ORBIFOLDS OF RESHETIKHIN–TURAEV TQFTS

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Abstract.   We construct three classes of generalised orbifolds of Reshetikhin–Turaev theory for a modular tensor category $\mathcal{C}$, using the language of defect TQFT from [CRS1]: (i) spherical fusion categories give orbifolds for the "trivial" defect TQFT associated to vect, (ii) G-crossed extensions of $\mathcal{C}$ give group orbifolds for any finite group G, and (iii) we construct orbifolds from commutative Δ-separable symmetric Frobenius algebras in $\mathcal{C}$. We also explain how the Turaev–Viro state sum construction fits into our framework by proving that it is isomorphic to the orbifold of case (i). Moreover, we treat the cases (ii) and (iii) in the more general setting of ribbon tensor categories. For case (ii) we show how Morita equivalence leads to isomorphic orbifolds, and we discuss Tambara–Yamagami categories as particular examples.

Contents

1. Introduction and summary

For any modular tensor category $\mathcal{C}$, Reshetikhin and Turaev [RT, Tu1] constructed a 3-dimensional topological quantum field theory $\mathcal{Z}^{\mathrm{RT},\mathcal{C}} \colon \widehat{\mathrm{Bord}}_3 \longrightarrow$ vect. This construction is intimately related to the connection between the representation theory of quantum groups and knot theory [Tu1], and rational conformal field theory [FRS2]. The symmetric monoidal functor $\mathcal{Z}^{\mathrm{RT},\mathcal{C}}$ acts on diffeomorphism classes of bordisms with embedded ribbons that are labelled with data from $\mathcal{C}$, hence it assigns topological invariants to ribbon embeddings into 3-manifolds. In [CRS2] we extended this by constructing a *Reshetikhin–Turaev defect TQFT* $\mathcal{Z}^{\mathcal{C}} \colon \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}}) \to$ vect that assigns invariants to equivalence classes of stratified bordisms whose 3-, 2- and 1-strata are respectively labelled by $\mathcal{C}$, certain Frobenius algebras in $\mathcal{C}$ and their cyclic modules. The original functor $\mathcal{Z}^{\mathrm{RT},\mathcal{C}}$ is

isomorphic to a restriction of $\mathcal{Z}^\mathcal{C}$, as ribbons can be modelled by a combination of 1- and 2-strata, cf. [CRS2, Rem. 5.9]. For an n-dimensional defect TQFT $\mathcal{Z}$ with $n \in \{2,3\}$, the labelled strata (or "defects", a term used in physics to refer to regions in spacetime with certain properties that distinguish them from their surroundings) of codimension j are known to correspond to j-cells in the n-category associated to $\mathcal{Z}$ [DKR, CMS]; this is also expected for $n \geqslant 4$. Defects in Reshetikhin–Turaev theory had previously been studied in [KS, FSV, CMS].

In the present paper we construct orbifolds of Reshetikhin–Turaev TQFTs. Inspired by earlier work on rational conformal field theory [FFRS], a (generalised) orbifold theory was developed for 2-dimensional TQFTs in [CR1], which we then further generalised to arbitrary dimensions in [CRS1]: Given an n-dimensional defect TQFT $\mathcal{Z}$ (i.e. a symmetric monoidal functor on decorated stratified n-dimensional bordisms, cf. Section 2.1) and an "orbifold datum" $\mathcal{A}$ (consisting of special labels for j-strata for all $j \in \{0, 1, \ldots, n\}$, cf. Section 2.3), the generalised orbifold construction produces a closed TQFT $\mathcal{Z}_\mathcal{A} \colon \mathrm{Bord}_n \to$ vect roughly as follows. On any given bordism, $\mathcal{Z}_\mathcal{A}$ acts by choosing a triangulation, decorating its dual stratification with the data $\mathcal{A}$, evaluating with $\mathcal{Z}$, and then applying a certain projector. The defining properties of orbifold data $\mathcal{A}$ are such that $\mathcal{Z}_\mathcal{A}$ is independent of the choice of triangulation.

In dimension $n = 2$, orbifold data turn out to be certain Frobenius algebras in the 2-category associated to $\mathcal{Z}$, and both state sum models [DKR] and ordinary group orbifolds [CR1, BCP] are examples of orbifold TQFTs $\mathcal{Z}_\mathcal{A}$. Here by "group orbifolds" we mean TQFTs $\mathcal{Z}_\mathcal{A}$, where $\mathcal{A}$ is obtained from an action of a finite symmetry group on $\mathcal{Z}$. (This is also the origin of our usage of the term "orbifold" TQFT: If $\mathcal{Z}^X$ is a TQFT obtained from a sigma model with target manifold X that comes with a certain action of a finite group G, then there is an orbifold datum $\mathcal{A}_G$ such that $(\mathcal{Z}^X)_{\mathcal{A}_G} \cong \mathcal{Z}^{X /\!\!/ G}$ where $\mathcal{Z}^{X /\!\!/ G}$ is a TQFT associated to the orbifold (in the geometric sense) $X /\!\!/ G$.) There are also interesting 2-dimensional orbifold TQFTs that go beyond these classes of examples, cf. [CRCR, NRC, RW].

For general 3-dimensional defect TQFTs we worked out the defining conditions on orbifold data in [CRS1]. In the present paper we focus on Reshetikhin–Turaev defect TQFTs $\mathcal{Z}^\mathcal{C}$ and reformulate their orbifold conditions internally to the modular tensor category $\mathcal{C}$. This is achieved in Proposition 3.5 which is the key technical result in our paper and is used to prove the two main theorems below.

Our first main result (stated as Proposition 4.4 and Theorem 4.8) concerns orbifolds of the "trivial" Reshetikhin–Turaev defect TQFT $\mathcal{Z}^\mathrm{triv} := \mathcal{Z}^\mathrm{vect}$, i.e. when the modular tensor category is simply vect. Recall that (as we review in Section 4.1) from every spherical fusion category $\mathcal{S}$ one can construct a 3-dimensional state sum TQFT called Turaev–Viro theory $\mathcal{Z}^{\mathrm{TV},\mathcal{S}}$ [TViro, BW].

Theorem A. For every spherical fusion category $\mathcal{S}$ there is an orbifold datum $\mathcal{A}^\mathcal{S}$ for $\mathcal{Z}^\mathrm{triv}$ such that $\mathcal{Z}^\mathrm{triv}_{\mathcal{A}^\mathcal{S}} \cong \mathcal{Z}^{\mathrm{TV},\mathcal{S}}$.

This result, appearing as Theorem 4.8 in the main text, vindicates the slogan "state

sum models are orbifolds of the trivial theory" in three dimensions. This can in fact be seen as a special case of the slogan "3-dimensional orbifold data are spherical fusion categories internal to 3-categories with duals", cf. Remark 4.5.

Our second main result concerns group extensions of tensor categories.[1] Recall that an extension of a tensor category $\mathcal{C}$ by a finite group G is a tensor category $\mathcal{B}$ which is graded by G with neutral component $\mathcal{B}_1 = \mathcal{C}$. To formulate our result we note that the nondegeneracy condition on a modular tensor category $\mathcal{C}$ is not needed to define orbifold data $\mathcal{A}$ for $\mathcal{Z}^{\mathcal{C}}$, and hence one can speak of orbifold data in arbitrary ribbon categories $\mathcal{B}$ (see Section 3.4 for details). In Section 5 we prove Theorem 5.1 (see e. g. [Tu2] for the notion of "ribbon crossed G-category"), which we paraphrase as follows:

Theorem B. Let $\mathcal{B}$ be a ribbon fusion category and let G be a finite group. Every ribbon crossed G-category $\widehat{\mathcal{B}} = \bigoplus_{g \in G} \mathcal{B}_g$, such that the component $\mathcal{B}_1$ labelled by the unit $1 \in G$ satisfies $\mathcal{B}_1 = \mathcal{B}$, gives rise to an orbifold datum for $\mathcal{B}$.

We will be particularly interested in the situation where $\mathcal{B} = \mathcal{B}_1$ is additionally a full ribbon subcategory of a modular tensor category $\mathcal{C}$, in which case an extension $\widehat{\mathcal{B}}$ provides orbifold data in $\mathcal{C}$. A special case of this is $\mathcal{B} = \mathcal{C}$ and where $\widehat{\mathcal{B}} = \mathcal{C}_G^{\times}$ is a G-crossed extension. An important source of examples for $G = \mathbb{Z}_2$ are Tambara–Yamagami categories, which are $\mathbb{Z}_2$-extensions of H-graded vector spaces for a finite abelian group H. This is explained in Section 5, where we also discuss orbifold data for the modular tensor categories associated to the affine Lie algebras $\widehat{\mathfrak{sl}}(2)_k$. Moreover, we prove a version of Theorem B that holds for certain non-semisimple ribbon categories $\mathcal{B}$, cf. Remark 5.6.

Taken together, Theorems A and B say that orbifolds unify state sum models and group actions in three dimensions.[2]

The orbifold data in Theorem B depend on certain choices, which are however all related by Morita equivalences that in turn lead to isomorphic orbifold TQFTs (when $\mathcal{B}$ is a subcategory of a modular category $\mathcal{C}$), as we explain in Section 3.8.

As a third source of orbifold data for the Reshetikhin–Turaev defect TQFT $\mathcal{Z}^{\mathcal{C}}$ we identify commutative Δ-separable Frobenius algebras in $\mathcal{C}$ in Section 3.18.

For the whole paper we fix an algebraically closed field $\Bbbk$ of characteristic zero, and we write the symmetric monoidal category of finite-dimensional $\Bbbk$-vector spaces simply as vect.

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[1]A more geometric approach to group orbifolds of Reshetikhin–Turaev TQFTs and more generally of 3-2-1-extended TQFTs has been given in [SW].

[2]The unification of state sum models and group orbifolds in two dimensions is a corollary of [DKR, CR1, BCP].

2. TQFTs with defects and orbifolds

In this section we briefly review the general notions of 3-dimensional defect TQFTs and their orbifolds from [CMS, CRS1], and the extension of Reshetikhin–Turaev theory to a defect TQFT with surface defects from [CRS2].

We start by recalling three types of tensor categories over $\Bbbk$ (see e. g. [Tu1, EGNO] for details). A *spherical fusion category* $\mathcal{S}$ is a semisimple $\Bbbk$-linear pivotal monoidal category with finitely many isomorphism classes of simple objects $i \in \mathcal{S}$, such that left and right traces coincide and $\mathrm{End}_{\mathcal{S}}(\mathbb{1}) = \Bbbk$. Pivotality implies that $\mathcal{S}$ has coherently isomorphic left and right duals, and sphericality implies that the associated left and right dimensions are equal. The *global dimension* of $\mathcal{S}$ is the sum $\dim \mathcal{S} = \sum_i \dim(i)^2$ over a choice of representatives i of the isomorphism classes of simple objects in $\mathcal{S}$. Since $\mathrm{char}(\Bbbk) = 0$ by assumption, we have that $\dim \mathcal{S} \neq 0$ [ENO]. A *ribbon fusion category* is a braided spherical fusion category. A *modular tensor category* is a ribbon fusion category with nondegenerate braiding.

2.1. RESHETIKHIN–TURAEV DEFECT TQFTs. Let $\mathcal{C}$ be a modular tensor category over $\Bbbk$. There is an associated (typically anomalous) 3-dimensional TQFT:

$$\mathcal{Z}^{\mathrm{RT},\mathcal{C}} \colon \widehat{\mathrm{Bord}_3} \longrightarrow \mathrm{vect} \,, \tag{1}$$

called Reshitikhin-Turaev theory. Here $\widehat{\mathrm{Bord}_3}$ is a certain extension of the symmetric monoidal category Bord_3 of 3-dimensional bordisms, which is needed to deal with the anomaly. For all details we refer to [RT, Tu1]; the constructions in the present paper do not require dealing with the anomaly in an explicit way.

In [CRS2] we constructed surface and line defects for $\mathcal{Z}^{\mathrm{RT},\mathcal{C}}$ from Δ-separable symmetric Frobenius algebras and their (cyclic) modules. We briefly recall the notion of a 3-dimensional defect TQFT from [CMS, CRS1], and the extension $\mathcal{Z}^{\mathcal{C}}$ of (1) to a full defect TQFT from [CRS2].

Conventions. We adopt the conventions from [CRS2], in particular we read string diagrams for $\mathcal{C}$ from bottom to top. For instance the braiding $c_{X,Y} \colon X \otimes Y \to Y \otimes X$ and its inverse are written as

$$c_{X,Y} = \vcenter{\hbox{\includegraphics}} \,, \quad c_{X,Y}^{-1} = \vcenter{\hbox{\includegraphics}} \,, \tag{2}$$

and we denote the twist isomorphism on an object $U \in \mathcal{C}$ by θ_U.

An algebra in $\mathcal{C}$ is an object $A \in \mathcal{C}$ together with morphisms $\mu \colon A \otimes A \to A$ and $\eta \colon \mathbb{1} \to A$ satisfying associativity and unit conditions. If (A_1, μ_1, η_1) and (A_2, μ_2, η_2) are algebras in $\mathcal{C}$ then the tensor product $A_1 \otimes A_2$ also carries an algebra structure; our

convention is that $A_1 \otimes A_2$ has the multiplication

$$\mu_{A_1 \otimes A_2} = (\mu_1 \otimes \mu_2) \circ (1_{A_1} \otimes c_{A_2,A_1} \otimes 1_{A_2}) = \qquad \qquad (3)$$

and unit $\eta_{A_1 \otimes A_2} = \eta_1 \otimes \eta_2$.

Let A, B be algebras in $\mathcal{C}$, let M be a right A-module and N a right B-module. From [CRS2, Expl. 2.13(ii)] we obtain that $M \otimes N$ is an $(A \otimes B)$-module with component actions

$$\qquad \qquad (4)$$

Analogously, $M \otimes N$ becomes a $(B \otimes A)$-module with the actions

$$\qquad \qquad (5)$$

3-dimensional defect TQFT. We recall from [CMS, CRS1] that a 3-dimensional defect TQFT is a symmetric monoidal functor

$$\mathcal{Z} \colon \operatorname{Bord}_3^{\operatorname{def}}(\mathbb{D}) \longrightarrow \operatorname{vect}, \qquad \qquad (6)$$

where the source category consists of stratified and decorated bordisms with orientations. For details we refer to [CRS1], but the main ingredients are as follows: A bordism $N \colon \Sigma \to \Sigma'$ between to stratified surfaces Σ, Σ' has 3-, 2- and 1-strata in the interior, while on the boundary also 0-strata are allowed. The possible decorations for the strata are specified by a set of 3-dimensional *defect data* $\mathbb{D}$ which is a tuple

$$\mathbb{D} = (D_3, D_2, D_1; s, t, j). \qquad \qquad (7)$$

Here D_i, $i \in \{1, 2, 3\}$, are sets whose elements label the i-dimensional strata of bordisms; the case $i = 0$ can naturally be added by a universal construction, see Remark 2.2. The *source, target* and *junction maps*

$$s, t \colon D_2 \to D_3 \text{ and } j \colon D_1 \to (\text{cyclic lists of elements of } D_2)$$

specify the adjacency conditions for the decorated strata. This is best described in an example:

$$\tag{8}$$
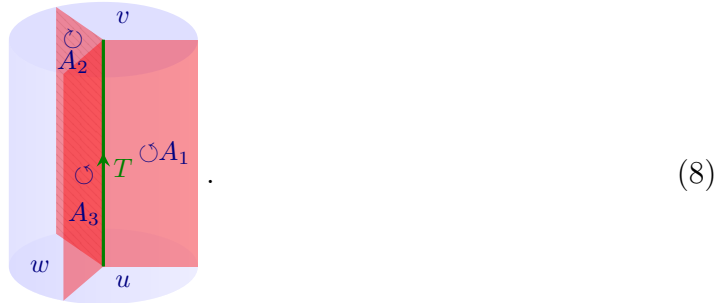
Here, $u, v, w \in D_3$ decorate 3-strata, $A_1, A_2, A_3 \in D_2$ decorate oriented 2-strata such that for example $s(A_1) = u$ and $t(A_1) = v$. Drawing a 2-stratum with a stripy pattern indicates that its orientation is opposite to that of the paper plane. To take also orientation reversal into account we extend the source and target maps to maps $s, t \colon D_2 \times \{\pm\} \to D_3$ and similarly for the junction map j, see [CRS2] for the full definition and more details. Finally $T \in D_1$ labels the 1-stratum, and the junction map applied to T is the cyclic set of the decorations of incident 2-strata, $j(T) = ((A_1, +), (A_2, +), (A_3, -))/\sim$.

