S$, the diagram below is a strict pullback of simplicial sets.

$$\begin{array}{ccc} \mathcal{P}\left(\begin{smallmatrix} B \sqcup \coprod R \\ \downarrow \\ S \end{smallmatrix}, \left(\left(\left(y_V\right), \left(x_R\right)\right); y_S\right)\right) & \xrightarrow{(\xi_R)^*} & \mathcal{P}\left(\begin{smallmatrix} B \sqcup \coprod A \\ \downarrow \\ S \end{smallmatrix}, \left(\left(\left(y_V\right), \left(\left(x_{R,U}\right)\right)\right); y_S\right)\right) \\ F \downarrow & & \downarrow F \\ \mathcal{Q}\left(\begin{smallmatrix} B \sqcup \coprod R \\ \downarrow \\ S \end{smallmatrix}, \left(\left(\left(F(y_V)\right), \left(F(x_R)\right)\right); F(y_S)\right)\right) & \xrightarrow{(F(\xi_R))^*} & \mathcal{Q}\left(\begin{smallmatrix} B \sqcup \coprod A \\ \downarrow \\ S \end{smallmatrix}, \left(\left(\left(F(y_V)\right), \left(\left(F(x_{R,U})\right)\right)\right); F(y_S)\right)\right) \end{array} \tag{5.3}$$

5.1.4. DEFINITION. *A map $F : \mathcal{P} \rightarrow \mathcal{Q}$ is called a genuine operadic op-fibration if F is a local fibration with cocartesian lifts: for any arity $(A \rightarrow R) \in \underline{\Sigma}^G$, sources $x_U \in \mathfrak{C}(\mathcal{P})_U$ for each $U \in A/G$, and operation $\psi \in \mathcal{Q}(A \rightarrow R, ((F(x_U)); y_R))$, there exists F -cocartesian $\xi \in \mathcal{P}(A \rightarrow R, ((x_U); x_R))$ such that $F(\xi) = \psi$.*

F is additionally q -split if we have a chosen system of cocartesian lifts, natural in $\underline{\Sigma}^G$; that is, fixed choices of colors and cocartesian arrows

$$(x_U)^{\otimes(A \rightarrow R)} \in \mathfrak{C}(\mathcal{P})_R, \quad \xi_{(x_U)} \in \mathcal{P}\left(\begin{smallmatrix} A \\ \downarrow \\ R \end{smallmatrix}, \left(\left(x_U\right); \left(x_U\right)^{\otimes(A \rightarrow R)}\right)\right),$$

such that for any arrow $q: (B \rightarrow S) \rightarrow (A \rightarrow R)$ in $\underline{\Sigma}^G$, we have $\xi_{q^(x_U)} = q^* \xi_{(x_U)}$.*

Lastly, if additionally the composite of chosen cocartesian arrows is again a chosen cocartesian arrow, then F is called fully split.

5.1.5. DEFINITION. $\mathcal{P} \in \mathbf{sOp}_G$ is (*q-split, fully split*) *op-fibrant* if the unique map to the terminal genuine equivariant operad \mathbf{Comm} is a (*q-split, fully split*) genuine operadic op-fibration⁹.

5.1.6. DEFINITION. Given two *q-split* genuine operadic op-fibrations $\mathcal{P}, \mathcal{P}' \rightarrow \mathcal{Q}$ over the same base, we say a functor $F : \mathcal{P} \rightarrow \mathcal{P}'$ is a map of op-fibrations if F preserves cocartesian arrows.

Following Definition 2.1.1, let $\mathbf{Fib}^q(\mathcal{Q}) \subseteq \mathbf{Fib}^f(\mathcal{Q})$ denote the subcategories of $\mathbf{Op}_G \downarrow \mathcal{Q}$ spanned by *q-split* and *fully-split* operadic op-fibrations over \mathcal{Q} , respectively, with maps of op-fibrations.

5.1.7. LEMMA. Suppose \mathcal{P} is op-fibrant. Then an operation $\xi \in \mathcal{P}(A \rightarrow R, ((x_U); x_R))$ is cocartesian iff the map

$$\mathcal{P}\left(\downarrow_S^R, (x_R; y_S)\right) \xrightarrow{\xi^*} \mathcal{P}\left(\downarrow_S^A, ((x_U); y_S)\right)$$

is an isomorphism for all S and $y_S \in \mathfrak{C}_S$.

PROOF. Given $(B \sqcup R \rightarrow S)$ and (y_V) as in (5.1), let ψ be a cocartesian arrow with source $((y_V), x_R)$ and arity $(B \sqcup R \rightarrow S)$. The result then follows from Lemma 5.1.3(ii) and 2-out-of-3 for isomorphisms. ■

We end this subsection by comparing the above notion with the original 1-categorical notions.

5.1.8. REMARK. When $G = *$, a functor $p : \mathcal{P} \rightarrow \mathcal{P}'$ between *discrete* operads is a genuine operadic op-fibration iff it is an operadic fibration in the sense of [Heu]. If \mathcal{P} and \mathcal{P}' are in fact categories, then the notions of *p-cocartesian* and (*q-split, fully-split*) genuine operadic op-fibrations correspond to the notions of *p-cocartesian* and (*q-split, fully-split*) Grothendieck op-fibrations as in Definition 2.1.1.

5.2. GENUINE EQUIVARIANT SYMMETRIC MONOIDAL CATEGORIES. We now quickly recall the main definitions from [BPd], namely a model for genuine equivariant symmetric monoidal categories. Further details, discussions, and examples, as well as comparisons to other models, can be found there.

A classic symmetric monoidal structure on a category \mathcal{V} encodes a way to multiply elements of \mathcal{V} together. In particular¹⁰, for any tuple (x_1, \dots, x_n) of objects of \mathcal{V} , there is an associated object $\otimes_i x_i$. For any bicomplete closed symmetric monoidal category \mathcal{V} , in particular $\mathcal{V} = \mathbf{sSet}$, there is also a \mathcal{V} -enriched notion, where the multiplication map (and associated natural transformations) are required to be \mathcal{V} -enriched.

Equivariantly, we make a similar definition, but starting with a *coefficient system* of simplicially enriched categories. By Definition 2.1.4, this is equivalent to a split simplicial

⁹This is a significantly stronger notion of fibrant than what is required to model genuine equivariant higher algebra: We expect there to be a (projective) model structure on \mathbf{sOp}_G Quillen equivalent to the model structures on $\mathbf{sOp}^G, \mathbf{dSet}^G$, etc. from [Per18, BPe], where \mathcal{P} is fibrant iff \mathcal{P} is locally fibrant.

¹⁰Here, we are using the “unbiased” definition of symmetric monoidal category, following e.g. [Lei04].

Grothendieck fibration $\underline{\mathcal{V}} \rightarrow \mathbf{O}_G$. First, we need to define an appropriate notion of “tuple” in this context. Our choice is the following, generalizing Definition 3.2.1.

5.2.1. DEFINITION. *Given a set or category $\underline{\mathcal{C}} \rightarrow \mathbf{O}_G$ over the orbit category, define $\Sigma_G \wr \underline{\mathcal{C}}$ to be the pullback*

$$\begin{array}{ccc} \Sigma_G \wr \underline{\mathcal{C}} & \longrightarrow & \mathbf{F}_s \wr \underline{\mathcal{C}} \\ \downarrow & & \downarrow \\ \underline{\Sigma}^{G,op} & \xrightarrow{\mathbf{L}_G} & \mathbf{F}_s \wr \mathbf{O}_G. \end{array} \tag{5.4}$$

Objects are tuples $(A \rightarrow R, (x_U))$ with $(A \rightarrow R) \in \underline{\Sigma}^G$ and for each $U \in A/G$, $x_U \in \underline{\mathcal{C}}_U$.

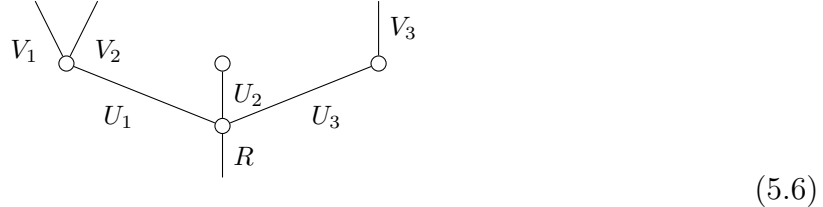
Giving \mathbf{O}_G and $\underline{\Sigma}^G$ the discrete simplicial enrichment, we define $\Sigma_G \wr \underline{\mathcal{C}}$ for any simplicial category $\underline{\mathcal{C}} \rightarrow \mathbf{O}_G$ over the orbit category to be the above pullback, taken in \mathbf{sCat} .

Unpacking, the mapping spaces (or hom-sets) are given by

$$\mathrm{Map}_{\Sigma_G \wr \underline{\mathcal{C}}} \left(\left(\begin{array}{c} B \\ \downarrow \\ S \end{array}, (y_V) \right), \left(\begin{array}{c} A \\ \downarrow \\ R \end{array}, (x_U) \right) \right) = \coprod_{(q,\bar{f}) \in \underline{\Sigma}^G(A,B)} \mathrm{Map}_{\mathcal{C}_V} (y_V, q_V^* x_{q(V)}) \tag{5.5}$$

where q, q_V are slight abuses of notation for the composite $q\bar{f}^{-1}$, resp. restricted to V .

5.2.2. EXAMPLE. Objects in $\Sigma_G \wr \Sigma_G$ are “height 2 G -trees”, an example of which is displayed below.



(5.6)

5.2.3. REMARK. [cf. Warning 3.2.6] If $\underline{\mathcal{C}} = \mathfrak{C}$ is a coefficient system of sets, then we have an isomorphism of fibrations over \mathbf{O}_G

$$\underline{\Sigma}_{\mathfrak{C}}^{G,op} \simeq \Sigma_G \wr \mathfrak{C} \times_{\mathbf{O}_G} \mathfrak{C}.$$

For any \mathfrak{C} -signature $\vec{\mathcal{C}}$ from (3.9), $\Sigma_G \wr \mathfrak{C}$ records the labelings $x_U \in \mathfrak{C}_U$ on the orbits of the source, while the remaining \mathfrak{C} records the label $x_R \in \mathfrak{C}_R$ on the target orbit.