A set of 3-dimensional defect data $\mathbb{D}$ yields the category $\mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D})$ of decorated 3-dimensional bordisms: The objects are stratified decorated surfaces, where each i-stratum, $i \in \{0, 1, 2\}$, is decorated by an element from D_{i+1} such that applying the maps s, t or j to the label of a given 1- or 0-stratum, respectively, gives the decorations for the incident 2- and 1-strata. A morphism $N \colon \Sigma \to \Sigma'$ between objects Σ, Σ' is a compact stratified 3-manifold N, with a decoration that is compatible with s, t, j and an isomorphism $\Sigma^{\mathrm{op}} \sqcup \Sigma' \to \partial N$ of decorated stratified 2-manifolds. Here, Σ^{op} is Σ with reversed orientation for all strata (but with the same decorations). The bordisms are considered up to isomorphism of stratified decorated manifolds relative to the boundary.

2.2. REMARK. There are two completion procedures for a defect TQFT $\mathcal{Z} \colon \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}) \to$ vect that will be important for us. First, one can also allow point defects in the interior of a bordism. The maximal set of possible decorations D_0 for such 0-strata turns out to be comprised of the elements in the vector space that $\mathcal{Z}$ assigns to a small sphere S around the given defect point, subject to an invariance condition (that will however be irrelevant for the present paper), see [CMS] and [CRS1, Sect. 2.4]. The resulting defect TQFT is called D_0-complete.

Second, one can allow for certain *point insertions* on strata (called "Euler defects" in [CRS1]). Point insertions are constructed from elements $\psi \in D_0$ that live on i-strata N_i for $i \in \{2, 3\}$ (which means that there are no 1-strata adjacent to the 0-stratum labelled ψ) and which are invertible with respect to a natural multiplication on the associated vector spaces $\mathcal{Z}(S)$. Evaluating $\mathcal{Z}$ on a bordism with point insertions is by definition given by inserting $\psi^{\chi_{\mathrm{sym}}(N_i)}$, where $\chi_{\mathrm{sym}}(N_i)$ is the "symmetric" Euler characteristic $2\chi(M_j) - \chi(\partial M_j)$, with χ the usual Euler characteristic, see [CRS1, Sect. 2.5].

Reshetikhin–Turaev defect TQFT. In [CRS2] we constructed a defect extension $\mathcal{Z}^{\mathcal{C}}$

of the Reshetikhin–Turaev TQFT $\mathcal{Z}^{\mathrm{RT},\mathcal{C}}$ for every modular tensor category $\mathcal{C}$. The associated defect data $\mathbb{D}^{\mathcal{C}} \equiv (D_1^{\mathcal{C}}, D_2^{\mathcal{C}}, D_3^{\mathcal{C}}, s, t, j)$ are as follows. We have $D_3^{\mathcal{C}} := \{\mathcal{C}\}$, meaning that all 3-strata are labelled by $\mathcal{C}$, and the label set for surface defects is

$$D_2^{\mathcal{C}} := \left\{ \Delta\text{-separable symmetric Frobenius algebras in } \mathcal{C} \right\}. \tag{9}$$

We recall that a Δ-*separable symmetric Frobenius algebra* A in $\mathcal{C}$ is a tuple $(A, \mu, \eta, \Delta, \varepsilon)$ consisting of an associative unital algebra (A, μ, η) and a coassociative counital coalgebra (A, Δ, ε) such that

$$\tag{10}$$

$$(\text{``}\Delta\text{-separable''}) \qquad (\text{``symmetric''}) \qquad (\text{``Frobenius''})$$

As decorations for the line defects we take

$$D_1^{\mathcal{C}} := \bigsqcup_{n \in \mathbb{Z}_{\geqslant 0}} L_n \,, \tag{11}$$

where $L_0 = \left\{ X \in \mathcal{C} \mid \theta_X = \mathrm{id}_X \right\}$, and, for $n > 0$,

$$L_n = \Big\{ \big((A_1, \varepsilon_1), (A_2, \varepsilon_2), \ldots, (A_n, \varepsilon_n), M\big) \mid A_i \in D_2^{\mathcal{C}}, \ \varepsilon_i \in \{\pm\},$$
$$M \text{ is a cyclic multi-module for } \big((A_1, \varepsilon_1), (A_2, \varepsilon_2), \ldots, (A_n, \varepsilon_n)\big) \Big\}.$$

A *multi-module* over $((A_1, \varepsilon_1), \ldots, (A_n, \varepsilon_n))$ is an $(A_1^{\varepsilon_1} \otimes \cdots \otimes A_n^{\varepsilon_n})$-module M, where $A_i^+ = A_i$ and A_i^- denotes the opposite algebra A_i^{op}. A multi-module is cyclic if it is equivariant with respect to cyclic permutations which leave the list $((A_1, \varepsilon_1), \ldots, (A_n, \varepsilon_n))$ invariant, see [CRS2, Def. 5.1] for the precise definition. The multi-modules that we consider in the present paper all have only trivial cyclic symmetry, so they are all automatically equivariant and there exists only one equivariant structure. Hence we will have no need to pay attention to this equivariance.

We furthermore have $s(A, \pm) \overset{\mathrm{def}}{=} \mathcal{C} \overset{\mathrm{def}}{=} t(A, \pm)$ for all $A \in D_2^{\mathcal{C}}$, and $j(M) \overset{\mathrm{def}}{=} \mathcal{C}$ for $M \in L_0$, while

$$j\big(((A_1, \varepsilon_1), \ldots, (A_n, \varepsilon_n), M)\big) \overset{\mathrm{def}}{=} ((A_1, \varepsilon_1), \ldots, (A_n, \varepsilon_n))/\sim \tag{12}$$

for $M \in L_n$ with $n > 0$, where as before $(\cdots)/\sim$ denotes cyclic sets.

It is shown in [CRS2, Thm. 5.8 & Rem. 5.9], that the TQFT $\mathcal{Z}^{\mathrm{RT},\mathcal{C}}$ is naturally extended to a 3-dimensional defect TQFT

$$\mathcal{Z}^{\mathcal{C}} \colon \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}}) \longrightarrow \mathrm{vect} \tag{13}$$

that we call *Reshetikhin–Turaev defect TQFT*. The definition of the functor $\mathcal{Z}^{\mathcal{C}}$ is roughly as follows. For a closed 3-bordism N pick an oriented triangulation of each 2-stratum
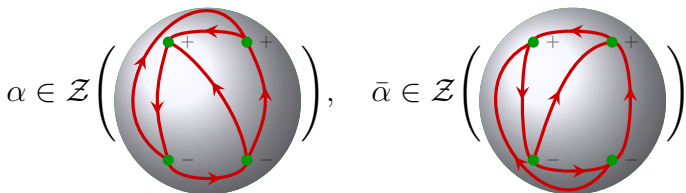
relative to its boundary. The Poincaré dual of the triangulation gives a ribbon graph in N that is decorated by the data of the corresponding Frobenius algebra. By definition, evaluating $\mathcal{Z}^{\mathcal{C}}$ on N is evaluating $\mathcal{Z}^{\mathrm{RT},\mathcal{C}}$ on the bordism which is N augmented by the ribbon graphs; this is independent of the choice of triangulation by the properties of Δ-separable symmetric Frobenius algebras. On objects and general bordisms our functor $\mathcal{Z}^{\mathcal{C}}$ is defined in terms of a standard limit construction which is detailed in [CRS2, Sect. 5].

2.3. ORBIFOLDS OF DEFECT TQFTs. As recalled in the introduction, there is a general notion of orbifolds of n-dimensional TQFTs for any $n \geqslant 1$. Already for $n = 3$, this produces a large list of axioms, and for practical purposes we define "special" 3-dimensional orbifold data to reduce the number of axioms, as recalled next.

Fix a 3-dimensional defect TQFT $\mathcal{Z}\colon \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{vect}$. A *special orbifold datum* $\mathcal{A}$ for $\mathcal{Z}$ is a list of elements $\mathcal{A}_j \in D_j$ for $j \in \{1, 2, 3\}$ as well as $\mathcal{A}_0^+, \mathcal{A}_0^- \in D_0$ together with point insertions ψ and ϕ for $\mathcal{A}_2$-labelled 2-strata and $\mathcal{A}_3$-labelled 3-strata, respectively (the "Euler defects" as recalled above), subject to the constraints below. In anticipation of our application to Reshetikhin–Turaev theory, we will use the notation
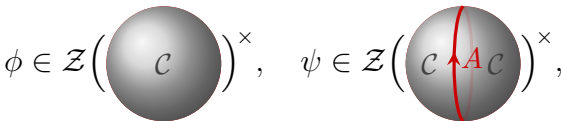
$$\mathcal{A}_3 = \mathcal{C}, \quad \mathcal{A}_2 = A, \quad \mathcal{A}_1 = T, \quad \mathcal{A}_0^+ = \alpha, \quad \mathcal{A}_0^- = \bar{\alpha}, \tag{14}$$

where "A" is for "algebra" and "T" is for "tensor product". The labels for 0-strata are elements in the vector space that $\mathcal{Z}$ assigns to a sphere around a 0-stratum which is dual to a 3-simplex (recall the D_0-completion mentioned in Remark 2.2),
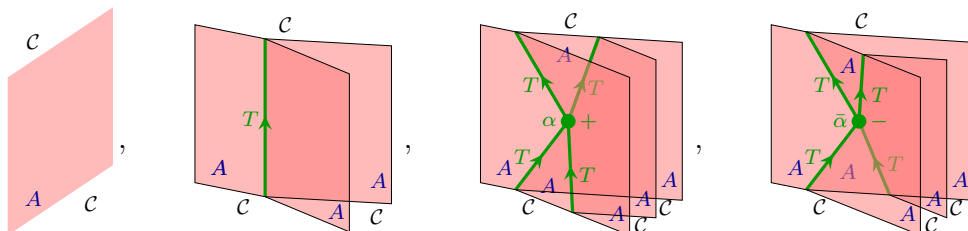
$$\alpha \in \mathcal{Z}\left(\vcenter{\hbox{}} \right), \quad \bar{\alpha} \in \mathcal{Z}\left(\vcenter{\hbox{}} \right) \tag{15}$$

where all 2-, 1- and 0-strata of the defect spheres are labelled by $\mathcal{C}$, A and T, respectively, while the point insertions are

$$\phi \in \mathcal{Z}\left(\vcenter{\hbox{}} \right)^{\times}, \quad \psi \in \mathcal{Z}\left(\vcenter{\hbox{}} \right)^{\times}, \tag{16}$$

and the neighbourhoods of defects labelled by the data (14) look as follows:

$$\vcenter{\hbox{}} \tag{17}$$

These data are subject to the axioms (which have to hold after applying the functor $\mathcal{Z}$ to invisible 3-balls surrounding the diagrams)

$$(18)$$

$$(19)$$

$$(20)$$

$$\tag{21}$$

$$\tag{22}$$

In (22) in the first picture, all 2-strata have the same orientation as the paper plane, while in the second and third picture the rear, respectively front, hemispheres have opposite orientation. We note that in the published version of [CRS1] the identities corresponding to (22) incorrectly feature insertions of ϕ and not the correct ϕ^2.

We remark that any 3-dimensional defect TQFT $\mathcal{Z}$ naturally gives rise to a Gray category with duals $\mathcal{T}_{\mathcal{Z}}$ as shown in [CMS]; in [CRS1] the definition of orbifold data is generalised to a notion internal to any such 3-category.

Given an orbifold datum $\mathcal{A}$ for a defect TQFT $\mathcal{Z}\colon \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}) \to \mathrm{vect}$, the associated $\mathcal{A}$-*orbifold theory* is a closed TQFT

$$\mathcal{Z}_{\mathcal{A}}\colon \mathrm{Bord}_3 \longrightarrow \mathrm{vect} \tag{23}$$

constructed in [CRS1, Sect. 3.2]. On an object Σ, $\mathcal{Z}_{\mathcal{A}}$ is evaluated by considering the cylinder bordism $\Sigma \times [0,1]$ and proceeding roughly as follows: For every triangulation τ with total order of Σ, denote the Poincaré dual stratification by Σ^{τ}. By decorating with the orbifold datum $\mathcal{A}$ we obtain an object $\Sigma^{\tau,\mathcal{A}} \in \mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D})$. For two such triangulations τ, τ' of Σ, the cylinder $C_{\Sigma} = \Sigma \times [0,1]$, regarded as a bordism $\Sigma \to \Sigma$ in Bord_3, has an oriented triangulation t extending the triangulations τ and τ' on the ingoing and outgoing boundaries, respectively. Again decorating the Poincaré dual C_{Σ}^t with the orbifold datum $\mathcal{A}$ we obtain a morphism $C_{\Sigma}^{t,\mathcal{A}}\colon \Sigma^{\tau,\mathcal{A}} \longrightarrow \Sigma^{\tau',\mathcal{A}}$. By triangulation independence we get a projective system

$$\mathcal{Z}(C_{\Sigma}^{t,\mathcal{A}})\colon \mathcal{Z}(\Sigma^{\tau,\mathcal{A}}) \longrightarrow \mathcal{Z}(\Sigma^{\tau',\mathcal{A}}) \tag{24}$$

whose limit is by definition $\mathcal{Z}_{\mathcal{A}}(\Sigma)$.

On a bordism $N\colon \Sigma_1 \to \Sigma_2$, the functor $\mathcal{Z}_{\mathcal{A}}$ is evaluated by (i) choosing oriented triangulations τ_1, τ_2 of Σ_1, Σ_2 and extending them to an oriented triangulation of N, (ii) decorating the Poincaré dual stratification with the data $\mathcal{A}$ to obtain a morphism $N^{t,\mathcal{A}}$ in

$\mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D})$, (iii) evaluating $\mathcal{Z}$ on $N^{t,\mathcal{A}}$, to obtain a morphism of projective systems

$$\mathcal{Z}(N^{t,\mathcal{A}}) \colon \mathcal{Z}(\Sigma_1^{\tau_1,\mathcal{A}}) \longrightarrow \mathcal{Z}(\Sigma_2^{\tau_2,\mathcal{A}}), \tag{25}$$

and (iv) taking the limit to make the construction independent of choices of triangulations. Note that by the construction in [CRS1, Sect. 2.5], for a bordism N with triangulation t, on each 2- and 3-stratum adjacent to the boundary of $M^{t,\mathcal{A}}$, there is one inserted point defect ψ and ϕ, respectively, while 2- and 3-strata in the interior have ψ^2- and ϕ^2-insertions.

For more details we refer to [CRS1], but we note that in the case of a closed 3-manifold $N \colon \emptyset \to \emptyset$ we have $\mathcal{Z}_{\mathcal{A}}(N) = \mathcal{Z}(N^{t,\mathcal{A}})$ for all triangulations: in this case step (iv) above is unnecessary since by the defining property of the orbifold datum $\mathcal{A}$ the value of $\mathcal{Z}$ on $N^{t,\mathcal{A}}$ is invariant under change of triangulation.

3. Orbifold data for Reshetikhin–Turaev theory

We now specialise to the case of the defect TQFT $\mathcal{Z}^{\mathcal{C}}$ from Section 2.1. Since the defect data for $\mathcal{Z}^{\mathcal{C}}$ are described internal to the given modular tensor category $\mathcal{C}$, it is desirable to describe also orbifold data and their constraints internal to $\mathcal{C}$. This internal formulation can be stated in any (not necessarily semisimple) ribbon category (Definition 3.6). We will show that commutative Δ-separable symmetric Frobenius algebras provide examples of orbifold data, and we describe relations (such as Morita equivalence) between orbifold data that lead to isomorphic orbifold TQFTs.

3.1. STATE SPACES FOR SPHERES. We will need to express the state spaces assigned to spheres with two and four 0-strata, respectively, and a network of 1-strata, in terms of Hom spaces in the modular tensor category $\mathcal{C}$. For two 0-strata, the following result was proven in [CRS2, Lem. 5.10].