The operation $\Sigma_G \wr (-)$ acts on the categories of Grothendieck fibrations from Definition 2.1.1.

5.2.4. PROPOSITION. [BPd] *The endofunctor $\Sigma_G \wr (-)$ from Definition 5.2.1 is a monad on the category of fully split simplicial Grothendieck fibrations over \mathbf{O}_G .*

In particular, we have a simplicially enriched functor

$$\Sigma_G \wr \Sigma_G \wr \underline{\mathcal{V}} \xrightarrow{\sigma^0} \Sigma_G \wr \underline{\mathcal{V}}, \quad \left(\begin{array}{c} A \\ \downarrow \\ R \end{array}, \left(\begin{array}{c} B_U \\ \downarrow \\ U \end{array}, (x_{U,V}) \right) \right) \mapsto \left(\begin{array}{c} B_U \\ \downarrow \\ R \end{array}, (x_{U,V}) \right).$$

We use this monad to define our algebraic structure.

5.2.5. DEFINITION. A simplicial q -split $E\Sigma_G$ -algebra is a fully split simplicial Grothendieck fibration $\underline{\mathcal{V}} \rightarrow \mathbf{O}_G$ equipped with the structure of a pseudo-algebra over the monad $\Sigma_G \wr (-)$ in the $(2,1)$ -category of fully split simplicial fibrations over \mathbf{O}_G and maps of split fibrations.

Unpacking, this is the data of a simplicially-enriched functor of split fibrations over \mathbf{O}_G

$$\Sigma_G \wr \underline{\mathcal{V}} \xrightarrow{\otimes} \underline{\mathcal{V}}$$

and a natural simplicially enriched associativity isomorphisms

$$\begin{array}{ccc} \Sigma_G \wr \Sigma_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \Sigma_G \wr \underline{\mathcal{V}} \\ \sigma^0 \downarrow & \swarrow \alpha & \downarrow \otimes \\ \Sigma_G \wr \underline{\mathcal{V}} & \xrightarrow{\otimes} & \underline{\mathcal{V}} \end{array}$$

which are unital and satisfy a “pentagon identity”.

If α is the identity, we say $\underline{\mathcal{V}}$ is fully split or G -permutative.

We will often abuse notation, and omit the adjectives “simplicial” and “ q -split”.

5.2.6. REMARK. What is written above differs from the more general definition given in [BPd]. However, when restricting to the q -strict case, i.e. when we require that \otimes is a map of split fibrations, the two definitions agree: functors $\underline{\mathcal{V}}^{op} \rightarrow \underline{\mathcal{W}}^{op}$ of split fibrations are the same data as functors $\underline{\mathcal{V}} \rightarrow \underline{\mathcal{W}}$ of split fibrations, even though they are not the same in general.

Definition 6.1.1 below provides a large class of examples: any symmetric monoidal category generates an $E\Sigma_G$ -algebra.

5.2.7. DEFINITION. A strong q -split monoidal functor between two q -split $E\Sigma_G$ -algebras $\underline{\mathcal{V}}$ and $\underline{\mathcal{W}}$ is a functor $F : \underline{\mathcal{V}} \rightarrow \underline{\mathcal{W}}$ of split fibrations over \mathbf{O}_G together with a natural isomorphism

$$\begin{array}{ccc} \Sigma_G \wr \underline{\mathcal{V}} & \xrightarrow{F} & \Sigma_G \wr \underline{\mathcal{W}} \\ \otimes \downarrow & \swarrow \rho & \downarrow \otimes \\ \underline{\mathcal{V}} & \xrightarrow{F} & \underline{\mathcal{W}} \end{array} \tag{5.7}$$

which is compatible with the associativity isomorphisms of $\underline{\mathcal{V}}$ and $\underline{\mathcal{W}}$.

We denote the category of simplicial q -split (resp. fully-split) $E\Sigma_G$ -algebras and strong q -split monoidal functors by $\mathbf{sSymMon}_G^q$ (resp. \mathbf{sPerm}_G), and $\mathbf{sSymMon}_{G,f}^q$ (resp. $\mathbf{sPerm}_{G,f}$) for the full subcategories spanned by the locally fibrant $\underline{\mathcal{V}}$.

In [Bon], we establish the following coherency result using an extension of Mac Lane’s construction, and as a consequence we have that “any diagram of associators commutes”.

5.2.8. THEOREM. The inclusion $\mathbf{sPerm}_{G,f} \hookrightarrow \mathbf{sSymMon}_{G,f}^q$ is an equivalence of categories.

We are now in position to extend the construction $\mathbf{SymMon} \hookrightarrow \mathbf{Op}(\mathbf{Set})$ into the genuine equivariant setting.

5.2.9. REMARK. Given any simplicial split Grothendieck fibration $\underline{\mathcal{V}}$, we have an additional “fiberwise” mapping space functor

$$\text{Map}_\bullet : \underline{\mathcal{V}} \times_{\mathbf{O}_G} \underline{\mathcal{V}}^{op} \longrightarrow \mathbf{sSet}^{op} \quad (U, x, y) \longmapsto \text{Map}_{\mathcal{V}_U}(x, y), \quad (5.8)$$

where $\underline{\mathcal{V}}^{op}$ is the “fiberwise opposite” category, i.e. the Grothendieck fibration associated to

$$\mathbf{O}_G^{op} \longrightarrow \mathbf{sCat}, \quad U \longmapsto \mathcal{V}_U^{op}. \quad (5.9)$$

A map $(V, \bar{x}, \bar{y}) \rightarrow (U, x, y)$ in $\underline{\mathcal{V}} \times_{\mathbf{O}_G} \underline{\mathcal{V}}^{op}$ is given by a map $q : V \rightarrow U$ in \mathbf{O}_G and maps $f : \bar{x} \rightarrow q^*x, g : q^*y \rightarrow \bar{y}$, and Map_\bullet sends this triple to the composite

$$\text{Map}_{\mathcal{V}_U}(x, y) \xrightarrow{q^*} \text{Map}_{\mathcal{V}_V}(q^*x, q^*y) \xrightarrow{f^*g^*} \text{Map}_{\mathcal{V}_V}(\bar{x}, \bar{y}); \quad (5.10)$$

an easy adjunction argument shows this functor is in fact enriched.

5.2.10. PROPOSITION. *There is a faithful functor $\mathcal{P}_{(-)} : \mathbf{sSymMon}_G^q \rightarrow \mathbf{sOp}_G$ from q -split $E\Sigma_G$ -algebras to multicolored genuine equivariant operads.*

PROOF. Fix an $E\Sigma_G$ -algebra $\underline{\mathcal{V}}$, and let \mathfrak{C} denote the coefficient system of objects. Define the \mathfrak{C} -colored G -symmetric sequence $\mathcal{P}_\underline{\mathcal{V}}$ to be the (opposite of the) following composite:

$$\mathcal{P}_\underline{\mathcal{V}}^{op} : \underline{\Sigma}_{\mathfrak{C}}^{G,op} \simeq \Sigma_G \wr \mathfrak{C} \times_{\mathbf{O}_G} \mathfrak{C}^{op} \rightarrow \Sigma_G \wr \underline{\mathcal{V}} \times_{\mathbf{O}_G} \underline{\mathcal{V}}^{op} \xrightarrow{\otimes} \underline{\mathcal{V}} \times_{\mathbf{O}_G} \underline{\mathcal{V}}^{op} \xrightarrow{\text{Map}_\bullet} \mathbf{sSet}^{op}. \quad (5.11)$$

Explicitly, $\mathcal{P}_\underline{\mathcal{V}}(A \rightarrow R, ((x_U); x_R)) := \text{Map}_{\underline{\mathcal{V}}_R}((x_U)^{\otimes A \rightarrow R}, x_R)$, and for arrows $f \in \underline{\Sigma}_{\mathfrak{C}}^G$ as in (3.10), define

$$\begin{aligned} \mathcal{P}_\underline{\mathcal{V}}(f) : \text{Map}_{\underline{\mathcal{V}}_R}((x_U)^{\otimes(A \rightarrow R)}, x_R) &\longrightarrow \text{Map}_{\underline{\mathcal{V}}_S}((q_V^* x_{q(V)})^{\otimes(B \rightarrow S)}, q^* x_R), \\ \varphi &\longmapsto \left((q_V^* x_{q(V)})^{\otimes(B \rightarrow S)} \xrightarrow{\cong} q^*((x_U)^{\otimes(A \rightarrow R)}) \xrightarrow{q^*(\varphi)} q^*(x_R) \right) \end{aligned}$$

where the first map in the image of φ is a bijection since \otimes is a map of split fibrations.

The composition and associativity of $\underline{\mathcal{V}}$ endow this genuine equivariant symmetric sequence with the structure of a genuine operad, via maps of the form

$$\text{Map}_{\underline{\mathcal{V}}_R}((x_U)^{\otimes(A \rightarrow R)}, x_R) \times \prod_U \text{Map}_{\underline{\mathcal{V}}_U}((x_{U,V})^{\otimes(B_U \rightarrow U)}, x_U) \rightarrow \text{Map}_{\underline{\mathcal{V}}_R}((x_{U,V})^{\otimes(\sqcup B_U \rightarrow R)}, x_R), \quad (5.12)$$

$$(\varphi, (\psi_U)) \mapsto \left((x_{U,V})^{\otimes(\sqcup B_U \rightarrow R)} \xrightarrow{\alpha} ((x_{U,V})^{\otimes(B_U \rightarrow U)})^{\otimes(A \rightarrow R)} \xrightarrow{(\psi_U)^{\otimes(A \rightarrow R)}} (x_U)^{\otimes(A \rightarrow R)} \xrightarrow{\varphi} x_R \right). \quad (5.13)$$

Associativity and unitality of $\mathcal{P}_\underline{\mathcal{V}}$ follow from the coherence of associativity and unitality of $\underline{\mathcal{V}}$.

Now, suppose we have a strong q -split map $(F, \rho) : (\underline{\mathcal{V}}, \otimes) \rightarrow (\underline{\mathcal{W}}, \otimes)$. Define $F : \mathcal{P}_{\underline{\mathcal{V}}} \rightarrow \mathcal{P}_{\underline{\mathcal{W}}}$ on an object $(A \rightarrow R, ((x_U); x_R)) \in \underline{\Sigma}_{\mathfrak{C}}^G$ by

$$\text{Map}_{\underline{\mathcal{V}}_R}((x_U)^{\otimes(A \rightarrow R)}, x_R) \xrightarrow{F} \text{Map}_{\underline{\mathcal{W}}_R}(F((x_U)^{\otimes(A \rightarrow R)}), F(x_R)) \xrightarrow{\varphi^*} \text{Map}_{\underline{\mathcal{W}}_R}((F(x_U))^{\otimes(A \rightarrow R)}, F(x_R)). \tag{5.14}$$

A simple diagram chase, using the fact that $\underline{\mathcal{V}}$, $\underline{\mathcal{W}}$, and F are all q -split, shows that this map is natural in $(A \rightarrow R, ((x_U); x_R))$. Moreover, F is a map of genuine equivariant operads: For any compatible collections as in (5.13), we have the diagram below.

$$\begin{array}{ccccccc} (F(x_{U,V}))^{\otimes(uBv \rightarrow R)} & \xrightarrow{\alpha} & ((F(x_{U,V}))^{\otimes(Bv \rightarrow U)})^{\otimes(A \rightarrow R)} & \xrightarrow{\rho} & (F((x_{U,V})^{\otimes(Bv \rightarrow U)}))^{\otimes(A \rightarrow R)} & \xrightarrow{\psi_U} & (F(x_U))^{\otimes(A \rightarrow R)} \xrightarrow{\rho} F((x_U)^{\otimes(A \rightarrow R)}) \xrightarrow{\varphi} F(x_R) \\ \rho \downarrow & & & & \downarrow \rho & & \downarrow \rho & \parallel \\ F((x_{U,V})^{\otimes(uBv \rightarrow R)}) & \xrightarrow{\alpha} & F(((x_{U,V})^{\otimes(Bv \rightarrow U)})^{\otimes(A \rightarrow R)}) & \xrightarrow{\psi_U} & F((x_U)^{\otimes(A \rightarrow R)}) & \xrightarrow{\varphi} & F(x_R) \end{array} \tag{5.15}$$

The left square is precisely the compatibility condition for ρ and hence commutes, while the middle square commutes by the naturality of ρ .

Finally, this functor is faithful, as the original map F can be recovered from $F : \mathcal{P}_{\underline{\mathcal{V}}} \rightarrow \mathcal{P}_{\underline{\mathcal{W}}}$ by its actions on the objects $(U \rightarrow U, ((x_U); y_U))$. ■

We record a result of this proof.

5.2.11. LEMMA. Fix an $E\Sigma_G$ -algebra $\underline{\mathcal{V}}$. Let $\vec{C} = (A \rightarrow R, ((x_U); (x_U)^{\otimes(A \rightarrow R)})) \in \underline{\Sigma}_{\mathfrak{C}}^G$ be a $\mathfrak{C}(\underline{\mathcal{V}})$ -signature, and

$$\xi \in \mathcal{P}_{\underline{\mathcal{V}}}(\vec{C}) = \text{Map}_{\underline{\mathcal{V}}_R}((x_U)^{\otimes(A \rightarrow R)}, (x_U)^{\otimes(A \rightarrow R)})$$

the identity. Then for all $q : S \rightarrow R$ in \mathbf{O}_G , $q^* : \mathcal{P}_{\underline{\mathcal{V}}}(\underline{C}) \rightarrow \mathcal{P}_{\underline{\mathcal{V}}}(q^*\underline{C})$ sends ξ to the identity.

5.2.12. REMARK. We note that the definition given of $\mathcal{P}_{\underline{\mathcal{V}}}$ is not well-defined unless $(\underline{\mathcal{V}}, \otimes)$ is q -split, and $F : \mathcal{P}_{\underline{\mathcal{V}}} \rightarrow \mathcal{P}_{\underline{\mathcal{W}}}$ is not even natural in $(A \rightarrow R) \in \underline{\Sigma}^G$ unless F itself was q -split.

5.3. PROOF OF THEOREM 1.0.3. In this section, we characterize the image of $\mathcal{P}_{(-)}$ in terms of operadic op-fibrations, and build an inverse functor. Specifically, we prove the following technical version of Theorem 1.0.3.

5.3.1. THEOREM. The faithful inclusion of categories

$$\text{sSymMon}_G^q \hookrightarrow \text{sOp}_G,$$

from Proposition 5.2.10 restricts to compatible isomorphisms of categories

$$\begin{array}{ccccc} \text{sPerm}_{G,f} & \xrightarrow[\cong]{\mathcal{P}_{(-)}} & \text{Fib}^f(\text{Comm}) & \hookrightarrow & \text{sOp}_{G,f} \\ \cong \downarrow & & \downarrow \cong & & \parallel \\ \text{sSymMon}_{G,f}^q & \xrightarrow[\mathcal{P}_{(-)}]{\cong} & \text{Fib}^q(\text{Comm}) & \hookrightarrow & \text{sOp}_{G,f}, \end{array} \tag{5.16}$$

where $\text{Fib}^f(\text{Comm}) \subseteq \text{Fib}^q(\text{Comm}) \subseteq \text{sOp}_G$ are defined as in Definition 5.1.6.

This is accomplished in three main steps: Proposition 5.3.3 establishes that the map $\mathcal{P}_{(-)}$ restricts as above, Proposition 5.3.6 proves that we have a well-defined map in the opposite direction, and the proof of Theorem 5.3.1 follows from identifying the compatibility between the two constructions.

First, we identify the cocartesian arrows in $\mathcal{P}_{\underline{\mathcal{V}}}$.

5.3.2. LEMMA. For all $(A \rightarrow R) \in \underline{\Sigma}^G$ and tuples of objects $(x_U)_{U \in A/G}$ with $x_U \in \underline{\mathcal{V}}_U$, an operation $\xi \in \mathcal{P}_{\underline{\mathcal{V}}}(A \rightarrow R, ((x_U); x_R))$ is cocartesian iff $\xi : (x_U)^{\otimes(A \rightarrow R)} \rightarrow x_R$ is an isomorphism in \mathcal{V}_R .

PROOF. This follows immediately from the composition structure of $\mathcal{P}_{\underline{\mathcal{V}}}$ from (5.13). ■

5.3.3. PROPOSITION. For any q -split (resp. fully split) $E\Sigma_G$ -algebra $\underline{\mathcal{V}}$, $\mathcal{P}_{\underline{\mathcal{V}}}$ is a q -split (resp. fully split) op-fibrant genuine equivariant operad.

PROOF. The identity map

$$id = \xi_{(x_U)} \in \mathcal{P}_{\underline{\mathcal{V}}}((x_U), (x_U)^{\otimes(A \rightarrow R)}) = \text{Map}_{\underline{\mathcal{V}}_R}((x_U)^{\otimes(A \rightarrow R)}, (x_U)^{\otimes(A \rightarrow R)})$$

is a cocartesian lift by Lemma 5.3.2. Moreover, Lemma 5.2.11 then implies that these choices are natural in $(A \rightarrow R) \in \underline{\Sigma}^G$. Finally, we observe that the composite of chosen cocartesian arrows is an instance of the natural isomorphism α , and thus these composites are all the identity iff α is the identity. ■

We will now show that these split op-fibrant genuine equivariant operads are precisely the image of $\text{sSymMon}_{G,f}^q$ by defining an inverse operation.

5.3.4. DEFINITION. Fix a q -split op-fibrant object $\mathcal{P} \in \text{sOp}_{G,f}$ with coefficient system of colors \mathfrak{C} . Define the coefficient system $\underline{\mathcal{V}} = \underline{\mathcal{V}}[\mathcal{P}]$ by setting $\underline{\mathcal{V}}_U$ to be the simplicial category with object set \mathfrak{C}_U and mapping spaces

$$\text{Map}_{\underline{\mathcal{V}}_U}(x, y) = \mathcal{P}\left(\begin{array}{c} U \\ \downarrow \\ U \end{array}, (x; y)\right).$$

Given $q : V \rightarrow U$ in O_G , define the restriction map $\underline{\mathcal{V}}_U \rightarrow \underline{\mathcal{V}}_V$ by

$$x \longmapsto q^*x, \quad \mathcal{P}\left(\begin{array}{c} U \\ \downarrow \\ U \end{array}, (x; y)\right) \xrightarrow{q^*} \mathcal{P}\left(\begin{array}{c} V \\ \downarrow \\ V \end{array}, (q^*x; q^*y)\right).$$

Given $(A \rightarrow R) \in \underline{\Sigma}^G$ and objects $x_U \in \mathfrak{C}_U$ for each $U \in A/G$, let $(x_U)^{\otimes(A \rightarrow R)}$ denote the codomain of the chosen cocartesian arrow associated to $(A \rightarrow R, (x_U))$, and denote the arrow itself by

$$\xi_{(x_U)} : (x_U) \longrightarrow (x_U)^{\otimes(A \rightarrow R)}, \quad \xi_{(x_U)} \in \mathcal{P}\left(\begin{array}{c} A \\ \downarrow \\ R \end{array}, \left((x_U); (x_U)^{\otimes(A \rightarrow R)}\right)\right).$$