3.2. LEMMA. *Let M, N be two cyclic multi-modules over a list $(A_1, \ldots, A_n)$ of Δ-separable symmetric Frobenius algebras A_i. The vector space $\mathcal{Z}^{\mathcal{C}}(S_{M,N})$ associated to the 2-sphere $S_{M,N}$ with M, N on its South and North pole, respectively, is given by the space of maps of multi-modules,*

$$\mathcal{Z}^{\mathcal{C}}(S_{M,N}) = \mathrm{Hom}_{A_1,\ldots,A_n}(M, N). \tag{26}$$

More complicated state spaces are obtained by invoking the tensor product over algebras in $\mathcal{C}$: Recall that for an algebra $A \in \mathcal{C}$, a right A-module (M_A, ρ_M) and a left A-module $(_A N, \rho_N)$, the tensor product over A, denoted by $M \otimes_A N$, is the coequaliser of

$$M \otimes A \otimes N \xrightarrow[\rho_M \,\otimes\, \mathrm{id}_N]{\mathrm{id}_M \,\otimes\, \rho_N} M \otimes N. \tag{27}$$

For a Δ-separable symmetric Frobenius algebra A, we can compute the tensor product

$M \otimes_A N$ as the image of the projector

$$p_{M,N} := (\rho_M \otimes \rho_N) \circ \big(\mathrm{id}_M \otimes (\Delta \circ \eta) \otimes \mathrm{id}_N\big) = \qquad . \tag{28}$$

It follows that for Δ-separable symmetric Frobenius algebras A, B and modules M_A, $_AN$, M'_B, $_BN'$ we have

$$\mathrm{Hom}_{\mathcal{C}}(M \otimes_A N, M' \otimes_B N')$$
$$= \big\{ f \in \mathrm{Hom}_{\mathcal{C}}(M \otimes N, M' \otimes N') \,\big|\, f \circ p_{M,N} = f = p_{M',N'} \circ f \big\}. \tag{29}$$

The proof of the next lemma is analogous to the proof of Lemma 3.2 in [CRS2, Lem. 5.10]. It basically amounts to the fact that in the definition of the defect TQFT $\mathcal{Z}^{\mathcal{C}}$ sketched in Section 2.1, the dual of a triangulation of an A-labelled 2-stratum produces projectors to tensor products over A.

3.3. LEMMA. *Let $A_1, \ldots, A_6$ be Δ-separable symmetric Frobenius algebras in $\mathcal{C}$, and let $_{A_4}K_{A_2 \otimes A_5}$, $_{A_1}L_{A_4 \otimes A_6}$, $_{A_3}M_{A_5 \otimes A_6}$ and $_{A_1}N_{A_2 \otimes A_3}$ be modules. The vector space $\mathcal{Z}^{\mathcal{C}}(\Sigma)$ associated to the defect 2-sphere*

$$\Sigma = \qquad \qquad \tag{30}$$
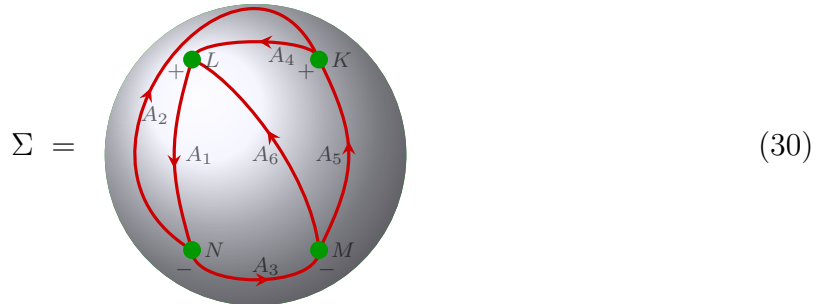
is isomorphic to

$$\mathrm{Hom}_{A_1, A_2 \otimes A_5 \otimes A_6} \big(_{A_1}N_{A_2 \otimes A_3} \otimes_{A_3} {_{A_3}M_{A_5 \otimes A_6}}, {_{A_1}L_{A_4 \otimes A_6}} \otimes_{A_4} {_{A_4}K_{A_2 \otimes A_5}}\big). \tag{31}$$

3.4. SPECIAL ORBIFOLD DATA INTERNAL TO RIBBON FUSION CATEGORIES. In this section we translate the orbifold data from Section 2.3 to data and axioms internal to a given modular tensor category $\mathcal{C}$. In fact we will find that this notion does not require the nondegeneracy of $\mathcal{C}$ and thus makes sense in arbitrary ribbon fusion categories (Definition 3.6), which will be relevant in our applications to G-extensions in Section 5.

Recall the notation introduced in (14). We will now describe the data $\mathcal{C}, A, T, \alpha, \bar{\alpha}, \psi, \phi$, and the conditions these need to satisfy, for special orbifold data in Reshetikhin–Turaev TQFTs. We start with the first three elements $\mathcal{C}, A, T$. According to the definition of $\mathbb{D}^{\mathcal{C}}$ as recalled in Section 2.1, we have

(i) $\mathcal{C}$ is a modular tensor category (from which the TQFT $\mathcal{Z}^{\mathrm{RT},\mathcal{C}}$ is constructed),

(ii) A is a Δ-separable symmetric Frobenius algebra in $\mathcal{C}$,

(iii) $T = {}_A T_{AA}$ is an $(A, A \otimes A)$-bimodule.

To keep track of the various A-actions, we sometimes denote the bimodule T as ${}_A T_{A_1 A_2}$; the corresponding 3-dimensional picture then is

$$\text{(figure)} \tag{32}$$

Consistently with the 3-dimensional picture, the right $(A \otimes A)$-action is equivalently described by two right A-actions on T, denoted with the corresponding number on the A-strings. These A-actions commute in the following sense:

$$\text{(figure)} \tag{33}$$

see [CRS2, Lemma 2.1]; of course both actions commute with the left A-action.

Next we turn to the data α and $\bar{\alpha}$. They correspond to certain maps of tensor products over A of multi-modules, and to keep track of the actions we enumerate the A-actions. From Lemma 3.3 we obtain that

(iv) α is a map of multi-modules

$$\alpha\colon\; {}_{A_1}\big({}_{A_1}T_{A_2 A_3} \otimes {}_{A_3}T_{A_5 A_6}\big)_{A_2 A_5 A_6} \longrightarrow {}_{A_1}\big({}_{A_1}T_{A_4 A_6} \otimes {}_{A_4}T_{A_2 A_5}\big)_{A_2 A_5 A_6} \tag{34}$$

such that $p_1 \circ \alpha = \alpha = \alpha \circ p_2$, where p_1 is the projector with respect to the action of A_4 in the second term, see Equation (28), while p_2 is the projector with respect to the action of A_3 in the first term.

(v) $\bar{\alpha}$ is a map of multi-modules

$$\bar{\alpha}\colon\; {}_{A_1}\big({}_{A_1}T_{A_4 A_6} \otimes {}_{A_4}T_{A_2 A_5}\big)_{A_2 A_5 A_6} \longrightarrow {}_{A_1}\big({}_{A_1}T_{A_2 A_3} \otimes {}_{A_3}T_{A_5 A_6}\big)_{A_2 A_5 A_6} \tag{35}$$

such that $p_2 \circ \bar{\alpha} = \bar{\alpha} = \bar{\alpha} \circ p_1$.

Here we used the conventions as in (4)–(5) for the actions on the three-fold tensor product $A_2 \otimes A_5 \otimes A_6$. The conditions for α and $\bar{\alpha}$ are more accessible when expressed graphically. In the pictures it is unambiguous to work only with labels $1, 2$ for the actions of ${}_A T_{A_1 A_2}$. The condition on $\alpha \colon T \otimes T \to T \otimes T$ to be a map of multi-modules reads

$$(36)$$

while the conditions involving the projectors is

$$(37)$$

and analogously for $\bar{\alpha}$.

From Lemma 3.3 we furthermore obtain:

(vi) The point insertion ψ is an invertible morphism $\psi \in \mathrm{End}_{AA}(A)$.

(vii) The point insertion ϕ is an invertible morphism $\phi \in \mathrm{End}_{\mathcal{C}}(\mathbb{1})$.

To express the axioms for the orbifold datum $\mathcal{A} \equiv (\mathcal{C}, A, T, \alpha, \bar{\alpha}, \psi, \phi)$ internal to $\mathcal{C}$, it is convenient to consider for an A-module ${}_A M$ the map

$$\mathrm{Hom}_{A,A}({}_A A_A, {}_A A_A) \longrightarrow \mathrm{Hom}_A({}_A M, {}_A M), \qquad (38)$$

which sends ψ to $\psi_0 := \rho_M \circ (\mathrm{id}_M \otimes (\psi \circ \eta_A))$, or graphically

$$(39)$$

When $M_{A_1 A_2}$ is a module over $A \otimes A$, we denote the images of ψ under the map analogous to (38) with respect to the A_i-action by ψ_i for $i \in \{1, 2\}$,

$$\tag{40}$$

The axioms for the data (i)–(vii) can now be formulated as follows:

3.5. PROPOSITION. *A special 3-dimensional orbifold datum* $\mathcal{A} \equiv (\mathcal{C}, A, T, \alpha, \bar{\alpha}, \psi, \phi)$ *for* $\mathcal{Z}^\mathcal{C}$ *consists of the data (i)–(vii) subject to the following conditions, expressed in terms of string diagrams in* $\mathcal{C}$:

$$\tag{41}$$

$$\tag{42}$$

$$\tag{43}$$

$$(44)$$

$$(45)$$

PROOF. As recalled in Section 2.1, to evaluate the defect TQFT $\mathcal{Z}^{\mathcal{C}}$ we need to pick triangulations for all 2-strata and use the data of the Frobenius algebra A to label the dual graphs. To verify the axioms, note that all 2-strata are discs, thus it is enough to consider one attached A-line per 2-stratum neighbouring a given 1-stratum. This translates the conditions (18)–(22) into those of (41)–(45). ∎

The data and conditions on a special orbifold datum can be formulated for general ribbon categories $\mathcal{B}$, without assumptions such as $\Bbbk$-linearity or semisimplicity. This is useful, as such orbifold data can then be placed in a modular tensor category via a ribbon functor. Let us describe this in more detail.

3.6. DEFINITION. *Let $\mathcal{B}$ be a ribbon category. A special orbifold datum in $\mathcal{B}$ is a tuple $(A, T, \alpha, \bar{\alpha}, \psi, \phi)$ as in (ii)–(vii) above (with $\mathcal{C}$ replaced by $\mathcal{B}$), subject to the conditions (41)–(45).*

We can use ribbon functors to transport special orbifold data. The following result is immediate and we omit its proof.

3.7. PROPOSITION. *Let $F \colon \mathcal{B} \to \mathcal{B}'$ be a ribbon functor and $\mathcal{A} = (A, T, \alpha, \bar{\alpha}, \psi, \phi)$ a special orbifold datum for $\mathcal{B}$. Then $F(\mathcal{A}) := (F(A), F(T), F(\alpha), F(\bar{\alpha}), F(\psi), \phi)$ is a special orbifold datum for $\mathcal{B}'$ (where we suppress the coherence isomorphism of F in the notation).*

3.8. MORITA EQUIVALENT ORBIFOLD DATA. In this section we investigate the interplay between Morita equivalence and special orbifold data in ribbon fusion categories. We explain how such orbifold data can be "transported along Morita equivalences", and we prove that the corresponding orbifold TQFTs are isomorphic.

Recall that two algebras A, B in a pivotal tensor category are *Morita equivalent* if there exists an A-B-bimodule X together with bimodule isomorphisms

$$X^* \otimes_A X \cong B, \quad X \otimes_B X^* \cong A. \tag{46}$$

By a *Morita equivalence* between algebras A, B we mean a choice of such a bimodule X (called *Morita module*) and isomorphisms as in (46).

3.9. NOTATION. Let A be a Δ-separable symmetric Frobenius algebra, M a left A-module and N a right A-module. We sometimes denote the projector $p_{M,N}$ of (28) string-diagrammatically by colouring the region between the M- and N-lines:

$$p_{M,N} = \quad\equiv\quad . \tag{47}$$

Moreover, we sometimes identify the right-hand side of (47) with $\mathrm{id}_{M\otimes_A N}$ or with $M\otimes_A N$ itself. For example, we employ this convention in (49) below. We stress that coloured regions always represent projectors of relative tensor products over Frobenius algebras; hence in the example of $X^* \otimes_A T \otimes_{AA} (X\otimes X)$ in (49), the rightmost coloured region represents a projector corresponding to the A-action on T and the right X-factor (and not between the two modules X on the right, as their product is not relative over A).

3.10. DEFINITION. *Let $\mathcal{A} = (A, T, \alpha, \bar{\alpha}, \psi, \phi)$ be a special orbifold datum in a ribbon fusion category $\mathcal{B}$, and let $B \in \mathcal{B}$ be an algebra that is Morita equivalent to A with Morita module X. The Morita transport of $\mathcal{A}$ along X is the tuple*

$$X(\mathcal{A}) := \left(B, T^X, \alpha^X, \bar{\alpha}^X, \psi^X, \phi\right) \tag{48}$$

where

$$T^X := X^* \otimes_A T \otimes_{AA} (X\otimes X) \equiv \tag{49}$$

$$\alpha^X := \quad , \quad \bar{\alpha}^X := \tag{50}$$

and $\psi^X \in \mathrm{End}_{BB}(B)$ is a choice of square root of

$$\left(\psi^X\right)^2 := \tag{51}$$

For algebras A, B as in Definition 3.10 we can choose a decomposition into *simple* Δ-separable symmetric Frobenius algebras A_i and B_i, respectively,

$$A \cong \bigoplus_i A_i, \quad B \cong \bigoplus_i B_i, \tag{52}$$

such that A_i and B_i are Morita equivalent with bimodule isomorphisms

$$X_i^* \otimes_{A_i} X_i \cong B_i\,, \quad X_i \otimes_{B_i} X_i^* \cong A_i\,, \quad X \cong \bigoplus_i X_i\,. \tag{53}$$
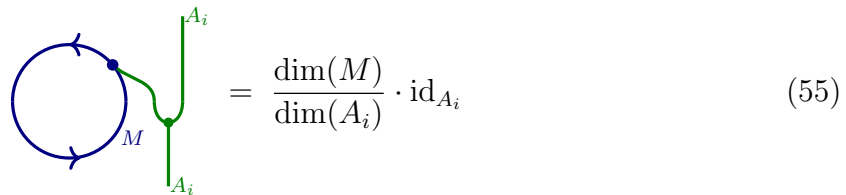
3.11. PROPOSITION. *Let* $\mathcal{B}, \mathcal{A}, B, X$ *be as in Definition 3.10 such that* $\dim(A_i) \neq 0 \neq \dim(B_i)$ *for all simple algebras in* (52). *Then* $X(\mathcal{A})$ *is a special orbifold datum in* $\mathcal{B}$.

PROOF. We have a decomposition $T \cong \bigoplus_{i,j,k} {}_kT_{i,j}$ of T into A_k-$(A_i \otimes A_j)$-bimodules, and up to the isomorphisms (52) the maps ψ, ψ^X are diagonal matrices with entries

$$\psi_{A_i} \in \mathrm{End}_{A_i A_i}(A_i) \cong \Bbbk\,, \quad \psi_{B_i} \equiv (\psi^X)_{B_i} \in \mathrm{End}_{B_i B_i}(B_i) \cong \Bbbk\,, \tag{54}$$

respectively.

To prove that $X(\mathcal{A})$ is a special orbifold datum we have to verify that the conditions (41)–(45) are satisfied. Using the above decompositions and identities of the form
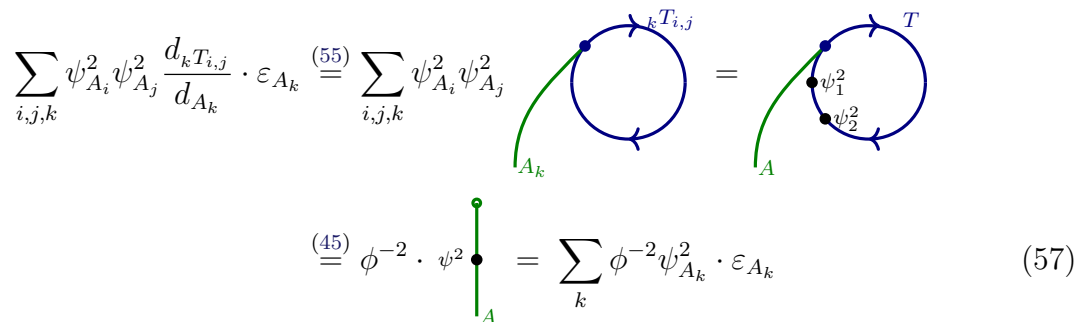

$$\tag{55}$$

for simple A_i and an A_i-module M, these checks mostly become tedious exercises in string diagram manipulations. We provide the details for the first condition in (42) and for (45), the remaining conditions are treated analogously.

We start by verifying condition (45). Abbreviating $d_U := \dim(U)$ for all $U \in \mathcal{B}$ we first note that (55) in particular implies

$$\frac{\psi_{B_i}^2}{\psi_{A_i}^2} = \frac{d_{X_i}}{d_{B_i}} = \frac{d_{A_i}}{d_{X_i}}\,. \tag{56}$$

Note that $d_{A_i} \neq 0 \neq d_{B_i}$ together with (53) and (55) implies that $d_{X_i} \neq 0$ for all i. Thus $\psi_{A_i} \neq 0 \neq \psi_{B_i}$ for all i. Moreover,

$$\tag{57}$$

and hence for all k:

$$\sum_{i,j} \psi_{A_i}^2 \psi_{A_j}^2 \frac{d_{\,_k T_{i,j}}}{d_{A_k}} = \phi^{-2} \psi_{A_k}^2 \, . \tag{58}$$

Now we compute

$$\stackrel{(55)}{=} \sum_{i,j,k} \psi_{B_i}^2 \psi_{B_j}^2 \frac{d_{X_i}}{d_{A_i}} \frac{d_{X_j}}{d_{A_j}} \frac{d_{\,_k T_{i,j}}}{d_{A_k}} \frac{d_{X_k}}{d_{B_k}} \cdot \Bigg|_{B_k}$$

$$\stackrel{(56)}{=} \sum_{k} \Bigg(\sum_{i,j} \psi_{A_i}^2 \psi_{A_j}^2 \frac{d_{\,_k T_{i,j}}}{d_{A_k}} \Bigg) \frac{d_{X_k}}{d_{B_k}} \cdot \varepsilon_{B_k}$$

$$\stackrel{(58)}{=} \sum_{k} \phi^{-2} \psi_{A_k}^2 \frac{d_{X_k}}{d_{B_k}} \cdot \varepsilon_{B_k} \stackrel{(56)}{=} \sum_{k} \phi^{-2} \psi_{B_k}^2 \cdot \varepsilon_{B_k} = \phi^{-2} \cdot \; {}_{(\psi^X)^2} \Bigg|_{B} \cdot \tag{59}$$

The other identities in (45) are checked analogously for $X(\mathcal{A})$.

Next we check the first identity in (42) for $X(\mathcal{A})$. Using (42) for $\mathcal{A}$ and (56), its left-hand side is

To see that this indeed equals the right-hand side of the first identity in (42) for $X(\mathcal{A})$,

i. e.

$$\tag{61}$$

we pre-compose both sides with

$$\tag{62}$$

and use (56) again to see that they are equal. ∎

3.12. REMARK. We note that the expressions for $T^X, \alpha^X, \bar\alpha^X, (\psi^X)^2$ in Definition 3.10 have a simple origin: they are obtained by the rule to "draw an X-line parallel to the T-lines on every 2-stratum in the neighbourhoods of $T, \alpha, \bar\alpha$ in (17)." This rule immediately produces (49) and (50), while (51) is motivated by wrapping an X-line around a ψ^2-insertion on an interior 2-stratum.

3.13. REMARK. The proof of Proposition 3.11 shows that the following more general result holds: Let $\mathcal{B}, \mathcal{A}, B, X$ be as above, except that $\mathcal{B}$ is not necessarily semisimple, but still the algebras A, B decompose into simple summands A_i, B_i of non-zero dimension as in (52), and the images of the projectors as in (47) for A_i, B_i exist in $\mathcal{B}$. Then $X(\mathcal{A})$ is a special orbifold datum in the (possibly non-semisimple) ribbon category $\mathcal{B}$.

3.14. PROPOSITION. *Let $\mathcal{B} = \mathcal{C}$ be a modular tensor category, and let $\mathcal{A}, X$ be as above. Then the orbifold TQFTs corresponding to $\mathcal{A}$ and $X(\mathcal{A})$ are isomorphic:*

$$(\mathcal{Z}^{\mathcal{C}})_{\mathcal{A}} \cong (\mathcal{Z}^{\mathcal{C}})_{X(\mathcal{A})}. \tag{63}$$

PROOF. We will construct a monoidal natural isomorphism $\nu \colon (\mathcal{Z}^{\mathcal{C}})_{\mathcal{A}} \to (\mathcal{Z}^{\mathcal{C}})_{X(\mathcal{A})}$. Recall from [CRS1, Sect. 3.2] and Section 2.3 that for $\Sigma \in \mathrm{Bord}_3$, the vector space $(\mathcal{Z}^{\mathcal{C}})_{\mathcal{A}}(\Sigma)$ is defined as the limit of a projective system that is built from $\mathcal{A}$-decorated dual triangulations of cylinders over Σ. Let τ be an oriented triangulation of Σ and decorate the dual stratification with the data $\mathcal{A}$ to obtain $\Sigma^{\tau, \mathcal{A}} \in \widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$. Extend τ to an oriented triangulation t of the cylinder $C_\Sigma = \Sigma \times [0, 1]$ and decorate the stratification dual to t with $\mathcal{A}$ to obtain a morphism $C_\Sigma^{t, \mathcal{A}} \colon \Sigma^{\tau, \mathcal{A}} \to \Sigma^{\tau, \mathcal{A}}$ in $\widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$. Then $(\mathcal{Z}^{\mathcal{C}})_{\mathcal{A}}(\Sigma) \cong \mathrm{Im}\, \mathcal{Z}^{\mathcal{C}}(C_\Sigma^{t, \mathcal{A}})$.

We will obtain the components ν_Σ by modifying $C_\Sigma^{t,\mathcal{A}}$ only near its outgoing boundary $\Sigma \times \{1\}$. The 2-strata in $C_\Sigma^{t,\mathcal{A}}$ have the topology of discs and are labelled by A. We will only be concerned with 2-strata that intersect the outgoing boundary component $\Sigma \times \{1\}$. Let D be such a 2-stratum.

In D insert a semi-circular 1-stratum which starts and ends on $\Sigma \times \{1\}$, which is oriented clockwise with respect to the orientation of D, and which is labelled by X. This splits D into two disc-shaped connected components D_{i} and D_{o} ("inner" and "outer"). The disc D_{i} is bounded by the X-labelled line and a single interval on the boundary $D \cap (\Sigma \times \{1\})$, while D_{o} is bounded by two disjoint intervals in $D \cap (\Sigma \times \{1\})$, as well as by X- and T-labelled 1-strata and 0-strata labelled by α or $\bar{\alpha}$. The 2-stratum D_{o} keeps its label A while the label of D_{i} is changed from A to B.

Note that by construction, each positively oriented T-labelled 0-stratum in the outgoing boundary $\Sigma \times \{1\}$ has one X^*- and two X-labelled 0-strata in its vicinity, and vice versa for a negatively oriented 0-stratum.

Recall from Section 2.1 the construction of the ribbon graph corresponding to this stratified bordism. This results in an A-network in D_{o} and a B-network in D_{i}. Furthermore, D_{i} gets an insertion of ψ_B, while D_{o} does not get any ψ_A-insertion since the corresponding Euler characteristic is zero.

We choose $\varepsilon > 0$ and enlarge the underlying cylinder C_Σ to $C_{\Sigma,\varepsilon} = \Sigma \times [0, 1+\varepsilon]$ and construct $\widetilde{C}_{\Sigma,\varepsilon}^{t,\mathcal{A},X}$ as follows: it is identical to the bordism with embedded ribbon graph constructed above along the interval $[0,1]$, and it is a cylinder along $[1, 1+\varepsilon]$ except that we insert the projectors $X^* \otimes T \otimes X \otimes X \twoheadrightarrow T^X$ and embeddings $T^X \hookrightarrow X^* \otimes T \otimes X \otimes X$ for all T-lines in $\Sigma \times [1, 1+\varepsilon]$, depending on the direction of T, such that all ribbons ending on $\Sigma \times \{1+\varepsilon\}$ are labelled either T^X or B.

In this way we obtain a bordism-with-ribbon-graph $\widetilde{C}_{\Sigma,\varepsilon}^{t,\mathcal{A},X}$ in $\widehat{\mathrm{Bord}}_3$ which represents a defect bordism $\Sigma^{\tau,\mathcal{A}} \to \Sigma^{\tau,X(\mathcal{A})}$ in $\widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$, and which (after applying $\mathcal{Z}^{\mathrm{RT},\mathcal{C}}$ and the projection to the limit) defines the component ν_Σ of the natural transformation ν.

To verify that this is indeed a natural transformation, one writes out the naturality square and notes that one can pass from one path to the other by replacing each ψ_A^2 inserted on an interior 2-stratum by a small circular 1-stratum labelled X and a ψ_B^2-insertion in its interior, using the identity (56). Furthermore, it follows directly from the construction that ν is monoidal (and hence an isomorphism, see e.g. [CR2, Lem. A.2]). ∎

Later in Section 5 we will have need to combine Morita transports with the following notion of isomorphisms of orbifold data:

3.15. DEFINITION. *Let* $\mathcal{A} = (A, T, \alpha, \bar{\alpha}, \psi, \phi)$ *and* $\widetilde{\mathcal{A}} = (A, \widetilde{T}, \widetilde{\alpha}, \widetilde{\bar{\alpha}}, \psi, \phi)$ *be special orbifold data in a ribbon fusion category. A T-compatible isomorphism from $\mathcal{A}$ to $\widetilde{\mathcal{A}}$ is an isomorphism $\rho\colon T \to \widetilde{T}$ of multi-modules such that*

$$(\rho \otimes \rho) \circ \alpha = \widetilde{\alpha} \circ (\rho \otimes \rho). \tag{64}$$

3.16. LEMMA. *A T-compatible isomorphism ρ from $\mathcal{A}$ to $\widetilde{\mathcal{A}}$ induces an isomorphism between the corresponding orbifold TQFTs:* $(\mathcal{Z}^{\mathcal{C}})_{\mathcal{A}} \cong (\mathcal{Z}^{\mathcal{C}})_{\widetilde{\mathcal{A}}}$.

PROOF. The construction of a monoidal natural isomorphism $\kappa \colon (\mathcal{Z}^{\mathcal{C}})_{\mathcal{A}} \to (\mathcal{Z}^{\mathcal{C}})_{\widetilde{\mathcal{A}}}$ is analogous to the construction in the proof of the previous proposition: Consider for $\Sigma \in \mathrm{Bord}_3$ a morphism $C_{\Sigma}^{t,\mathcal{A}} \colon \Sigma^{\tau,\mathcal{A}} \to \Sigma^{\tau,\mathcal{A}}$ in $\widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$ as above. Again we enlarge the cylinder to $C_{\Sigma,\varepsilon} = \Sigma \times [0, 1+\varepsilon]$ and construct a morphism $C_{\Sigma,\varepsilon}^{t,\mathcal{A},\widetilde{\mathcal{A}}} \colon \Sigma^{\tau,\mathcal{A}} \to \Sigma^{\tau,\widetilde{\mathcal{A}}}$ in $\widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$ as $C_{\Sigma}^{t,\mathcal{A}}$ along $[0,1]$ and as a cylinder along $[1, 1+\varepsilon]$, where we insert the isomorphism ρ on each outwards oriented 1-stratum and the isomorphism $(\rho^{-1})^*$ on each inward oriented 1-stratum. Thus $C_{\Sigma,\varepsilon}^{t,\mathcal{A},\widetilde{\mathcal{A}}} \colon \Sigma^{\tau,\mathcal{A}} \to \Sigma^{\tau,\widetilde{\mathcal{A}}}$ is a well-defined morphism in $\widehat{\mathrm{Bord}}_3^{\mathrm{def}}(\mathbb{D}^{\mathcal{C}})$ and we define κ_Σ to be the map from $(\mathcal{Z}^{\mathcal{C}})_{\mathcal{A}}(\Sigma)$ to $(\mathcal{Z}^{\mathcal{C}})_{\widetilde{\mathcal{A}}}(\Sigma)$ that is induced by $\mathcal{Z}^{\mathcal{C}}(C_{\Sigma,\varepsilon}^{t,\mathcal{A},\widetilde{\mathcal{A}}})$. It is monoidal by construction, and for a bordism $M \colon \Sigma \to \Sigma'$ with triangulation and $\mathcal{A}$-decoration $M^{t',\mathcal{A}}$ we can replace each α by $(\rho^{-1} \otimes \rho^{-1}) \circ \widetilde{\alpha} \circ (\rho \otimes \rho)$ without changing $\mathcal{Z}^{\mathcal{C}}(M^{t',\mathcal{A}})$. On each inner T-line we can then cancel the ρ with the ρ^{-1} decoration to obtain a decoration by $\widetilde{\mathcal{A}}$ in the interior composed with a cylinder $C_{\Sigma,\varepsilon}^{t,\mathcal{A},\widetilde{\mathcal{A}}}$ as above and its inverse on the boundaries. After evaluating with $\mathcal{Z}^{\mathcal{C}}$ this shows the naturality of κ. ∎

3.17. COROLLARY. *Let $\mathcal{A}$, $\mathcal{B}$, X be as in Definition 3.10. Let $\mathcal{C}$ be a modular tensor category and $F \colon \mathcal{B} \to \mathcal{C}$ a ribbon functor. Then $(\mathcal{Z}^{\mathcal{C}})_{F(\mathcal{A})} \cong (\mathcal{Z}^{\mathcal{C}})_{F(X(\mathcal{A}))}$.*

PROOF. This follows from Lemma 3.16 and Propositions 3.7 and 3.14, together with the observation that there is a T-compatible isomorphism $F(X(\mathcal{A})) \cong F(X)(F(\mathcal{A}))$. ∎

3.18. COMMUTATIVE FROBENIUS ALGEBRAS GIVE ORBIFOLD DATA. A simple example of orbifold data can be obtained as follows. Let $\mathcal{B}$ be a ribbon category and let A be a commutative Δ-separable symmetric Frobenius algebra in $\mathcal{B}$. Commutativity and symmetry together imply that the twist on A is trivial, $\theta_A = \mathrm{id}_A$. For the bimodule $_AT_{AA}$ we take $T = A$ with all actions given by the multiplication on A. For the remaining data we choose

$$\alpha = \bar\alpha = \Delta \circ \mu, \quad \psi = \mathrm{id}_A, \quad \phi = 1. \tag{65}$$

3.19. PROPOSITION. *The tuple $(A, T, \alpha, \bar\alpha, \psi, \phi)$ described in (65) is a special orbifold datum in $\mathcal{B}$.*

PROOF. Note that the commutativity of A is needed for A to be a right $(A \otimes A)$-module. The fact that the conditions in Proposition 3.5 are all satisfied follows from commutativity and the Δ-separable symmetric Frobenius algebra properties of A. We will check the first identity in (42) and condition (43) as examples.

For the first condition (42), the left-hand side becomes $\Delta \circ \mu \circ \Delta \circ \mu = \Delta \circ \mu$, using Δ-separability. The right-hand side can be rewritten as $(\mathrm{id} \otimes \mu) \circ (\Delta \otimes \mathrm{id})$, and the required equality is then just the Frobenius property.

As for (43), using symmetry one can replace the duality maps on A by $\varepsilon \circ \mu$ and $\Delta \circ \eta$, thereby also replacing the A^*-labelled strand by its orientation-reversed version (which is then labelled A). After this, the computation boils down to using symmetry and commutativity a number of times, as well as Δ-separability. ∎

In the case $\mathcal{B}$ is a modular tensor category and A is in addition simple, the orbifold theory for an orbifold datum as in Proposition 3.19 is equivalent to the Reshetikhin–Turaev TQFT obtained from the category of local A-modules in $\mathcal{B}$, which is again modular [KO, Thm. 4.5]. Based on the work [MR] this result will be explained in [CMRSS].

4. Turaev–Viro theory

In this section we explain how "state sum models are orbifolds of the trivial theory" in three dimensions: We start in Section 4.1 by reviewing Turaev–Viro theory (which constructs a state sum TQFT $\mathcal{Z}^{\mathrm{TV},\mathcal{S}}$ for every spherical fusion category $\mathcal{S}$ over an algebraically closed field $\Bbbk$), in the formulation of Turaev–Virelizier. Then, independently of Section 4.1, in Section 4.3 we define the "trivial" 3-dimensional defect TQFT $\mathcal{Z}^{\mathrm{triv}}$, and for every spherical fusion category $\mathcal{S}$ we construct a special orbifold datum $\mathcal{A}^{\mathcal{S}}$ for $\mathcal{Z}^{\mathrm{triv}}$. In Section 4.6 we prove that $\mathcal{Z}^{\mathrm{TV},\mathcal{S}} \cong (\mathcal{Z}^{\mathrm{triv}})_{\mathcal{A}^{\mathcal{S}}}$. Altogether, this establishes our first main result (Theorem A from the introduction).