We define the genuine monoidal product $\Sigma_G \wr \underline{\mathcal{V}} \xrightarrow{\otimes} \underline{\mathcal{V}}$ on objects by $(A \rightarrow R, (x_U)) \mapsto (x_U)^{\otimes(A \rightarrow R)}$. Given an arrow¹¹, $(q, (f_V)) : (B \rightarrow S, (y_V)) \rightarrow (A \rightarrow R, (x_U))$ in $\underline{\Sigma}_{\mathfrak{C}}^{G,op}$ define the associated arrow in $\underline{\mathcal{V}}$ by

$$(y_V)^{\otimes(B \rightarrow S)} \xrightarrow{(f_V)^{\otimes(B \rightarrow S)}} (q_V^* x_{q(V)}) = q^* ((x_U)^{\otimes(A \rightarrow R)}) \xrightarrow{q} (x_U)^{\otimes(A \rightarrow R)},$$

where $(f_V)^{\otimes(B \rightarrow S)}$ is the unique operation (via Lemma 5.1.3(iii)) such that the following commutes

$$\begin{array}{ccc} (y_V) & \xrightarrow{(f_V)} & (q_V^* x_{q(V)}) \xrightarrow{\xi_{q^*(x_U)}} (q_V^* x_{q(V)})^{\otimes D} \\ \downarrow \xi_{(y_V)} & \nearrow \exists! (f_v)^{\otimes D} & \\ (y_V)^{\otimes D}, & & \end{array} \tag{5.17}$$

and we know $(q_V^* x_{q(V)}) = q^* ((x_U)^{\otimes C})$ since \mathcal{P} is q -split.

5.3.5. LEMMA. The above multiplication map $\otimes : \Sigma_G \wr \underline{\mathcal{V}}[\mathcal{P}] \rightarrow \underline{\mathcal{V}}[\mathcal{P}]$ is functorial, and moreover a map of split fibrations over \mathbf{O}_G .

PROOF. The ‘‘moreover’’ statement follows by the naturality of the chosen cocartesian arrows. Given composable maps

$$(E, (z_W)) \xrightarrow{(p,(g_W))} (D, (y_V)) \xrightarrow{(q,(f_V))} (C, (x_U))$$

in $\Sigma_G \wr \underline{\mathcal{V}}[\mathcal{P}]$, the first claim holds since the following diagram commutes,

$$\begin{array}{ccccc} (z_W) & \xrightarrow{(g_W)} & (y_W) = p^*(y_V) & \xrightarrow{p^*((f_V))} & p^*((x_V)) = (x_W) \\ \xi_{(z_W)} \downarrow & & \xi_{(y_W)} \downarrow p^* \xi_{(y_V)} & & p^* \xi_{(x_V)} \downarrow \xi_{(x_W)} \\ (z_W)^{\otimes E} & \xrightarrow{(g_W)^{\otimes E}} & (y_W)^{\otimes E} = p^*((y_V)^{\otimes D}) & \xrightarrow{p^*((f_V)^{\otimes D})} & p^*((x_V)^{\otimes D}) = (x_W)^{\otimes E}, \end{array} \tag{5.18}$$

where

$$x_V = q_V^* x_{q(V)}, \quad x_W = p_W^* x_{p(W)}, \quad y_W = p_W^* y_{p(W)}.$$

■

5.3.6. PROPOSITION. For $\mathcal{P} \in \mathbf{sOp}_{G,f}$ q -split (resp. fully split) op-fibrant, $(\underline{\mathcal{V}}[\mathcal{P}], \otimes)$ is a q -split (resp. fully split) $E\Sigma_G$ -algebra

PROOF. Let \mathfrak{C} be the coefficient system of colors associated to \mathcal{P} . Consider an element $(A \rightarrow R, (B_U \rightarrow U, (x_{U,V}))) \in \Sigma_G \wr \Sigma_G \wr \underline{\mathcal{V}}$, so $x_{U,V} \in \mathfrak{C}_V$ for all $V \in B_U/G$ and all $U \in \mathbf{A}/G$, and

¹¹We warn that this is now the reverse of what we saw in (3.10).

let $B = \sqcup B_U$. We will build a natural isomorphisms $(x_V)^{\otimes(B \rightarrow R)} \rightarrow ((x_V)^{\otimes(B_U \rightarrow U)})^{\otimes(A \rightarrow R)}$ and it's inverse. Let

$$\begin{aligned} \xi_B &: (x_V)_{V \in \mathbf{B}/G} \longrightarrow (x_V)^{\otimes(B \rightarrow R)}, \\ \xi_U &: (x_V)_{V \in \mathbf{B}_U/G} \longrightarrow (x_V)^{\otimes(B_U \rightarrow U)}, \\ \xi_A &: ((x_V)^{\otimes(B_U \rightarrow U)})_{U \in \mathbf{A}/G} \longrightarrow ((x_V)^{\otimes(B_U \rightarrow U)})^{\otimes(A \rightarrow R)} \end{aligned}$$

denote the chosen cocartesian arrows.

First, define $\alpha : (x_V)^{\otimes(B \rightarrow R)} \rightarrow ((x_V)^{\otimes(B_U \rightarrow U)})^{\otimes(A \rightarrow R)}$ in $\mathcal{P}(A \rightarrow R)$ to be the unique operation (since ξ_B is cocartesian) such that $\alpha \circ \xi_B = \xi_A \circ (\xi_U)$.

$$\begin{array}{ccc} (x_V) & \xrightarrow{(\xi_U)} & ((x_V)^{\otimes(B_U \rightarrow U)}) \xrightarrow{\xi_A} ((x_V)^{\otimes(B_U \rightarrow U)})^{\otimes(A \rightarrow R)} \\ \downarrow \xi_B & \dashrightarrow \exists! \alpha & \\ (x_V)^{\otimes(B \rightarrow R)} & & \end{array} \tag{5.19}$$

Conversely, by Lemma 5.1.3(iii) there exists a unique $\beta : ((x_V)^{\otimes(B_U \rightarrow U)}) \rightarrow (x_V)^{\otimes(B \rightarrow R)}$ in $\mathcal{P}(A \rightarrow R)$ such that $\beta \circ (\xi_U) = \xi_B$. Then

$$\alpha \circ \beta \circ (\xi_U) = \alpha \circ \xi_B = \xi_A \circ (\xi_U),$$

and hence Lemma 5.1.3(iii) implies $\alpha \circ \beta = \xi_A$.

Now, let $\gamma : ((x_V)^{\otimes(B_U \rightarrow U)})^{\otimes(A \rightarrow R)} \rightarrow (x_V)^{\otimes(B \rightarrow R)}$ in $\mathcal{P}(A \rightarrow R)$ denote the unique operation such that $\gamma \circ \xi_A = \beta$. We claim α and γ are inverse natural isomorphisms. We observe that

$$\gamma \circ \alpha \circ \xi_B = \gamma \circ \xi_A \circ (\xi_U) = \beta \circ (\xi_U) = \xi_B,$$

and hence uniqueness implies $\gamma \circ \alpha = id$. Similarly,

$$\alpha \circ \gamma \circ \xi_A \circ (\xi_U) = \alpha \circ \beta \circ (\xi_U) = \alpha \circ \xi_B = \xi_A \circ (\xi_U),$$

and again Lemma 5.1.3(iii) implies that $\alpha \circ \gamma = id$.

Second, naturality of α and γ follow from the naturality of the chosen cocartesian arrows ξ .

Third, unitality and the ‘‘pentagon identity’’ for $(\underline{\mathcal{V}}[\mathcal{P}], \otimes, \alpha)$ follow from analogous arguments as above, using the uniqueness of these factorizations involving the cocartesian arrows.

Finally, if the composite of chosen cocartesian arrows is a chosen cocartesian arrow, then by considering (5.19) we conclude that α must be the identity. ■

We may now prove Theorem 5.3.1.

PROOF OF THEOREM 5.3.1. On objects, $\mathcal{P}_{(-)}$ and $\mathcal{V}[-]$ are inverses by (5.1) and by unpacking definitions and using the fact that our chosen lifts in $\mathcal{P}_{\mathcal{V}}$ are the identities.

On arrows, for any q -split strong monoidal $F: \underline{\mathcal{V}} \rightarrow \underline{\mathcal{W}}$, \mathcal{P}_F sends cocartesian morphisms to their composite with the associated component of the natural isomorphism ρ , which is again cocartesian by Lemma 5.3.2. Conversely, any $F: \mathcal{P}_{\underline{\mathcal{V}}} \rightarrow \mathcal{P}_{\underline{\mathcal{W}}}$ induces a map of coefficient systems $\underline{\mathcal{V}} \rightarrow \underline{\mathcal{W}}$, and if F also preserves cocartesian arrows, we define $\rho_{(x_U)}$ to be the image under F of the chosen cocartesian (identity) maps $\xi_{(x_U)}$. It is straightforward to check that this produces a strong q -split monoidal simplicial functor, and that these operations are inverse on hom-sets. ■

5.3.7. NOTATION. By abuse of notation, we will use $(\underline{\mathcal{V}}, \otimes)$ to denote either a $E\Sigma_G$ -algebra or its image in \mathbf{sOp}_G .

5.4. PROOF OF THEOREM 1.0.2. We will now show that the subcategory $\mathbf{sSymMon}_G^q$ in $\mathbf{sOp}_{G,f}$ maps under N^\otimes to the (1)-subcategory of G -symmetric monoidal G - ∞ -categories $\mathbf{SymMon}_{\infty,G}$ inside the (1)-category of \mathbf{Op}_G - ∞ -operads. The bulk of the work is in Proposition 5.4.3, which translates fibration information in \mathbf{sOp}_G to fibration information in \mathbf{sCat} .

First, recalling Definition 3.2.9(iii) and Remark 3.2.10, we consider the following.