4.1. TURAEV–VIRELIZIER CONSTRUCTION. Let $\mathcal{S} \equiv (\mathcal{S}, \otimes, \mathbb{1})$ be a spherical fusion category. We choose a set I of representatives of the isomorphism classes of simple objects in $\mathcal{S}$ such that $\mathbb{1} \in I$, and we denote their quantum dimensions as

$$d_i := \dim(i) \in \mathrm{End}_{\mathcal{S}}(\mathbb{1}) = \Bbbk \qquad \text{for all } i \in I. \tag{66}$$

For all $i, j, k \in I$ we say that two bases Λ of $\mathrm{Hom}_{\mathcal{S}}(i \otimes j, k)$ and $\widehat{\Lambda}$ of $\mathrm{Hom}_{\mathcal{S}}(k, i \otimes j)$ are *dual* to each other if they are dual with respect to the trace pairing

$$\mathrm{Hom}_{\mathcal{S}}(i \otimes j, k) \times \mathrm{Hom}_{\mathcal{S}}(k, i \otimes j) \supset \Lambda \times \widehat{\Lambda} \ni (\lambda, \widehat{\mu}) \longmapsto \raisebox{-1em}{} \in \Bbbk. \tag{67}$$

By useful abuse of notation we then write

$$\raisebox{-1em}{} = \delta_{\lambda,\mu} \qquad \text{for all } \lambda \in \Lambda, \, \widehat{\mu} \in \widehat{\Lambda} \tag{68}$$

and we will always denote the dual basis element of λ by $\widehat{\lambda}$. Note that the simple objects i, j, k are suppressed in the notation for the basis elements $\lambda, \widehat{\mu}$, and we will infer the former from the context.
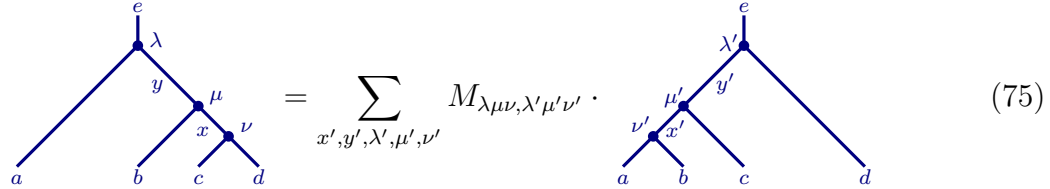
4.2. LEMMA.

(i) *For all $i, j, j', k, k' \in I$ we have*

$$\raisebox{-1.5em}{} = \frac{1}{d_k} \delta_{\lambda,\mu} \delta_{k,k'} \cdot \raisebox{-1.5em}{}, \qquad \raisebox{-1.5em}{} = \frac{1}{d_j} \delta_{\lambda,\mu} \delta_{j,j'} \cdot \raisebox{-1.5em}{}. \tag{69}$$

(ii) *For all $i, j, a, b \in I$ we have*

$$\sum_{k,\lambda} d_k \cdot \quad = \quad \Bigg|\Bigg| \quad , \qquad \sum_{l,\mu} d_l \cdot \quad = \quad \Bigg|\Bigg| \tag{70}$$

where the first sum is over all $k \in I$ and (for fixed k) all elements λ of a chosen basis of $\mathrm{Hom}_{\mathcal{S}}(i \otimes j, k)$, and similarly for the second sum.

(iii) *For all $i, j, k \in I$, $\Gamma \in \mathrm{Hom}_{\mathcal{S}}(\mathbb{1}, k^* \otimes i \otimes j)$ and $\Gamma' \in \mathrm{Hom}_{\mathcal{S}}(k^* \otimes i \otimes j, \mathbb{1})$ we have*

$$\sum_{\lambda} \quad = \quad . \tag{71}$$

In parts (i)–(iii), the vertically reflected versions of the identities hold as well.

PROOF. All these identities follow from simple manipulations with bases: for part (i) take quantum traces on both sides; in part (ii) post-compose both sides with the same basis vector and use part (i); part (iii) follows from inserting the first identity in (ii) applied to $i \otimes j$ on the right-hand side and then using the second identity in (ii) together with the observation that $\mathrm{Hom}_{\mathcal{S}}(\mathbb{1}, l) = \{0\}$ unless $l = \mathbb{1}$, and that the $l = \mathbb{1}$ summand in (ii) can be written as $\frac{1}{d_a} \delta_{a,b} \, \mathrm{coev}_a \circ \widetilde{\mathrm{ev}}_a$. ∎

We define the *F-matrix* elements $F_{\mu\mu'}^{\lambda\lambda'}$ in terms of the chosen bases as follows:

$$\quad = \quad \sum_{d,\lambda',\mu'} F_{\mu\mu'}^{\lambda\lambda'} \cdot \quad , \tag{72}$$

$$\quad = \quad \sum_{c,\lambda'',\mu''} (F^{-1})_{\mu'\mu''}^{\lambda'\lambda''} \cdot \quad . \tag{73}$$

Using Lemma 4.2(i), these can be expressed in terms of closed string diagrams as

$$F_{\mu\mu'}^{\lambda\lambda'} = d_d \cdot \quad , \qquad (F^{-1})_{\mu'\mu''}^{\lambda'\lambda''} = d_c \cdot \quad . \tag{74}$$

The pentagon identity satisfied by the associator translates into an identity for F-matrix elements as follows. One computes the change-of-basis matrix $M_{\lambda\mu\nu,\lambda'\mu'\nu'}$ in

$$\sum_{\delta} F^{\lambda\delta}_{\mu\nu'} F^{\delta\lambda'}_{\nu\mu'} = \sum_{z,\varepsilon,\phi,\kappa} F^{\mu\varepsilon}_{\nu\phi} F^{\lambda\lambda'}_{\varepsilon\kappa} F^{\kappa\mu'}_{\phi\nu'}, \tag{76}$$

where the indices take values as prescribed by (75), and z labels the edge between the vertices with labels ϕ and ε.

In the remainder of Section 4.1 we review the Turaev–Virelizier construction [TVire] of the Turaev–Viro TQFT

$$\mathcal{Z}^{\mathrm{TV},\mathcal{S}} \colon \mathrm{Bord}_3 \longrightarrow \mathrm{vect} \tag{77}$$

for a spherical fusion category $\mathcal{S}$. We only provide the details we need for the proof of Theorem 4.8 in Section 4.6.

Let $\Sigma \in \mathrm{Bord}_3$ and let M be a 3-bordism. Recall from [TVire, Ch. 11] the notions of an *oriented graph* Γ in Σ, and of an *oriented stratified 2-polyhedron* P in M. We will exclusively consider the special cases where Γ is the Poincaré dual of a triangulation of Σ with chosen orientations for the 1-strata of Γ (called edges), and where P is dual to a triangulation of M with chosen orientations for the 2-strata of P (called regions). We will denote the sets of j-strata of Γ and P by Γ_j and P_j, respectively.

For an oriented graph Γ in Σ as above, let c be an *$\mathcal{S}$-colouring* of Γ, i.e. a map $c\colon \Gamma_1 \to I$. For a vertex $v \in \Gamma_0$ consider the cyclic set of edges $(e_1, \ldots, e_n)$ incident on v as determined by the opposite orientation of Σ. Set $\varepsilon(e_i) = +$ if e_i is oriented towards v, and $\varepsilon(e_i) = -$ otherwise, and then

$$H_{e_i} = \mathrm{Hom}_{\mathcal{S}}\left(\mathbb{1}, c(e_i)^{\varepsilon(e_i)} \otimes \cdots \otimes c(e_n)^{\varepsilon(e_n)} \otimes c(e_1)^{\varepsilon(e_1)} \otimes \cdots \otimes c(e_{i-1})^{\varepsilon(e_{i-1})}\right) \tag{78}$$

where we use the notation $u^+ = u$ and $u^- = u^*$ for all $u \in \mathcal{S}$.

The duality morphisms of $\mathcal{S}$ induce isomorphisms $\{f\}$ between the H_{e_i} which form a projective system, and in [TVire] the vector space $H(E_c^v)$ assigned to the data $E_c^v = ((e_1, \ldots, e_n), c, \varepsilon)$ is defined to be the projective limit. One can also use the duality morphisms of $\mathcal{S}$ to obtain isomorphisms $\{g\}$ that move tensor factors between the arguments of $\mathrm{Hom}_{\mathcal{S}}(-,-)$, for example

$$\mathrm{Hom}_{\mathcal{S}}\left(\mathbb{1}, c(e_k)^+ \otimes c(e_j)^- \otimes c(e_i)^-\right) \cong \mathrm{Hom}_{\mathcal{S}}\left(c(e_i) \otimes c(e_j), c(e_k)\right). \tag{79}$$

The projective limit of the system $\{f, g\}$ is uniquely isomorphic to the limit of $\{f\}$, hence we can and will work with the former as $H(E_c^v)$. In terms of these we set

$$H(\Gamma, c; \Sigma) = \bigotimes_{v \in \Gamma_0} H(E_c^v) \tag{80}$$

for Σ, Γ, c as above, and we note that there is a canonical isomorphism $H(\Gamma^{\mathrm{op}}, c; \Sigma^{\mathrm{op}})^* \cong H(\Gamma, c; \Sigma)$, where $(-)^{\mathrm{op}}$ signifies opposite orientation. For example in (78) for $i = 1$ one pairs $\mathrm{Hom}_{\mathcal{S}}(\mathbb{1}, c(e_1)^{\varepsilon(e_1)} \otimes \cdots \otimes c(e_n)^{\varepsilon(e_n)})$ with $\mathrm{Hom}_{\mathcal{S}}(\mathbb{1}, c(e_n)^{-\varepsilon(e_n)} \otimes \cdots \otimes c(e_1)^{-\varepsilon(e_1)})$ using the duality morphisms.

Let now $\Sigma = S^2$ be endowed with an oriented graph Γ with an $\mathcal{S}$-colouring c. Using the cone isomorphisms $\{f\}$ and sphericality one can associate to these data a functional (see [TVire, Sect. 12.2])

$$\mathbb{F}_{\mathcal{S}}(\Gamma, c) \in H(\Gamma, c; \Sigma)^* . \tag{81}$$

The idea is that locally around every vertex of Γ one can interpret it as a slot into which one can insert elements of the tensor factors in (80). This tensor product is then evaluated to the corresponding string diagram in $\mathcal{S}$. We will have need for only two types of graph Γ, to which we turn next; the associated functionals will be defined in (84) below.

Consider a 3-bordism M with an oriented stratified 2-polyhedron P that comes from a triangulation of M as discussed above. All vertices in P correspond to oriented tetrahedra, hence locally they look like

$$\tag{82}$$
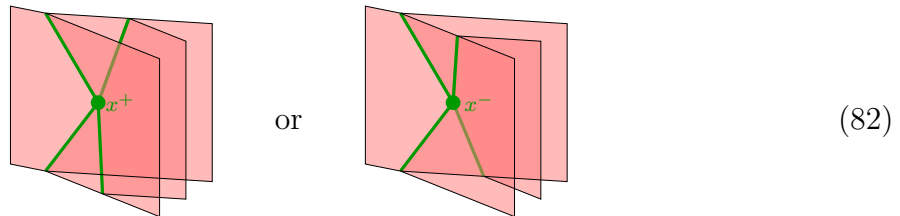
where only the 2-dimensional regions are oriented (all inducing the counter-clockwise orientation in the paper plane). On the boundary $\partial B_x \cong S^2$ of a small ball B_x around such vertices, oriented as induced from $M \setminus B_x$ according to the construction of [TVire], the above stratifications induce the oriented graphs

$$\tag{83}$$
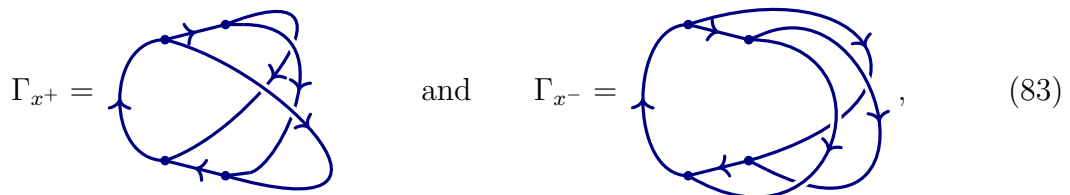
respectively. For every $\mathcal{S}$-colouring c of the edges in the graphs $\Gamma_{x^\pm}$, we obtain functionals $\mathbb{F}_{\mathcal{S}}(\Gamma_{x^\pm}, c) \in H(\Gamma_{x^\pm}, c; S^2)^*$. Denoting the values of the colouring by $i, j, k, l, m, n \in I$ we

may employ the cone isomorphisms of the projective system defining $H(\Gamma_{x\pm}, c; S^2)$ to obtain the explicit functionals (where here and below we suppress the choice of i, j, k, l, m, n in the notation)

$$\mathbb{F}_{\mathcal{S}}(\Gamma_{x^+}) = \qquad , \qquad \mathbb{F}_{\mathcal{S}}(\Gamma_{x^-}) = \qquad \qquad (84)$$

which are respectively elements of

$$\left(\mathrm{Hom}_{\mathcal{S}}(j \otimes i, k) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{S}}(n \otimes l, i) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{S}}(m, j \otimes n) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{S}}(k, m \otimes l)\right)^*,$$
$$\left(\mathrm{Hom}_{\mathcal{S}}(i \otimes j, k) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{S}}(l \otimes n, i) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{S}}(m, n \otimes j) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{S}}(k, l \otimes m)\right)^*. \qquad (85)$$

The final ingredient for the Turaev–Viro invariant in the construction of [TVire] are the contraction maps $*_e$. We describe them for the case of interest to us: Let M, P be as before. Choose an $\mathcal{S}$-colouring $c\colon P_2 \to I$ and decorate each region $r \in P_2$ with the object $c(r)$. Hence every internal edge e of P looks like

$$(86)$$

for some $i, j, k \in I$. Recall that regions of P are oriented, but edges are not. The region coloured by k in (86) induces an orientation on the edge e (upwards in the diagram in our convention), and we denote the corresponding *oriented* edge by e^+. The oppositely oriented edge is denoted e^-. To these the construction of [TVire] associates vector spaces

$$H_c(e^+) \cong \mathrm{Hom}_{\mathcal{S}}(i \otimes j, k), \qquad H_c(e^-) \cong \mathrm{Hom}_{\mathcal{S}}(k, i \otimes j). \qquad (87)$$

Since the pairing $H_c(e^+) \otimes_{\Bbbk} H_c(e^-) \to \Bbbk$ of (67) is nondegenerate, there is a unique dual copairing $\gamma\colon \Bbbk \to H_c(e^-) \otimes_{\Bbbk} H_c(e^+)$ given by $\gamma(1) = \sum_\lambda \widehat{\lambda} \otimes \lambda$. The *contraction map* $*_e\colon H_c(e^+)^* \otimes_{\Bbbk} H_c(e^-)^* \to \Bbbk$ is defined to be the dual map γ^* composed with the canonical isomorphisms $\Bbbk \cong \Bbbk^*$ and $(V \otimes_{\Bbbk} W)^* \cong W^* \otimes_{\Bbbk} V^*$ for all $V, W \in \mathrm{vect}_{\Bbbk}$. Thus we have (for basis elements $\lambda, \mu \in \mathrm{Hom}_{\mathcal{S}}(i \otimes j, k)$)

$$*_e\colon H_c(e^+)^* \otimes_{\Bbbk} H_c(e^-)^* \longrightarrow \Bbbk, \qquad \lambda^* \otimes \widehat{\mu}^* \longmapsto \delta_{\lambda,\mu}, \qquad (88)$$

where we use

$$\lambda^* := \begin{pmatrix} \boxed{-} \\ \widehat{\lambda} \end{pmatrix}, \qquad \widehat{\mu}^* := \begin{pmatrix} \mu \\ \boxed{-} \end{pmatrix}. \tag{89}$$

We can now describe the Turaev–Viro invariants $\mathcal{Z}_{\mathcal{C}}^{\mathrm{TV}}(M) \in \Bbbk$ for closed 3-manifolds M [TVire]:

$$\mathcal{Z}^{\mathrm{TV},\mathcal{S}}(M) = \big(\dim(\mathcal{S})\big)^{-|P_3|} \sum_{c\colon P_2\to I} \Big(\prod_{r\in P_2} d_{c(r)}^{\chi(r)}\Big)\Big(\bigotimes_{e\in P_1} *_e\Big)\Big(\bigotimes_{x\in P_0} \mathbb{F}_{\mathcal{S}}(\Gamma_x)\Big) \tag{90}$$

for any oriented stratified 2-polyhedron P of M, where P_3 denotes the set of connected components in $M \setminus P$ and $\chi(r)$ denotes the Euler characteristic of the 2-stratum r. Note that for every edge $e \in P_1$ its two oriented versions e^+, e^- correspond to precisely two tensor factors in $\bigotimes_{x\in P_0} \mathbb{F}_{\mathcal{S}}(\Gamma_x)$, since every edge has two endpoints in P_0, and for every $x \in P_0$ its incident edges are all treated as outgoing in the definition of $\mathbb{F}_{\mathcal{S}}(\Gamma_x)$,

On objects and arbitrary morphisms the functor $\mathcal{Z}^{\mathrm{TV},\mathcal{S}}$ is defined along the same lines. For a surface Σ with an embedded oriented graph Γ as above, we set

$$\big|\Gamma;\Sigma\big|^0 = \bigoplus_{c\colon \Gamma_1\to I} H(\Gamma, c; \Sigma) \tag{91}$$

with $H(\Gamma, c; \Sigma)$ as in (80). For a bordism $M\colon \emptyset \to \Sigma$ we choose an extension of the graph Γ to an oriented stratified 2-polyhedron P of M. Let $\mathrm{Col}(P,c)$ be the set of $\mathcal{S}$-colourings $\widetilde{c}\colon P_2 \to I$ with $\widetilde{c}(r_e) = c(e)$ for the region r_e with $e \subset r_e \cap \partial M$ and all edges e in Γ. We write P_1^{int} for the set of edges in P without endpoints in ∂M, and P_1^{∂} for the set of edges in P with precisely one endpoint in ∂M. Such endpoints correspond to vertices $v \in \Gamma_0$ and we have

$$\bigotimes_{e\in P_1^{\partial}} H_{\widetilde{c}}(e^{\mathrm{out}})^* \cong H(\Gamma^{\mathrm{op}}, c; \Sigma^{\mathrm{op}}) \tag{92}$$

where e^{out} denotes the edge e with orientation towards ∂M. Thus by contracting along interior edges $e \in P_1^{\mathrm{int}}$ we obtain a vector

$$\big|M;\Gamma, c\big|^0 = \big(\dim(\mathcal{S})\big)^{-|P_3|} \sum_{\widetilde{c}\in\mathrm{Col}(P,c)} \Big(\prod_{r\in P_2} d_{\widetilde{c}(r)}^{\chi(r)}\Big)\Big(\bigotimes_{e\in P_1^{\mathrm{int}}} *_e\Big)\Big(\bigotimes_{x\in P_0} \mathbb{F}_{\mathcal{S}}(\Gamma_x)\Big) \tag{93}$$

in $H(\Gamma^{\mathrm{op}}, c; \Sigma^{\mathrm{op}})^* \cong H(\Gamma, c; \Sigma)$, generalising (90).

If $(\Sigma, \Gamma) = (\Sigma'^{\mathrm{op}}, \Gamma'^{\mathrm{op}}) \sqcup (\Sigma'', \Gamma'')$ we can view M as a bordism $\Sigma' \to \Sigma''$. Writing

$$\Upsilon\colon H(\Gamma', c; \Sigma')^* \otimes_{\Bbbk} H(\Gamma''^{\mathrm{op}}, c; \Sigma''^{\mathrm{op}})^* \xrightarrow{\cong} \mathrm{Hom}_{\Bbbk}\big(H(\Gamma', c; \Sigma'), H(\Gamma'', c; \Sigma'')\big) \tag{94}$$

for the canonical isomorphism, we obtain a linear map

$$\big|M;\Sigma',\Gamma',\Sigma'',\Gamma'',c\big|^0 = \frac{\big(\dim(\mathcal{S})\big)^{|\Sigma''\backslash\Gamma''|}}{\prod_{e\in\Gamma''_1} d_{c(e)}} \cdot \Upsilon\Big(\big|M;\Gamma,c\big|^0\Big), \tag{95}$$

where $|\Sigma'' \backslash \Gamma''|$ denotes the number of components of $\Sigma'' \backslash \Gamma''$. Restricting to $\Sigma' = \Sigma''$ and the cylinder $M = \Sigma' \times [0,1]$, by summing over all $\mathcal{S}$-colourings $c\colon (\Gamma' \sqcup \Gamma'')_1 \to I$, we obtain a projective system

$$p(\Gamma',\Gamma'')\colon \big|\Gamma',\Sigma'\big|^0 \longrightarrow \big|\Gamma'',\Sigma'\big|^0. \tag{96}$$

Then by definition

$$\mathcal{Z}^{\mathrm{TV},\mathcal{S}}(\Sigma') = \varprojlim p(\Gamma',\Gamma'') \tag{97}$$

and $\mathcal{Z}^{\mathrm{TV},\mathcal{S}}$ acts on arbitrary bordism classes as the induced linear maps.

4.3. ORBIFOLD DATA FOR THE TRIVIAL RESHETIKHIN–TURAEV THEORY. By the (3-dimensional) *trivial defect TQFT* $\mathcal{Z}^{\mathrm{triv}}\colon \mathrm{Bord}_3(\mathbb{D}^{\mathrm{triv}}) \to \mathrm{vect}$ we mean the Reshetikhin–Turaev defect TQFT constructed (in Section 2.1) from the "trivial" modular tensor category vect:

$$\mathcal{Z}^{\mathrm{triv}} := \mathcal{Z}^{\mathrm{vect}}. \tag{98}$$

Hence $\mathcal{Z}^{\mathrm{triv}}(\Sigma) = \Bbbk$ for every unstratified surface $\Sigma \in \mathrm{Bord}_3(\mathbb{D}^{\mathrm{triv}})$, while 2- and 1-strata of bordisms in $\mathrm{Bord}_3(\mathbb{D}^{\mathrm{triv}})$ are labelled by Δ-separable symmetric Frobenius $\Bbbk$-algebras and their cyclic modules in vect, respectively. In this section we will construct orbifold data for $\mathcal{Z}^{\mathrm{triv}}$:

4.4. PROPOSITION. *Given a spherical fusion category $\mathcal{S}$, the following is an orbifold datum for $\mathcal{Z}^{\mathrm{triv}}$, denoted $\mathcal{A}^{\mathcal{S}}$:*

$$\mathcal{C} := \mathrm{vect}, \tag{99}$$

$$A := \bigoplus_{i\in I} \Bbbk \qquad (\textit{direct sum of trivial Frobenius algebras } \Bbbk), \tag{100}$$

$$T := \bigoplus_{i,j,k\in I} \mathrm{Hom}_{\mathcal{S}}(i \otimes j, k), \tag{101}$$

$$\alpha\colon \lambda \otimes \mu \longmapsto \sum_{d,\lambda',\mu'} d_d^{-1} F_{\mu\mu'}^{\lambda\lambda'} \cdot \lambda' \otimes \mu', \tag{102}$$

$$\bar{\alpha}\colon \lambda' \otimes \mu' \longmapsto \sum_{c,\lambda'',\mu''} d_c^{-1} (F^{-1})_{\mu'\mu''}^{\lambda'\lambda''} \cdot \lambda'' \otimes \mu'', \tag{103}$$

$$\psi^2 := diag(d_1, d_2, \ldots, d_{|I|}) \qquad (\psi \textit{ is a choice of square root}), \tag{104}$$

$$\phi^2 := \Big(\sum_{i\in I} d_i^2\Big)^{-1} = (\dim \mathcal{S})^{-1} \qquad (\phi \textit{ is a choice of square root}), \tag{105}$$

where the basis elements and sums in (102) *and* (103) *are as in* (72) *and* (73) *while* $\alpha(\lambda \otimes \mu) \stackrel{def}{=} 0 \stackrel{def}{=} \bar{\alpha}(\lambda' \otimes \mu')$ *if* λ, μ *and* λ', μ' *are not compatible as in* (72) *and* (73), *respectively, and the action of* A *on* $\mathrm{Hom}_\mathcal{S}(i \otimes j, k)$ *in* T *is such that only the* k-th *summand* $\Bbbk_k$ *acts non-trivially from the left, and only* $\Bbbk_i \otimes \Bbbk_j$ *acts non-trivially from the right. Different choices of square root in* (104) *give equivalent orbifold data in the sense of Definition* 3.10.

As preparation for the proof of Proposition 4.4 we spell out composition and adjunctions for 2-morphisms in $\mathcal{T}_{\mathcal{Z}^{\mathrm{triv}}}$. Using the isomorphism

$$\mathrm{Hom}_\mathcal{S}(k, i \otimes j) \stackrel{\cong}{\longrightarrow} \mathrm{Hom}_\mathcal{S}(i \otimes j, k)^*, \qquad \widehat{\lambda} \longmapsto \boxed{} \tag{106}$$

we exhibit the $(A \otimes A)$-A-bimodule

$$T^\dagger := \bigoplus_{i,j,k \in I} \mathrm{Hom}_\mathcal{S}(k, i \otimes j) \tag{107}$$

as the adjoint of T via the maps

$$\mathrm{ev}_T \colon T^\dagger \otimes_A T = \bigoplus_{a,b,i,j,k} \mathrm{Hom}_\mathcal{S}(k, a \otimes b) \otimes_\Bbbk \mathrm{Hom}_\mathcal{S}(i \otimes j, k) \longrightarrow A \otimes_\Bbbk A$$

$$\mathrm{Hom}_\mathcal{S}(k, a \otimes b) \otimes_\Bbbk \mathrm{Hom}_\mathcal{S}(i \otimes j, k) \ni \widehat{\mu} \otimes \lambda \longmapsto \delta_{a,i}\, \delta_{b,j}\, \delta_{\lambda,\mu} \cdot 1_i \otimes 1_j$$

and

$$\mathrm{coev}_T \colon A \longrightarrow T \otimes_{A \otimes_\Bbbk A} T^\dagger = \bigoplus_{i,j,k} \mathrm{Hom}_\mathcal{S}(i \otimes j, k) \otimes_\Bbbk \mathrm{Hom}_\mathcal{S}(k, i \otimes j) \tag{108}$$

$$1_k \longmapsto \sum_{i,j,\lambda} \lambda \otimes \widehat{\lambda} \tag{109}$$

where 1_k denotes $1 \in \Bbbk$ in the k-th copy of $\Bbbk$ in A. Note how tensor products over the direct sum algebra A turn into tensor products over $\Bbbk$ of matching summands. Similarly, we have adjunction maps

$$\widetilde{\mathrm{ev}}_T \colon T \otimes_{A \otimes_\Bbbk A} T^\dagger \longrightarrow A\,, \qquad \widetilde{\mathrm{coev}}_T \colon A \otimes_\Bbbk A \longrightarrow T^\dagger \otimes_A T\,. \tag{110}$$

PROOF PROOF OF PROPOSITION 4.4. We will show that the data $(\mathrm{vect}, A, T, \alpha, \bar{\alpha}, \psi, \phi)$ satisfy the constraints in Proposition 3.5.

The constraints (41) and (42) reduce to the pentagon axiom for $\mathcal{S}$ (expressed in terms of F-symbols) and to the fact that up to dimension factors, α is the inverse of $\bar{\alpha}$. For example, writing c for the symmetric braiding of vect, the left-hand side of (41) becomes

$$
\begin{array}{ccc}
T \otimes T \otimes T & & \displaystyle\sum_{x',y',\lambda',\mu',\nu'} d_{x'}^{-1} d_{y'}^{-1} \sum_{\delta} F_{\mu\nu'}^{\lambda\delta} F_{\nu\mu'}^{\delta\lambda'} \cdot \lambda' \otimes \mu' \otimes \nu' \\
\big\uparrow {\scriptstyle \alpha \otimes \mathrm{id}} & & \big\uparrow \\
T \otimes T \otimes T & & \displaystyle\sum_{x',\nu',\delta} d_{x'}^{-1} F_{\mu\nu'}^{\lambda\delta} \cdot \delta \otimes \nu \otimes \nu' \\
\big\uparrow {\scriptstyle \mathrm{id}_T \otimes c_{T,T}^{-1}} & & \big\uparrow \\
T \otimes T \otimes T & & \displaystyle\sum_{x',\nu',\delta} d_{x'}^{-1} F_{\mu\nu'}^{\lambda\delta} \cdot \delta \otimes \nu' \otimes \nu \\
\big\uparrow {\scriptstyle \alpha \otimes \mathrm{id}} & & \big\uparrow \\
T \otimes T \otimes T & & \lambda \otimes \mu \otimes \nu
\end{array}
\tag{111}
$$

which is the left-hand side of (76), up to the dimension factors which cancel against corresponding factors on the right-hand side of (41) together with the factors coming from ψ^2.

We now turn to the first condition of (43), which is an identity of linear maps on

$$
\begin{aligned}
T \otimes_A T^\dagger &= \bigoplus_{i,j,k,l,m} \mathrm{Hom}_{\mathcal{S}}(m \otimes k, l) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{S}}(i, m \otimes j) \\
&= \bigoplus_{i,j,k,l,m} {}_l T_{m,k} \otimes_{\Bbbk} ({}_i T_{m,j})^\dagger .
\end{aligned}
\tag{112}
$$

Here and below we use the abbreviations

$$
{}_k T_{i,j} := \mathrm{Hom}_{\mathcal{S}}(i \otimes j, k), \quad ({}_k T_{i,j})^\dagger := \mathrm{Hom}_{\mathcal{S}}(k, i \otimes j).
\tag{113}
$$

Hence it is sufficient to show that for all fixed $i,j,k,l,m \in I$ and for all basis elements $\lambda \in \mathrm{Hom}_{\mathcal{S}}(m \otimes k, l)$, $\widehat{\widetilde{\lambda}} \in \mathrm{Hom}_{\mathcal{S}}(i, m \otimes j)$ the left-hand side of (43) acts as the identity times d_m^{-1} on $\lambda \otimes \widehat{\widetilde{\lambda}}$. This action on $\lambda \otimes \widehat{\widetilde{\lambda}}$ is computed in Figure 1, where the Roman summation indices a,b,x,y,z' range over I while Greek indices range over chosen bases elements:

$$
\begin{aligned}
\mu &\in \mathrm{Hom}_{\mathcal{S}}(j \otimes x, k), \\
\lambda' &\in \mathrm{Hom}_{\mathcal{S}}(y \otimes x, l), \\
\mu' &\in \mathrm{Hom}_{\mathcal{S}}(m \otimes j, y),
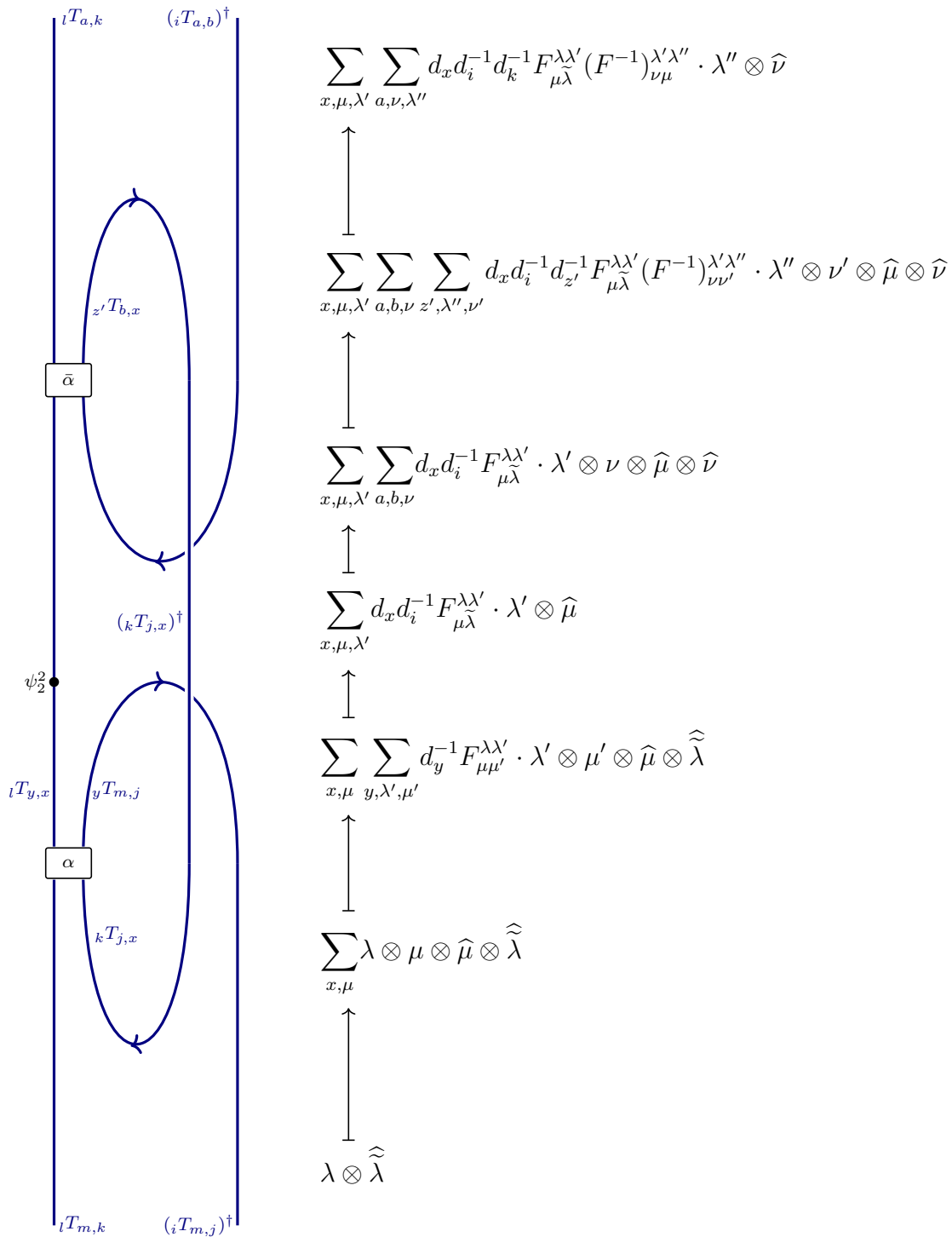\end{aligned}
$$