5.4.1. DEFINITION. [Nar17],[BDG⁺] *A G -symmetric monoidal G - ∞ -category is an ∞ -category \mathcal{C} equipped with a map $F: \mathcal{C} \rightarrow \underline{\mathbf{F}}_*^G$ which is a cocartesian fibration in \mathbf{sSet} of Segal type. A monoidal functor between G -symmetric monoidal G - ∞ -categories is a map of fibrations over $\underline{\mathbf{F}}_*^G$, i.e. it preserves cocartesian arrows. We denote this (1)-category by $\mathbf{SymMon}_{\infty,G}$.*

Moving back to the category of (1)-categories briefly, we make the following definition.

5.4.2. DEFINITION. [cf. Definition 3.2.9(iii), Remark 3.2.10] *Let $\mathcal{C} \rightarrow \underline{\mathbf{F}}_*^G$ be a (split) simplicial Grothendieck op-fibration of categories. We say \mathcal{C} is of Segal type if for all objects $(A \rightarrow R) \in \underline{\mathbf{F}}_*^G$, the product of the maps induced by the (chosen) cocartesian liftings against the inert projection maps $\pi_U \in \underline{\mathbf{F}}_*^G(A \rightarrow R, U \rightarrow R)$*

$$\mathcal{C}_{\langle A \rightarrow R \rangle} \xrightarrow{(\pi_U)} \prod_{U \in A/G} \mathcal{C}_{\langle U \rightarrow R \rangle} \tag{5.20}$$

is an equivalence of simplicial categories.

Extending Definition 2.1.1, we write $\mathbf{Fib}_{\text{Segal}}^f(\underline{\mathbf{F}}_*^G) \subseteq \mathbf{Fib}_{\text{Segal}}^q(\underline{\mathbf{F}}_*^G)$ for the full subcategories of $\mathbf{Fib}^f(\underline{\mathbf{F}}_*^G) \subseteq \mathbf{Fib}^q(\underline{\mathbf{F}}_*^G)$ spanned by op-fibrations of Segal type.

We have the following.

5.4.3. PROPOSITION. *Let $F: \mathcal{P} \rightarrow \mathbf{Comm}$ be a map in \mathbf{sOp}_G . Then F is a (q -split, fully split) genuine operadic op-fibration if and only if $F^\otimes: \mathcal{P}^\otimes \rightarrow \mathbf{Comm}^\otimes = \underline{\mathbf{F}}_*^G$ is a (q -split, fully split) Grothendieck op-fibration of Segal type.*

PROOF. Suppose F^\otimes is a (q -split, fully split) genuine operadic op-fibration. Fix $(A \rightarrow R)$, $(B \rightarrow S)$ in \underline{F}_*^G , an arrow $f = (q, \bar{f}) \in \underline{F}_*^G(A, B)$, and an object $(A \rightarrow R, (x_U)) \in \mathcal{P}^\otimes$ over $(A \rightarrow R)$. For each $V \in B/G$, let

$$\xi_V : (x_{\bar{U}})_{\bar{U} \in \bar{f}^{-1}(V)/G} \longrightarrow x_V, \quad x_{\bar{U}} = q_{\bar{U}}^* x_{q(\bar{U})},$$

denote a (chosen) cocartesian lift in \mathcal{P} with arity $(f^{-1}(V) \rightarrow V)$ and the given source. Then the collection (ξ_V) is in fact a lift of f in \mathcal{P}^\otimes with source $(A \rightarrow R, (x_U))$.

Moreover, we claim it is F^\otimes -cocartesian. Given another object $(C \rightarrow T)$, we can identify the composition map $(\xi_V)^*$ as in the diagram below.

$$\begin{array}{ccc} \mathcal{P}^\otimes \left(\left(\downarrow_S, (x_V) \right), \left(\downarrow_T, (z_W) \right) \right) & \xrightarrow{(\xi_V)^*} & \mathcal{P}^\otimes \left(\left(\downarrow_R, (x_U) \right), \left(\downarrow_T, (z_W) \right) \right) \\ \parallel & & \parallel \\ \coprod_{(p,\bar{g})} \prod_{W \in C/G} \mathcal{P} \left(\begin{array}{c} \bar{g}^{-1}(W) \\ \downarrow \\ W \end{array}, \left((p_{\bar{V}}^* x_{p(\bar{V})}); z_W \right) \right) & \xrightarrow{\prod p_{\bar{V}}^* \xi_{p(\bar{V})}} & \coprod_{(p,\bar{g})} \prod_{W \in C/G} \mathcal{P} \left(\begin{array}{c} \bar{g}\bar{f}^{-1}(W) \\ \downarrow \\ W \end{array}, (x_{\bar{U}}); z_W \right) \end{array}$$

where for each $\bar{U} \in (qp)^* A/G$, we define the color $x_{\bar{U}}$ in $\mathfrak{C}_{\bar{U}} = \mathfrak{C}(\mathcal{P})_{\bar{U}}$ to be the image of $x_{qp(\bar{U})}$ under either map below (cf. (4.7)).

$$\begin{array}{ccc} \mathfrak{C}_{qp(\bar{U})} & \xrightarrow{q_{p(\bar{U})}^*} & \mathfrak{C}_{p^*(\bar{U})} \\ & \searrow (qp)_{\bar{U}}^* & \swarrow p_{\bar{U}}^* \\ & \mathfrak{C}_{\bar{U}} & \end{array}$$

These maps are all well-defined by (3.4), each $p_{\bar{V}}^* \xi_{p(\bar{V})}$ is cocartesian since ξ_V is cocartesian, and thus the map is an isomorphism by Lemma 5.1.3(iii).

Conversely, if F^\otimes is a simplicial Grothendieck op-fibration, then the (chosen) cocartesian arrow of \mathcal{P}^\otimes over the canonical map $(A \rightarrow R) \rightarrow (R \rightarrow R)$ with source $(A \rightarrow R, (x_U))$ is precisely an operation, cocartesian by Lemma 5.1.7, for \mathcal{P} with source (x_U) and arity $(A \rightarrow R)$.

Lastly, naturality and composite stability of chosen lifts in fully split op-fibrant $\mathcal{P} \in \mathbf{Op}_G$ exactly correspond to naturality and composite stability of chosen lifts in \mathcal{P}^\otimes . ■

5.4.4. PROPOSITION. *Suppose $p : \mathcal{C}^\otimes \rightarrow \underline{F}_{G,*}$ is a Grothendieck op-fibration of Segal type. Then the homotopy coherent nerve $N(p)$ of p in \mathbf{sSet} is a cocartesian fibration of Segal type.*

PROOF. Since Grothendieck op-fibrations are in particular local fibrations, $N(p)$ is a cocartesian fibration by [Lur09, Lemma 2.4.1.10(ii)]. As N is right adjoint, it preserves pullbacks, products, and equivalences, and hence translates one Segal type condition to the other. ■

Putting these pieces together, we have our proof.

PROOF OF THEOREM 1.0.2. Since the category $\mathbf{sSymMon}_{G,f}^q$ is isomorphic to the category of q -split op-fibrant objects in \mathbf{Op}_G , the composite

$$\mathbf{sSymMon}_{G,f}^q \hookrightarrow \mathbf{sOp}_{G,f} \xrightarrow{N^\otimes} \mathbf{Op}_{\infty,G}$$

factors through $\mathbf{SymMon}_{\infty,G}$ by combining Propositions 5.4.3 and 5.4.4. ■

We end this section by showing that the above functor also preserves the underlying categories.

5.4.5. DEFINITION. Given $\underline{\mathcal{V}} \in \mathbf{sSymMon}_{G,f}^q$, the underlying \mathbf{O}_G -category is the underlying Grothendieck fibration $\underline{\mathcal{V}} \rightarrow \mathbf{O}_G$.

Given a G -symmetric monoidal G - ∞ -category $\underline{\mathcal{V}}^\otimes$, the underlying \mathbf{O}_G - ∞ -category is the cocartesian fibration given by the left pullback square below, while the underlying symmetric monoidal ∞ -category is given by the right pullback square.

$$\begin{array}{ccccc} \underline{\mathcal{V}} & \longrightarrow & \underline{\mathcal{V}}^\otimes & \longleftarrow & \mathcal{V}^\otimes \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{O}_G^{op} & \longrightarrow & \underline{\mathbf{F}}_*^G & \longleftarrow & \mathbf{F}_* \end{array}$$

Unpacking definitions, the following is clear.

5.4.6. LEMMA. If $\mathcal{C} \rightarrow \mathcal{B}$ is a fully split Grothendieck fibration, then $\mathcal{C}^{op,op} \rightarrow \mathcal{B}^{op}$ is the associated dual fully split Grothendieck op-fibration (cf. Remark 5.2.9).

5.4.7. COROLLARY. Fix $(\underline{\mathcal{V}}, \otimes) \in \mathbf{sSymMon}_{G,f}^q$. Then $N(\underline{\mathcal{V}}^{op,op})$ is an \mathbf{O}_G - ∞ -category.

Moreover, the underlying \mathbf{O}_G - ∞ -category associated to the G -symmetric monoidal \mathbf{O}_G - ∞ -category $N^\otimes(\underline{\mathcal{V}}, \otimes)$ is equivalent to $N(\underline{\mathcal{V}}^{op,op})$, and the underlying symmetric monoidal ∞ -category is equivalent to $N^\otimes(\underline{\mathcal{V}}_{G/G}, \otimes)$, the non-equivariant operadic nerve of the symmetric monoidal simplicial category $(\underline{\mathcal{V}}_{G/G}, \otimes)$.

PROOF. The first claim follows by Lemma 5.4.6 and [Lur09, Lemma 2.4.1.10(ii)], while the moreover follows from the straightforward check that the squares below are pullbacks of simplicial categories.

$$\begin{array}{ccccc} \underline{\mathcal{V}}^{op,op} & \longrightarrow & (\underline{\mathcal{V}}, \otimes)^\otimes & \longleftarrow & (\underline{\mathcal{V}}_{G/G}, \otimes)^\otimes \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{O}_G^{op} & \longrightarrow & \underline{\mathbf{F}}_*^G & \longleftarrow & \mathbf{F}_* \end{array}$$

■

6. Examples and Algebras

As indicated in [BPc, Cor. 4.40], the usual notion of equivariant simplicial operads form a reflexive subcategory of genuine equivariant simplicial operads. Thus Theorem 1.0.1 provides a means to convert our favorite G -operads into \mathbf{O}_G - ∞ -operads. In this section, we unpack this for four prominent examples of single-colored equivariant operads.

6.0.1. DEFINITION. [BPc, §4.3] *Given $\mathcal{O} \in \mathbf{sOp}^G$ with a single color, define $i_*\mathcal{O} \in \mathbf{sOp}_G$ by*

$$i_*\mathcal{O} \left(\begin{matrix} A \\ \downarrow \\ R \end{matrix} \right) = \left(\prod_{r \in R} \mathcal{O}(A_r) \right)^G \simeq \mathcal{O}(|A_{r_0}|)^{\Gamma_{A_{r_0}}}, \tag{6.1}$$

where A_r is the inverse image of $r \in R$, r_0 any fixed element of R , and $\Gamma_{A_r} = \Gamma(\alpha_r)$ the graph of the homomorphism structure map $\alpha_r : H_r \rightarrow \Sigma_{|A_r|}$ encoding the H_r -action on A_r .

6.0.2. DEFINITION. *We recall that a subgroup $\Gamma \leq G \times \Sigma_n$ is called a graph subgroup if $\Gamma \cap \Sigma_n = \{e\}$. This is equivalent to the condition that Γ is the graph of some homomorphism $G \geq H \rightarrow \Sigma_n$.*

A simplicial G -operad $\mathcal{O} \in \mathbf{sOp}^G$ with a single color is called G -graph fibrant if for all $n \geq 0$ and all graph subgroups $\Gamma \leq G \times \Sigma_n$, $\mathcal{O}(n)^\Gamma$ is a fibrant simplicial set.

The main result of [BPc] states that the inclusion $i_* : \mathbf{sOp}^G \rightarrow \mathbf{sOp}_G$ is a Quillen equivalence between the G -graph model structure on \mathbf{sOp}^G , where weak equivalences and fibrations are detected on graph-subgroup fixed points, and the projective model structure on \mathbf{sOp}_G .

6.0.3. COROLLARY. *Suppose $\mathcal{O} \in \mathbf{sOp}^G$ is a G -graph-fibrant simplicial operad with a single color. Then $i_*\mathcal{O} \in \mathbf{sOp}_G$ is locally fibrant, and thus there exists an associated \mathbf{O}_G - ∞ -operad $N^\otimes(\mathcal{O})$.*

Moreover, composition in $\mathcal{O}^\otimes = (i_*\mathcal{O})^\otimes$ is defined just as in the non-equivariant category of operators: the functor i_* can be thought of as an encapsulation of the fact that composition in \mathcal{O} is well-defined when restricted to the fixed-point subspaces of this form.

6.0.4. EXAMPLE. Following Example 4.1.4, for $\mathcal{O} = \mathbf{Comm} \in \mathbf{sOp}^G$, the associated \mathbf{O}_G - ∞ -operad is simply the identity on $N(\mathbf{F}_*^G)$.

6.0.5. EXAMPLE. Let V be a finite-dimensional real orthogonal G -representation. The little V -disks operad \mathcal{D}_V has n -ary operations the space of affine embeddings $\mathbf{Emb}^{\mathbf{Aff}}(\underline{n} \times D(V), D(V))$.

Let T be an (ordered) H -set with n -elements, and $\alpha : H \rightarrow \Sigma_n$ the associated structure map. Then

$$\mathcal{D}_V(n)^{\Gamma_T} = \mathcal{D}_V(n)^{\Gamma(\alpha)} = \mathcal{D}_V(T)^H = \mathbf{Emb}^{\mathbf{Aff}, H}(T \times D(V), D(V)), \tag{6.2}$$

and moreover this space is homotopy equivalent to the space of H -equivariant embeddings $\mathbf{Emb}^H(T, V)$ (for more discussion, see e.g. [GM17, Lemma 1.2], [BH15, Thm. 4.19]).

Thus, for any two objects $(A \rightarrow G/H)$ and $(B \rightarrow G/K)$ in \underline{F}_*^G , we see that

$$\mathcal{D}_V^\otimes \left(\begin{array}{c} A & B \\ \downarrow & \downarrow \\ G/H & G/K \end{array} \right) \simeq \coprod_{f:A \rightarrow B} \prod_{Gb \in B/G} \text{Emb}^{\text{Aff}, G_b}(\bar{f}^{-1}(b) \times D(V), D(V)) \sim \coprod_{f:A \rightarrow B} \prod_{Gb \in B/G} \text{Emb}^{G_b}(\bar{f}^{-1}(b), V). \tag{6.3}$$

Now, we say that a map $\bar{f} : A_f \rightarrow B$ of G -spaces is \mathcal{D}_V -admissible if for all $b \in B$, $\bar{f}^{-1}(b)$ has a $\text{Stab}_G(b)$ -equivariant embedding into V . Given an arrow $f = (q, A_f, \bar{f})$, we note that if \bar{f} is not \mathcal{D}_V -admissible, then the f -component of $\mathcal{D}_V^\otimes(A, B)$ is empty.

Asaf Horev has constructed a completely \mathbf{O}_G - ∞ -categorical model for the framed little V -disks operad, and has shown it is equivalent to $N^\otimes(\mathcal{D}_V)$ as an \mathbf{O}_G - ∞ -operad [Hor, §3.9], with applications to genuine equivariant factorization homology. Additional uses of the N^\otimes construction will appear in upcoming work of Horev, Inbar Klang, and Foling Zou.

6.0.6. EXAMPLE. [cf. [GM17, Defn 1.2]] Let V be a finite-dimensional real orthogonal G -representation. Let $\text{Emb}_V(n)$ denote the G -space of embeddings $\text{Emb}(\underline{n} \times V, V)$. With the obvious composition, these assemble into the V -embeddings operad Emb_V .

Now, let $R_V \subseteq E_V(1)$ denote the subspace of distance reducing embeddings. A Steiner path is a map $h : I \rightarrow R_V$ with $h(1) = id$; let P_V denote the G -space of Steiner paths. There is a natural “evaluation at 0” map $\epsilon_0 : P_V \rightarrow R_V$. Let $\mathcal{K}_V(n)$ denote the G -space of ordered n -tuples of Stein paths (h_i) such that $\epsilon_0(h_i)$ are all distinct. With composition defined by amalgamation of paths pointwise, these form the V -Steiner operad \mathcal{K}_V .

We observe that for all H -sets T with n -objects and associated structure map $\alpha : H \rightarrow \Sigma_n$,

$$\mathcal{K}_V(n)^{\Gamma_T} = \mathcal{K}_V(n)^{\Gamma(\alpha)} = \mathcal{K}_V(T)^H \tag{6.4}$$

is equal to the set of “ H -stable T -tuples of ϵ_0 -distinct Steiner paths”; that is, T -indexed tuples of Steiner paths (h_t) with distinct ϵ_0 -values and $g.h_{g^{-1}t} = h_t$ for all $g \in H$ and $t \in T$.

Additionally, by [GM17, Lemma 1.5], we have a G -graph equivalence of operads $\mathcal{D}_V \rightarrow \mathcal{K}_V$, and so $\mathcal{K}_V(n)^{\Gamma_T} \sim \text{Emb}^H(T, V)$.

6.0.7. EXAMPLE. For any G -set A , let EA denote the associated chaotic G -category, with object G -set A and a unique morphism between any two objects. Now let \mathcal{P}_G denote the equivariant Barratt-Eccles operad from [GM17, GMM17], with $\mathcal{P}_G(n) = \text{Cat}(EG, E\Sigma_n) = E\text{Set}(G, \Sigma_n)$. Then for any graph subgroup $\Gamma \leq G \times \Sigma_n$, $\mathcal{P}_G(n)^\Gamma \simeq E(\text{Set}(G, \Sigma_n)^\Gamma)$, and so

$$(\mathcal{P}_G)^\otimes \left(\begin{array}{c} A & B \\ \downarrow & \downarrow \\ R & S \end{array} \right) \simeq \coprod_{(q, \bar{f})} \prod_{V \in B/G} E \left(\text{Set}(G, \Sigma_{|\bar{f}^{-1}(v_0)|})^{\Gamma_{\bar{f}^{-1}(v_0)}} \right).$$

6.1. G -SYMMETRIC MONOIDAL G - ∞ -CATEGORY OF STRICT G -OBJECTS. We investigate the effects of these constructions on a fundamental class of $E\Sigma_G$ -algebras.