$$\sum_{x,\mu,\lambda'} \sum_{a,\nu,\lambda''} d_x d_i^{-1} d_k^{-1} F_{\mu\widetilde{\lambda}}^{\lambda\lambda'} (F^{-1})_{\nu\mu}^{\lambda'\lambda''} \cdot \lambda'' \otimes \widehat{\nu}$$

$$\sum_{x,\mu,\lambda'} \sum_{a,b,\nu} \sum_{z',\lambda'',\nu'} d_x d_i^{-1} d_{z'}^{-1} F_{\mu\widetilde{\lambda}}^{\lambda\lambda'} (F^{-1})_{\nu\nu'}^{\lambda'\lambda''} \cdot \lambda'' \otimes \nu' \otimes \widehat{\mu} \otimes \widehat{\nu}$$

$$\sum_{x,\mu,\lambda'} \sum_{a,b,\nu} d_x d_i^{-1} F_{\mu\widetilde{\lambda}}^{\lambda\lambda'} \cdot \lambda' \otimes \nu \otimes \widehat{\mu} \otimes \widehat{\nu}$$

$$\sum_{x,\mu,\lambda'} d_x d_i^{-1} F_{\mu\widetilde{\lambda}}^{\lambda\lambda'} \cdot \lambda' \otimes \widehat{\mu}$$

$$\sum_{x,\mu} \sum_{y,\lambda',\mu'} d_y^{-1} F_{\mu\mu'}^{\lambda\lambda'} \cdot \lambda' \otimes \mu' \otimes \widehat{\mu} \otimes \widehat{\widetilde{\lambda}}$$

$$\sum_{x,\mu} \lambda \otimes \mu \otimes \widehat{\mu} \otimes \widehat{\widetilde{\lambda}}$$

$$\lambda \otimes \widehat{\widetilde{\lambda}}$$

Diagram labels: $_lT_{a,k}$, $(_iT_{a,b})^\dagger$, $_{z'}T_{b,x}$, $\bar\alpha$, $(_kT_{j,x})^\dagger$, ψ_2^2, $_lT_{y,x}$, $_yT_{m,j}$, α, $_kT_{j,x}$, $_lT_{m,k}$, $(_iT_{m,j})^\dagger$.

Figure 1: Computing the left-hand side of condition (43) for $\mathcal{A}^\mathcal{S}$. Note that this is a string diagram in vect, so there is no need to distinguish between over- and under-crossings, but we prefer to keep the notation from Proposition 3.5.

$$\nu \in \operatorname{Hom}_{\mathcal{S}}(a \otimes b, i),$$
$$\lambda'' \in \operatorname{Hom}_{\mathcal{S}}(a \otimes z', l),$$
$$\nu' \in \operatorname{Hom}_{\mathcal{S}}(b \otimes x, z'). \tag{114}$$

The outcome

$$\sum_{x,\mu,\lambda'} \sum_{a,\nu,\lambda''} d_x \big(d_i^{-1} F_{\mu\widetilde{\lambda}}^{\lambda\lambda'}\big) \cdot \big(d_k^{-1}(F^{-1})_{\nu\mu}^{\lambda'\lambda''}\big) \cdot \lambda'' \otimes \widehat{\nu} \tag{115}$$

of the computation in Figure 1 can be further simplified:

Hence we have shown that the first identity in (43) holds. The other identity is checked similarly.

Next we turn to the first constraint in (44). Verifying that it holds for our orbifold data is similar to the case of (43): We have to show that the left-hand side of (44) acts as the identity times d_k^{-1} on $\widehat{\widetilde{\lambda}} \otimes \lambda$ for all elements $\lambda \in \operatorname{Hom}_{\mathcal{S}}(i \otimes j, k)$ and $\widehat{\widetilde{\lambda}} \in \operatorname{Hom}_{\mathcal{S}}(k, a \otimes b)$ of chosen bases for all $a, b, i, j, k \in I$. This action is computed in Figure 2 to produce

$$\sum_{c,z,\nu,\nu',\mu,\mu'} d_c \big(d_a^{-1} F_{\mu\mu'}^{\lambda\widetilde{\lambda}}\big) \cdot \big(d_j^{-1}(F^{-1})_{\mu'\mu}^{\nu\nu''}\big) \cdot \widehat{\nu} \otimes \nu''. \tag{116}$$

$$\sum_{c,\mu,\mu'}\sum_{z,\nu}\sum_{\nu''} d_c d_a^{-1} d_j^{-1} F_{\mu\mu'}^{\lambda\widetilde{\lambda}}(F^{-1})_{\mu'\mu}^{\nu\nu''} \cdot \widehat{\nu}\otimes\nu''$$

$$\sum_{c,\mu,\mu'}\sum_{z,\nu}\sum_{z'',\nu'',\mu''} d_c d_a^{-1} d_{z''}^{-1} F_{\mu\mu'}^{\lambda\widetilde{\lambda}}(F^{-1})_{\mu'\mu''}^{\nu\nu''} \\ \cdot \widehat{\nu}\otimes\nu''\otimes\mu''\otimes\widehat{\mu}$$

$$\sum_{c,\mu,\mu'}\sum_{z,\nu} d_c d_a^{-1} F_{\mu\mu'}^{\lambda\widetilde{\lambda}}\cdot \widehat{\nu}\otimes\nu\otimes\mu'\otimes\widehat{\mu}$$

$$\sum_{c,\mu,\mu'} d_c d_a^{-1} F_{\mu\mu'}^{\lambda\widetilde{\lambda}}\cdot \mu'\otimes\widehat{\mu}$$

$$\sum_{c,d,\mu}\sum_{y',\lambda',\mu'} d_{y'}^{-1} F_{\mu\mu'}^{\lambda\lambda'}\cdot \widehat{\widetilde{\lambda}}\otimes\lambda'\otimes\mu'\otimes\widehat{\mu}$$

$$\sum_{c,d,\mu}\widehat{\widetilde{\lambda}}\otimes\lambda\otimes\mu\otimes\widehat{\mu}$$

$$\widehat{\widetilde{\lambda}}\otimes\lambda$$
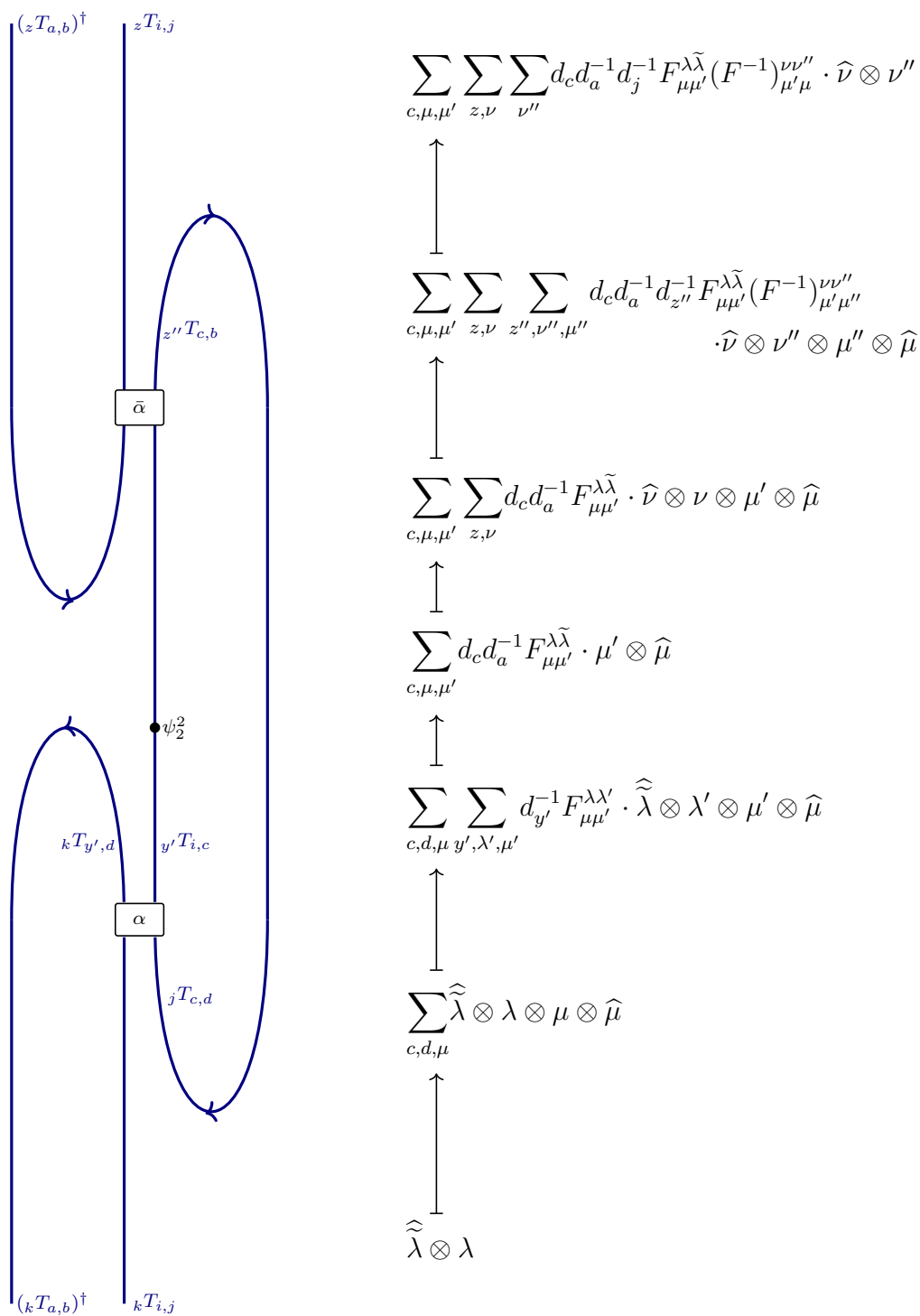
Figure 2: Computing the left-hand side of condition (44) for $\mathcal{A}^{\mathcal{S}}$.

This can be simplified to

$$(116) = \sum_{c,z,\nu,\nu'',\mu,\mu'} d_c \cdot$$

$$\cdot \widehat{\nu} \otimes \nu''$$

$$\overset{(*)}{=} \sum_{c,z,\nu,\nu'',\mu'} d_c \cdot$$ $$\cdot \widehat{\nu} \otimes \nu'' = \sum_{z,\nu,\nu''}$$ $$\cdot \widehat{\nu} \otimes \nu''$$

$$= \sum_{z,\nu,\nu''} \frac{1}{d_k} \delta_{\nu,\widetilde{\lambda}} \, \delta_{k,z} \cdot$$ $$\cdot \widehat{\nu} \otimes \nu'' = \frac{1}{d_k} \sum_{\nu}$$ $$\cdot \widehat{\widetilde{\lambda}} \otimes \nu''$$

$$= \frac{1}{d_k} \cdot \widehat{\widetilde{\lambda}} \otimes \lambda \,,$$

where in $(*)$ first the basis element λ in the left diagram and the element $\widehat{\nu''}$ in the right diagram are "taken around" by using the cyclicity of the trace, and then (71) is used. The second identity in (44) follows analogously.

It remains to verify the constraints in (45). Writing again 1_k for the unit in the k-th copy of $\Bbbk$ in $A = \bigoplus_{k \in I} \Bbbk$, the left-hand side of the first identity in (45) is

$$
\begin{array}{ccccc}
A & \xrightarrow{\psi_1^2 \psi_2^2 \mathrm{coev}_T} & T \otimes_{A \otimes_\Bbbk A} T^\dagger & \xrightarrow{\ \widetilde{\mathrm{coev}}_T\ } & A \\[2mm]
1_k & \longmapsto & \displaystyle\sum_{i,j,\lambda} d_i d_j \cdot \lambda \otimes \widehat{\lambda} & \longmapsto & \displaystyle\sum_{i,j} d_i d_j N_{ij}^k \cdot \mathrm{id}_{A_k}
\end{array}
\tag{117}
$$

where $N_{ij}^k := \dim_\Bbbk \mathrm{Hom}_{\mathcal{S}}(i \otimes j, k)$. We further compute

$$\sum_{i,j} d_i d_j N_{ij}^k = \sum_{i,j} d_i d_{j^*} N_{ik^*}^{j^*} = \sum_i d_i d_i d_k = \phi^{-2} \cdot (\eta_A \circ \psi^2)\big|_{A_k}\,, \tag{118}$$

where in the first step we used that $N_{ij}^k = N_{ik^*}^{j^*}$ and that $d_j = d_{j^*}$. The second step is $\dim(i \otimes k^*) = \sum_l N_{ik^*}^l \dim(l)$. ∎

4.5. REMARK. The orbifold datum $\mathcal{A}^{\mathcal{S}}$ of Definition 4.4 constructed from a spherical fusion category $\mathcal{S}$ is expressed internally to the modular tensor category vect, in line with the general setup of Section 3.4. Equivalently, $\mathcal{A}^{\mathcal{S}}$ can be described internal to the 3-category with duals $\mathrm{Bimod}_{\Bbbk}$ of spherical fusion categories, bimodule categories with module traces, bimodule functors and their natural transformations (studied in [Sc]), along the general lines of [CRS1, Sect. 4.2]. In this formulation, 3-, 2-, 1- and 0-strata are labelled by vect, the vect-vect-bimodule $\mathcal{S}$, the functor $\otimes\colon \mathcal{S}\boxtimes\mathcal{S} \to \mathcal{S}$ and natural transformations constructed from the associator, respectively. Similarly, the ϕ- and ψ-insertions are also natural transformations; for example, one can compute the right quantum dimension $\dim_{\mathrm{r}}(\otimes)$ to be $\dim\mathcal{S}$ times the identity, which fixes ϕ^2 to be $(\dim\mathcal{S})^{-1} \cdot \mathrm{id}_{\mathrm{Id}_{\mathrm{vect}}}$.

In this way the orbifold datum $\mathcal{A}^{\mathcal{S}}$ is a spherical fusion category internal to the 3-category with duals $\mathcal{T}_{\mathcal{Z}^{\mathrm{triv}}} \subset \mathrm{Bimod}_{\Bbbk}$ constructed from $\mathcal{Z}^{\mathrm{triv}}$ as in [CMS]. A related idea to use (spherical fusion categories viewed as) "2-algebras" to construct Turaev–Viro theory was outlined in [BL].

In general, we can think of orbifold data for a 3-dimensional defect TQFT $\mathcal{Z}$ as spherical fusion categories internal to $\mathcal{T}_{\mathcal{Z}}$.

4.6. TURAEV–VIRO THEORY IS AN ORBIFOLD. In this section we prove that for every spherical fusion category $\mathcal{S}$, Turaev–Viro theory $\mathcal{Z}^{\mathrm{TV},\mathcal{S}}$ and the orbifold theory $\mathcal{Z}^{\mathrm{triv}}_{\mathcal{A}^{\mathcal{S}}}$ are isomorphic as TQFTs.