6.1.1. DEFINITION. [cf. [BPd]] Let (\mathcal{V}, \square) be an (unbiased) symmetric monoidal simplicial category, and let $\mathbf{O}_G \wr \mathcal{V} \rightarrow \mathbf{O}_G$ denote the simplicial Grothendieck fibration associated to the functor

$$\mathbf{O}_G^{op} \rightarrow \text{sCat}, \quad U \mapsto \mathcal{V}^{G \times U},$$

where $G \ltimes U$ denotes the action groupoid of G on U . This is naturally a simplicial $E\Sigma_G$ -algebra, denoted $\mathcal{O}_G \wr \mathcal{V}^\square$, via the composition

$$\Sigma_G \wr (\mathcal{O}_G \wr \mathcal{V}) \xrightarrow{\cong} \mathcal{O}_G \wr (\Sigma \wr \mathcal{V}) \xrightarrow{\mathcal{O}_G \wr \square} \mathcal{O}_G \wr \mathcal{V}. \tag{6.5}$$

Explicitly, an object in the source is equivalent to the data

$$\left(\begin{matrix} A \\ \downarrow \\ R \end{matrix}, G \ltimes A \xrightarrow{X} \mathcal{V} \right),$$

and the composite (6.5) is given on objects by

$$(A \rightarrow R, X)^{\otimes(A \rightarrow R)} = C_*^\square X, \quad C_*^\square X(r) = \bigotimes_{A_r} x_a,$$

where $C: A \rightarrow R$ is as given, and C_*^\square is the indexed monoidal product of [HHR16, §A.3.2]. On mapping spaces, this is given by

$$\begin{aligned} \text{Map}_{\Sigma_G \wr \mathcal{O}_G \wr \mathcal{V}} \left(\begin{pmatrix} B \\ \downarrow \\ S \end{pmatrix}, Y \right), \begin{pmatrix} A \\ \downarrow \\ R \end{pmatrix}, X \Big) &\xrightarrow{\otimes} \text{Map}_{\mathcal{O}_G \wr \mathcal{V}} \left((S, D_*^\square Y), (R, C_*^\square X) \right) \\ \parallel & \parallel \\ \coprod_{(q, \bar{f})} \text{Map}_{\mathcal{V}^{G \ltimes B}} (Y, q^* X) &\xrightarrow{\coprod D_*^\square} \coprod_q \text{Map}_{\mathcal{V}^{G \ltimes R}} (D_*^\square Y, D_*^\square(q^* X)), \end{aligned} \tag{6.6}$$

where $C: A \rightarrow R$ and $D: B \rightarrow S$ are as given, the equality is given by adapting (5.5) to our case of $\underline{\mathcal{C}} = \mathcal{O}_G \wr \mathcal{V}$, and $D_*^\square(q^* X)$ is naturally isomorphic to $q^*(C_*^\square X)$ by [HHR16, Prop. A.31].

Now, the associated genuine category of operators $(\mathcal{O}_G \wr \mathcal{V}^\square, \otimes)^\otimes$ has objects $(A \rightarrow R, G \ltimes A \xrightarrow{X} \mathcal{V})$, and, following (4.4) and (6.6), mapping spaces of the form

$$\text{Map} \left(\begin{pmatrix} A \\ \downarrow \\ R \end{pmatrix}, G \ltimes A \xrightarrow{X} \mathcal{V} \right), \begin{pmatrix} B \\ \downarrow \\ S \end{pmatrix}, G \ltimes B \xrightarrow{Y} \mathcal{V} \Big) = \coprod_{(q, \bar{f}) \in \mathbb{F}_*^G(A, B)} \prod_{V \in B/G} \text{Map}_{\mathcal{V}^{G \ltimes B}} \left((\bar{f}_V)_*^\square f^* X, Y \right)$$

where

$$f^* X: G \ltimes \bar{f}^{-1}(B) \rightarrow G \ltimes q^* A \xrightarrow{q} G \ltimes A \xrightarrow{X} \mathcal{V}, \quad \bar{f}_V: \bar{f}^{-1}(B) \rightarrow V,$$

and we are using that the following square commutes up to natural isomorphism for all covering categories $p: I \rightarrow J$ and $q: I' \rightarrow J'$.

$$\begin{array}{ccc} \mathcal{V}^I \times \mathcal{V}^J & \xrightarrow{\cong} & \mathcal{V}^{I \sqcup J} \\ p_*^\otimes \times q_*^\otimes \downarrow & & \downarrow (p \sqcup q)_*^\otimes \\ \mathcal{V}^{I'} \times \mathcal{V}^{J'} & \xrightarrow{\cong} & \mathcal{V}^{I' \sqcup J'} \end{array}$$

To ensure that all of our mapping spaces are in fact Kan complexes, we need an additional assumption on \mathcal{V} .

6.1.2. DEFINITION. We say that a symmetric monoidal simplicial category is globally fibrant if the simplicial category of strict G -objects \mathcal{V}^G is locally fibrant for every finite group G .

We note that any symmetric monoidal topological category is globally fibrant.

6.1.3. DEFINITION. Let (\mathcal{V}, \square) be a globally fibrant symmetric monoidal simplicial category. We define the G -symmetric monoidal G - ∞ -category of strict G -objects in (\mathcal{V}, \square) , denoted $\underline{\mathcal{V}}_{\infty, G}^{\square}$, to be $N^{\otimes}(\mathbf{O}_G \wr \mathcal{V}^{\square}, \otimes)$ the genuine operadic nerve of the genuine equivariant operad associated to the simplicial $E\Sigma_G$ -algebra $\mathbf{O}_G \wr \mathcal{V}^{\square}$.

Since action groupoids $G \ltimes B$ are equivalent to disjoint unions of groups $\coprod_{[b] \in B/G} G_b$, the genuine equivariant operad $(\mathbf{O}_G \wr \mathcal{V}^{\square}, \otimes)$ is locally fibrant, and thus, by Theorem 1.0.2, $\underline{\mathcal{V}}_{\infty, G}^{\square}$ is in fact a G -symmetric monoidal G - ∞ -category.

We elaborate on this construction for a particular example.

6.1.4. EXAMPLE. Let $(\mathcal{V}, \otimes) = (\mathbf{Top}, \sqcup)$ denote the category of compactly-generated spaces (with compactly-generated mapping spaces). Then for G -sets U , functors $G \ltimes U \xrightarrow{X} \mathbf{Top}$ are equivalent to maps of G -spaces $X \rightarrow U$, and under this presentation, the $E\Sigma_G$ -algebra structure on $\mathbf{O}_G \wr \mathbf{Top}^{\sqcup}$ takes the form

$$\left(\left(\begin{array}{c} X_U \\ \downarrow \\ U \end{array} \right)_{U \in A/G} \right)^{\otimes(A \rightarrow R)} = \left(\begin{array}{c} X_U \\ \downarrow \\ R \end{array} \right).$$

Let \mathcal{P} denote the associated genuine equivariant operad, with colors and mapping spaces

$$\mathfrak{C}_U = \mathbf{Top}^G \downarrow U, \quad \mathcal{P} \left(\begin{array}{c} A \\ \downarrow \\ R \end{array}, \left(\begin{array}{c} X_U \\ \downarrow \\ U \end{array}; \begin{array}{c} Y \\ \downarrow \\ R \end{array} \right) \right) = \text{Map}_{\mathbf{Top}^G \downarrow R} \left(\begin{array}{c} X_U \\ \downarrow \\ R \end{array}, \begin{array}{c} Y \\ \downarrow \\ R \end{array} \right).$$

We see that the genuine category of operators $(\mathbf{O}_G \wr \mathbf{Top}^{\sqcup})^{\otimes} = \mathcal{P}^{\otimes}$ has objects of the form

$$\left(\begin{array}{c} A \\ \downarrow \\ R \end{array}, (G \ltimes U \xrightarrow{X_U} \mathbf{Top}) \right), \quad \text{or equivalently} \quad \left(\begin{array}{c} X \\ \downarrow \\ A \\ \downarrow \\ R \end{array} \right),$$

and mapping spaces

$$\text{Map} \left(\begin{array}{c} X \\ \downarrow \\ A \\ \downarrow \\ R \end{array}, \begin{array}{c} Y \\ \downarrow \\ B \\ \downarrow \\ S \end{array} \right) = \coprod_{(q, \bar{f})} \prod_{V \in B/G} \text{Map}_{\mathbf{Top}^G \downarrow V} \left(\begin{array}{c} X_V \\ \downarrow \\ V \end{array}, \begin{array}{c} Y_V \\ \downarrow \\ V \end{array} \right), \tag{6.7}$$

where X_V is the fiber of q^*X over $\bar{f}^{-1}(V)$, and Y_V is the fiber of Y over V . Unpacking further, we see that the vertices of (6.7) are given by triples (q, \bar{f}, \bar{F}) , such that the following diagram commutes, where the two left-most and the top-middle squares are

pullbacks.

$$\begin{array}{ccccccc}
 X & \longleftarrow & q^*X & \longleftrightarrow & X_f & \xrightarrow{\bar{F}} & Y \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 A & \longleftarrow & q^*A & \longleftrightarrow & A_f & \xrightarrow{\bar{f}} & B \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 Rq & \xleftarrow{q} & S & \xlongequal{\quad} & S & \xlongequal{\quad} & S
 \end{array}$$

This construction recovers the G -symmetric monoidal G - ∞ -category of G -spaces under disjoint union, as found in [Hor].

6.1.5. **REMARK.** Let \mathcal{V} be any globally fibrant symmetric monoidal simplicial category. Following Corollary 5.4.7, we note that:

- the underlying \mathbf{O}_G - ∞ -category of $\underline{\mathcal{V}}_{\infty, G}^{\square}$ is $N((\mathbf{O}_G \wr \mathcal{V})^{op, op})$, without its monoidal structure, and in particular the fiber over $G/G = G/G$ is simply $N(\mathcal{V}^G)$, the coherent nerve of the category of strict G -objects in \mathcal{V} .
- The underlying symmetric monoidal ∞ -category of $\underline{\mathcal{V}}_{\infty, G}^{\square}$ is $N^{\otimes}((\mathbf{O}_G \wr \mathcal{V})_{G/G}, \otimes) = N^{\otimes}(\mathcal{V}^G, \otimes)$.

6.1.6. **EXAMPLE.** For $(V, \square) = (F_*, \sqcup)$, the underlying \mathbf{O}_G - ∞ -category of $\underline{F}_{*, \infty, G}^{\sqcup}$ is precisely \underline{F}_*^G .

6.1.7. **REMARK.** There is a similarly named construction in parametrized higher category theory, the G - ∞ -category of G -objects from [BDG⁺, Defn. 7.4]. Given any ∞ -category \mathcal{D} , BDGNS define an \mathbf{O}_G - ∞ -category $\underline{\mathcal{D}}_G \rightarrow \mathbf{O}_G^{op}$ whose fiber over G/G is equivalent to the functor ∞ -category $\text{Fun}(\mathbf{O}_G^{op}, \mathcal{D})$.

We warn that $\underline{\mathcal{V}}_{\infty, G}^{\square}$ is distinct from this notion applied to the infinity category $\mathcal{D} = N(\mathcal{V})$, even after forgetting the monoidal structure. Specifically, consider the fibers over G/G , $N(\mathcal{V}^G)$ and $\text{Fun}(\mathbf{O}_G^{op}, N(\mathcal{V}))$. The objects in these categories differ in two important ways:

- (i) Objects in $N(\mathcal{V}^G)$ are simply objects with G -action, while those in $\text{Fun}(\mathbf{O}_G^{op}, N(\mathcal{V}))$ are *genuine* G -objects; and
- (ii) Objects in $N(\mathcal{V}^G)$ have a *strict* G -action, while those in $\text{Fun}(\mathbf{O}_G^{op}, N(\mathcal{V}))$ have a *homotopy coherent* \mathbf{O}_G^{op} -action.

6.2. **ALGEBRAS OVER OPERADS.** An algebra in a closed symmetric monoidal simplicial category (\mathcal{V}, \square) over a simplicial operad $\mathcal{O} \in \mathbf{sOp}$ can be recovered as a functor of simplicial operads $\mathcal{O} \rightarrow (\mathcal{V}, \square)$, where we identify (\mathcal{V}, \square) with its image under the inclusion $\mathbf{sSymMon} \hookrightarrow \mathbf{sOp}$ that we extend in Proposition 5.2.10.

Similarly, an algebra in the simplicial category of G -objects \mathcal{V}^G over a simplicial G -operad $\mathcal{O} \in \mathbf{sOp}^G$ can be recovered as a functor of simplicial G -operads $\mathcal{O} \rightarrow (\mathcal{V}_G, \square)$,

where \mathcal{V}_G is the G -enriched variation on (\mathcal{V}, \square) : objects are G -objects in \mathcal{V} , with mapping G -spaces of *all* arrows, with G acting via conjugation.

In this short subsection, we prove Theorem 1.0.4, which translates algebras over operads from the equivariant and simplicially-enriched setting to the G - ∞ -categorical one.

We first define the categories in question.

6.2.1. DEFINITION. *Given an equivariant simplicial operad $\mathcal{O} \in \mathbf{sOp}^G$ and a symmetric monoidal simplicial category \mathcal{V} , define the simplicial category of \mathcal{O} -algebras in \mathcal{V}^G , denoted $\mathbf{Alg}_{\mathcal{O}}(\mathcal{V}^G)$, to be the simplicial category of functors $\mathbf{Fun}_{\mathbf{sOp}^G}(\mathcal{O}, (\mathcal{V}_G, \square))$, with objects maps $F : \mathcal{O} \rightarrow (\mathcal{V}_G, \square)$ in \mathbf{sOp}^G , and mapping spaces*

$$\mathbf{Nat}(F, G) \subseteq \prod_{x \in \mathfrak{C}(\mathcal{O})} \mathbf{Map}_{\mathcal{V}}(F(x), G(x))$$

the subcomplex generated by the vertices (Φ_x) which form operadic natural transformations. i.e. for all $\varphi \in \mathcal{O}(x_1, \dots, x_n; x_0)$, the diagram below commutes.

$$\begin{array}{ccc} \square_n F(x_i) & \xrightarrow{\Phi} & \square_n G(x_i) \\ F(\varphi) \downarrow & & \downarrow G(\varphi) \\ F(x_0) & \xrightarrow{\Phi_{x_0}} & G(x_0) \end{array}$$

For $\mathcal{P} \in \mathbf{sOp}_G$ and $E\Sigma_G$ -algebra $\underline{\mathcal{V}}$, we analogously define simplicial categories

$$\mathbf{Alg}_{\mathcal{P}}(\underline{\mathcal{V}}) = \mathbf{Fun}_{\mathbf{sOp}_G}(\mathcal{P}, (\underline{\mathcal{V}}, \otimes)), \quad \mathbf{Alg}_{\mathcal{P}^\otimes}(\underline{\mathcal{V}}^\otimes) \subseteq \mathbf{Fun}_{\mathbf{sCat}_{\downarrow \mathbb{F}_*^G}}(\mathcal{P}^\otimes, \underline{\mathcal{V}}^\otimes).$$

We note that in the case where \mathcal{V} or $\underline{\mathcal{V}}$ is locally fibrant, so are these simplicial categories. Finally, essentially by construction, we have simplicially-enriched comparison maps

$$\mathbf{Alg}_{\mathcal{O}}(\mathcal{V}^G) \longrightarrow \mathbf{Alg}_{i_* \mathcal{O}}(i_* \mathcal{V}_G), \quad \mathbf{Alg}_{\mathcal{P}}(\underline{\mathcal{V}}) \longrightarrow \mathbf{Alg}_{\mathcal{P}^\otimes}(\underline{\mathcal{V}}^\otimes).$$

6.2.2. DEFINITION. *For \mathcal{O}^\otimes a \mathbf{O}_G - ∞ -operad and \mathcal{C} a G -symmetric monoidal G - ∞ -category, define the ∞ -category of \mathcal{O}^\otimes -algebras in \mathcal{C} , denoted $\mathbf{Alg}_{\mathcal{O}^\otimes}(\mathcal{C})$, to be the full subcomplex of $\mathbf{Map}_{\mathbf{sSet}_{\downarrow \mathbb{F}_*^G}}(\mathcal{O}^\otimes, \mathcal{C})$ spanned by the maps of \mathbf{O}_G - ∞ -operads.*

The following observation is the key step in the proof of Theorem 1.0.4.

6.2.3. LEMMA. *There exists a natural transformation*

$$\begin{array}{ccccc} \mathbf{sSymMon} & \xleftarrow{(-)^G} & \mathbf{sSymMon}^G & \xrightarrow{\quad} & \mathbf{sOp}^G \\ \mathbf{O}_G \wr (-) \downarrow & & \swarrow & & \downarrow i_* \\ \mathbf{sSymMon}_G & \xleftarrow{\quad} & & \xrightarrow{\quad} & \mathbf{sOp}_G \end{array}$$

PROOF. This follows from unpacking definitions. Let (\mathcal{V}, \square) be a symmetric monoidal simplicial category. The genuine equivariant operad $i_*(\mathcal{V}_G, \square)$ has object coefficient system the constant system at $\text{Ob}(\mathcal{V}^G)$, and we define our natural transformation on objects by sending a pair (U, X) to the diagram

$$\Delta_U X : G \times U \rightarrow \mathcal{V}, \quad u \mapsto X, \quad (u \rightarrow g.u) \mapsto (X \xrightarrow{g} X).$$

By [BH20, Prop. 5.2], there is a natural isomorphism of mapping spaces

$$\begin{aligned} \text{Map}_{i_*(\mathcal{V}_G, \square)} \left(\underset{R}{\downarrow}^A, ((X_U); Y) \right) &\longrightarrow \text{Map}_{\mathcal{O}_G \wr \mathcal{V}} \left(\underset{R}{\downarrow}^A, ((\Delta_U X_U); \Delta_R Y) \right) \\ \parallel & \\ \left(\prod_{r \in R} \text{Map}_{\mathcal{V}} \left(\square_{C(a)=r} X_U, Y \right) \right)^G &\xrightarrow{\simeq} \text{Map}_{\mathcal{V}^{G \times R}} \left(C_*^\otimes(\Delta_A X); \Delta_R Y \right). \end{aligned}$$

where $X_U, Y \in \mathcal{V}^G$ for each $U \in A/G$. The result follows. ■

PROOF OF THEOREM 1.0.4. First, given an algebra $\mathcal{O} \rightarrow (\mathcal{V}_G, \square)$, we have an associated composite of locally finite genuine equivariant operads

$$i_* \mathcal{O} \longrightarrow i_*(\mathcal{V}_G, \square) \longrightarrow (\mathcal{O}_G \wr \mathcal{V}^\square, \otimes).$$

By functoriality, this induces a map of \mathcal{O}_G - ∞ -operads $N^\otimes(\mathcal{O}) \rightarrow \underline{\mathcal{V}}_{\infty, G}^\square$.

Second, using Definition 6.2.1 and Lemma 6.2.3, we have simplicial functors

$$\text{Alg}_{\mathcal{O}}(\mathcal{V}^G) \longrightarrow \text{Alg}_{i_* \mathcal{O}}(i_* \mathcal{V}_G) \longrightarrow \text{Alg}_{i_* \mathcal{O}}(\mathcal{O}_G \wr \mathcal{V}^\square) \longrightarrow \text{Alg}_{(i_* \mathcal{O})^\otimes}((\mathcal{O}_G \wr \mathcal{V})^\otimes).$$

Third, for any simplicial categories \mathcal{C} and \mathcal{D} , we have a canonical map of simplicial sets

$$N(\text{Fun}(\mathcal{C}, \mathcal{D})) \longrightarrow \text{Fun}(N\mathcal{C}, N\mathcal{D})$$

produced over two adjoints via the composite

$$\tau(N\text{Fun}(\mathcal{C}, \mathcal{D}) \times N\mathcal{C}) \rightarrow \tau N\text{Fun}(\mathcal{C}, \mathcal{D}) \times \tau N\mathcal{C} \xrightarrow{\epsilon} \text{Fun}(\mathcal{C}, \mathcal{D}) \times \mathcal{C} \xrightarrow{ev} \mathcal{D},$$

where $\tau : \mathbf{sSet} \rightarrow \mathbf{sCat}$ is the left adjoint of the homotopy coherent nerve N .

Combining these with Definition 6.2.2, we produce a functor of ∞ -categories as desired.

$$N\text{Alg}_{\mathcal{O}}(\mathcal{V}^G) \longrightarrow \text{Alg}_{N^\otimes \mathcal{O}}(\underline{\mathcal{V}}_{\infty, G}^\square)$$

■

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