We first show that $\mathcal{Z}^{\mathrm{TV},\mathcal{S}}$ and $\mathcal{Z}^{\mathrm{triv}}_{\mathcal{A}^{\mathcal{S}}}$ assign *identical* invariants to closed 3-manifolds. Let M be such a closed manifold, and let t be an oriented triangulation of M. As recalled from [CRS1] in Section 2.3, by decorating the Poincaré dual stratification with the orbifold datum $\mathcal{A}^{\mathcal{S}}$ from Definition 4.4, we obtain a morphism $M^{t,\mathcal{A}^{\mathcal{S}}}\colon \emptyset \to \emptyset$ in $\mathrm{Bord}_3^{\mathrm{def}}(\mathbb{D}^{\mathrm{triv}})$. By definition,

$$\mathcal{Z}^{\mathrm{triv}}_{\mathcal{A}^{\mathcal{S}}}(M) = \mathcal{Z}^{\mathrm{triv}}\big(M^{t,\mathcal{A}^{\mathcal{S}}}\big). \tag{119}$$

To compute the right-hand side of (119), we will denote the set of j-strata of $M^{t,\mathcal{A}^{\mathcal{S}}}$ by M_j for $j \in \{1,2,3\}$, while the sets of positively and negatively oriented 0-strata are denoted M_0^+ and M_0^-, respectively. By construction, the invariant $\mathcal{Z}^{\mathrm{triv}}(M^{t,\mathcal{A}^{\mathcal{S}}})$ is a single string diagram D in vect. Using the decompositions $A = \bigoplus_i \Bbbk$ and $T = \bigoplus_{i,j,k} \mathrm{Hom}_{\mathcal{S}}(i \otimes j, k)$, the diagram D can be written as a sum of string diagrams whose strings are labelled by simple objects in I. The morphisms in these diagrams are either point insertions ψ^2, ϕ^2, or duality maps

$$\mathrm{Hom}_{\mathcal{S}}(l, a \otimes b) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{S}}(i \otimes j, k) \ni \widehat{\lambda'} \otimes \lambda \longmapsto \delta_{a,i}\delta_{b,j}\delta_{k,l}\delta_{\lambda,\lambda'}\,, \tag{120}$$

$$1_k \longmapsto \sum_\lambda \lambda \otimes \widehat{\lambda} \in \mathrm{Hom}_{\mathcal{S}}(i \otimes j, k) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{S}}(k, i \otimes j)\,, \tag{121}$$

or their tilded versions

$$\mathrm{Hom}_{\mathcal{S}}(i \otimes j, k) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{S}}(l, a \otimes b) \ni \lambda \otimes \widehat{\lambda'} \longmapsto \delta_{a,i}\delta_{b,j}\delta_{k,l}\delta_{\lambda,\lambda'}\,, \tag{122}$$

$$1_k \longmapsto \sum_\lambda \widehat{\lambda} \otimes \lambda \in \mathrm{Hom}_{\mathcal{S}}(k, i \otimes j) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{S}}(i \otimes j, k) \tag{123}$$

as in Section 4.1, or the component maps

$$\alpha\colon \lambda\otimes\mu \longmapsto \sum_{d,\lambda',\mu'} d_d^{-1} F_{\mu\mu'}^{\lambda\lambda'} \cdot \lambda'\otimes\mu' , \tag{124}$$

$$\bar\alpha\colon \lambda'\otimes\mu' \longmapsto \sum_{c,\lambda'',\mu''} d_c^{-1} (F^{-1})_{\mu'\mu''}^{\lambda'\lambda''} \cdot \lambda''\otimes\mu'' \tag{125}$$

of Definition 4.4, corresponding to 0-strata in $M_0^\pm$.

As we will explain in the following, $\mathcal{Z}^{\mathrm{triv}}(M^{t,\mathcal{A}^{\mathcal{S}}})$ is equal to

$$\phi^{2|M_3|} \cdot \sum_{\mathcal{I}=(i_1,\dots,i_{|M_2|})\in I^{|M_2|}} d_{i_1}\dots d_{i_{|M_2|}} \sum_{\lambda_e^{\mathcal{I}}} \Big(\prod_{x\in M_0} \mathbb{F}_{\mathcal{C}}(\Gamma_x)\Big(\bigotimes_{e\in E_x} \lambda_e^{\mathcal{I}}\Big)\Big). \tag{126}$$

To arrive at this expression, first note that each 3-stratum in $M^{t,\mathcal{A}^{\mathcal{S}}}$ carries a ϕ^2-insertion, leading to the global factor $\phi^{2|M_3|}$. Secondly, each A-labelled 2-stratum carries an insertion of $\psi^2 = \mathrm{diag}(d_1,\dots,d_{|I|})$, leading to $\sum_{\mathcal{I}} d_{i_1}\dots d_{i_{|M_2|}}$ when decomposing $A=\bigoplus_i \Bbbk$. Thirdly, for fixed $\mathcal{I}\in I^{|M_2|}$ and $e\in M_1$, $\lambda_e^{\mathcal{I}}$ ranges over a basis of $\mathrm{Hom}_{\mathcal{S}}(i\otimes j,k)$ if a neighbourhood of e looks like[3]

$$\tag{127}$$

Fourthly, for $x\in M_0$ we write E_x for the list of edges incident on x. Then, if for a fixed colouring $\mathcal{I}$ the neighbourhood of $x\in M_0^+$ looks like

$$\tag{128}$$

we have $E_x=(e_1,e_2,e_3,e_4)$, and for fixed $\lambda_{e_1}^{\mathcal{I}},\lambda_{e_2}^{\mathcal{I}},\lambda_{e_3}^{\mathcal{I}},\lambda_{e_4}^{\mathcal{I}}$ we have

$$\mathbb{F}_{\mathcal{C}}(\Gamma_x)\Big(\bigotimes_{e\in E_x}\lambda_e^{\mathcal{I}}\Big) = \quad\text{}\quad . \tag{129}$$

---

[3]When we say that a 2-stratum is labelled with $i\in I$, here and below we mean that we consider the contribution of the i-th copy of $\Bbbk$ in A.

Similarly, for $y \in M_0^-$ we have that $\mathbb{F}_{\mathcal{C}}(\Gamma_y)(\bigotimes_{e \in E_y} \lambda_e^{\mathcal{I}})$ is given by an appropriate evaluation of a functional as in (84), i.e. a diagram of the form

$$\text{(130)}$$

In summary, the invariant $\mathcal{Z}_{\mathcal{A}^{\mathcal{S}}}^{\mathrm{triv}}(M)$ has the form

$$\big(\dim(\mathcal{S})\big)^{-|M_3|} \sum_{M_2} d_{i_1} \ldots d_{i_{|M_2|}} \sum_{M_1} \sum_{\{\lambda\}} \Big(\prod_{M_0^+}\Big) \Big(\prod_{M_0^-}\Big). \tag{131}$$

4.7. PROPOSITION. *We have $\mathcal{Z}_{\mathcal{A}^{\mathcal{S}}}^{triv}(M) = \mathcal{Z}^{TV,\mathcal{S}}(M)$ for all closed 3-manifolds M.*

PROOF. Recall from (90) that $\mathcal{Z}^{\mathrm{TV},\mathcal{S}}(M)$ is given by

$$\mathcal{Z}^{\mathrm{TV},\mathcal{S}}(M) = \big(\dim(\mathcal{S})\big)^{-|P_3|} \sum_{c: P_2 \to I} \Big(\prod_{r \in P_2} d_{c(r)}^{\chi(r)}\Big)\Big(\bigotimes_{e \in P_1} *_e\Big)\Big(\bigotimes_{x \in P_0} \mathbb{F}_{\mathcal{S}}(\Gamma_x)\Big), \tag{132}$$

where we choose the oriented stratified 2-polyhedron P associated to $M^{t,\mathcal{A}^{\mathcal{S}}}$. In this case we have $\chi(r) = 1$ for all $r \in P_2 = M_2$, so what remains to be verified is that for fixed $c: P \to I$ (and $\mathcal{I} \in I^{|M_2|}$) the number

$$\Big(\bigotimes_{e \in P_1} *_e\Big)\Big(\bigotimes_{x \in P_0} \mathbb{F}_{\mathcal{S}}(\Gamma_x)\Big) \tag{133}$$

is indeed the sum over all decorations of the string diagrams as in (129) and (130) corresponding to all vertices of P.

We first note that for a vertex $x \in M_0^+$ and a fixed colouring of the edges of Γ_x, we have

$$\mathbb{F}_{\mathcal{S}}(\Gamma_x) = \sum_{\lambda,\lambda',\mu,\mu'} \quad \cdot \lambda^* \otimes \mu^* \otimes \widehat{\mu'}^* \otimes \widehat{\lambda'}^*, \tag{134}$$

with $\lambda^*, \widehat{\mu'}^*$ defined in (89). If $y \in M_0^-$ is a negatively oriented vertex, there is an analogous expression for $\mathbb{F}_{\mathcal{S}}(\Gamma_y)$. Each basis element $\lambda, \lambda', \mu, \mu'$ above corresponds to one of the edges incident on x. For example, if λ corresponds to an edge e which has x as one endpoint and some vertex $z \in P_0$ as the other endpoint, and if the basis element corresponding to e at z is $\widehat{\kappa}$, then the contraction map $*_e$ of (88) acts as $\lambda \otimes \widehat{\kappa} \mapsto \delta_{\lambda,\kappa}$. Hence for a given $\mathcal{S}$-colouring c, $(\bigotimes_{e \in P_1} *_e)(\bigotimes_{x \in P_0} \mathbb{F}_{\mathcal{S}}(\Gamma_x))$ is the sum, over all elements of a basis which can be inserted at the vertices of all Γ_x, of the product of the respective evaluations of all $\mathbb{F}_{\mathcal{S}}(\Gamma_x)$. Thus we see that indeed $\mathcal{Z}_{\mathcal{A}^{\mathcal{S}}}^{\mathrm{triv}}(M) = \mathcal{Z}^{\mathrm{TV},\mathcal{S}}(M)$. ∎

Let now M be an arbitrary 3-bordism. By comparing $\mathcal{Z}^{\mathrm{triv}}_{\mathcal{A}^{\mathcal{S}}}(M)$ and $\mathcal{Z}^{\mathrm{TV},\mathcal{S}}(M)$ analogously to the above discussion, one finds that the two constructions are identical, except for how they treat 2- and 3-strata of $M^{t,\mathcal{A}^{\mathcal{S}}}$ (or the corresponding 2-polyhedron P) which intersect with the boundary ∂M: while the orbifold construction $\mathcal{Z}^{\mathrm{triv}}_{\mathcal{A}^{\mathcal{S}}}$ treats incoming and outgoing boundaries on an equal footing (leading to factors of $d_i^{1/2}$ and $(\dim \mathcal{S})^{-1/2}$ for 2- and 3-strata, respectively), the construction $\mathcal{Z}^{\mathrm{TV},\mathcal{S}}$ of [TVire] involves contributions only from the incoming boundary (leading to factors of d_i and $(\dim \mathcal{S})^{-1}$).

This mismatch can be formalised in terms of Euler defect TQFTs, see [Qu, CRS1]. Indeed, in the language of [CRS1, Ex. 2.14] the choices for $\mathcal{Z}^{\mathrm{TV},\mathcal{S}}$ favouring the incoming boundary correspond to the choice $\lambda = 1$ for the Euler TQFT, while the choice for $\mathcal{Z}^{\mathrm{triv}}_{\mathcal{A}^{\mathcal{S}}}$ corresponds to $\lambda = \frac{1}{2}$. Since both Euler TQFTs are isomorphic [Qu], Lemma 2.30 and Remark 3.14 of [CRS1] imply that this isomorphism lifts directly to $\mathcal{Z}^{\mathrm{TV},\mathcal{S}}$ and $\mathcal{Z}^{\mathrm{triv}}_{\mathcal{A}^{\mathcal{S}}}$.

To describe the isomorphism in detail, let $M\colon \Sigma' \to \Sigma''$ be as in (95) with embedded graphs Γ', Γ'' on Σ', Σ''. Let t be an oriented triangulation of M extending the duals of the graphs on Σ' and Σ'', and for any surface Σ with embedded graph Γ set $f(\Sigma, \Gamma) := (\dim(\mathcal{S}))^{|\Sigma\backslash\Gamma|/2} \prod_{e\in\Gamma_1} d_{c(e)}^{-1/2}$. Then by construction

$$\mathcal{Z}(M^{t,\mathcal{A}}) = \frac{f(\Sigma',\Gamma')}{f(\Sigma'',\Gamma'')}\, p(\Gamma',\Gamma'')\,, \tag{135}$$

so the factors $f(\Sigma,\Gamma)$ form an isomorphism between the projective system (96) for $\mathcal{Z}^{\mathrm{TV},\mathcal{S}}$ and the corresponding projective system (25) for $\mathcal{Z}^{\mathrm{triv}}_{\mathcal{A}^{\mathcal{S}}}$. Thus we obtain an isomorphism between the corresponding limits, which by (135) is the Σ-component of a natural isomorphism $\mathcal{Z}^{\mathrm{triv}}_{\mathcal{A}^{\mathcal{S}}} \to \mathcal{Z}^{\mathrm{TV},\mathcal{S}}$. Since the map f is multiplicative under disjoint union by definition, the natural isomorphism is also monoidal. We have thus shown:

4.8. THEOREM. *For any spherical fusion category $\mathcal{S}$, there is a monoidal natural isomorphism between the Turaev–Viro TQFT $\mathcal{Z}^{TV,\mathcal{S}}$ and the $\mathcal{A}^{\mathcal{S}}$-orbifold of the trivial 3-dimensional defect TQFT:*

$$\mathcal{Z}^{triv}_{\mathcal{A}^{\mathcal{S}}} \cong \mathcal{Z}^{TV,\mathcal{S}}\,. \tag{136}$$

5. Group extensions of modular tensor categories

In this section we show that for every suitable G-extension of a ribbon fusion category $\mathcal{B}$ there is a corresponding orbifold datum for $\mathcal{B}$ in the sense of Definition 3.6. One type of such extensions are G-extensions of modular tensor categories $\mathcal{C}$. Another interesting situation is when we have a ribbon functor $F\colon \mathcal{B} \to \mathcal{C}$ and a G-crossed extension of $\mathcal{B}$, as this gives orbifold data in $\mathcal{C}$ (by Proposition 3.7). We consider examples of this where F is the embedding of a subcategory of $\mathcal{C}$.

In fact our second main result, Theorem 5.1 (which is Theorem B in the introduction), also holds for certain non-fusion (e. g. non-semisimple) ribbon categories, see Remark 5.6.

We fix a finite group G, a ribbon fusion category $\mathcal{B}$, and a *ribbon crossed G-category* $\widehat{\mathcal{B}} = \bigoplus_{g\in G} \mathcal{B}_g$ such that $\mathcal{B} = \mathcal{B}_1$ and $\mathcal{B}_g \neq 0$ for all $g \in G$. Roughly, this means that

the tensor product of $\widehat{\mathcal{B}}$ is compatible with the G-grading, there is a monoidal functor $\varphi\colon \underline{G} \to \mathrm{Aut}_\otimes(\widehat{\mathcal{B}})$ (where $\underline{G}$ is G viewed as a discrete monoidal category), and the twist and braiding of $\widehat{\mathcal{B}}$ are "twisted" by the G-action φ. For details we refer to [Tu2, Sect. VI.2] from which we deviate in that for us G acts from the right, i.e.

$$\varphi(g)(\mathcal{B}_h) \subset \mathcal{B}_{g^{-1}hg} \quad \text{for all } g,h \in G, \tag{137}$$

and the G-twisted braiding has components

$$c_{X,Y} \equiv \quad : X \otimes Y \xrightarrow{\cong} Y \otimes \varphi(h)(X) \quad \text{if } Y \in \mathcal{B}_h,$$

$$\tilde{c}_{Y,X} \equiv \quad : Y \otimes X \xrightarrow{\cong} \varphi(h^{-1})(X) \otimes Y \quad \text{if } Y \in \mathcal{B}_h. \tag{138}$$

Here we wrote $\tilde{c}$ for the braiding describing the opposite crossing. Up to coherence isomorphism from the group action, the inverse $c_{X,Y}^{-1}$ of the braiding is given by $\tilde{c}_{Y,\varphi(h)(X)}$.

For every $g \in G$, we now choose a simple object $m_g \in \mathcal{B}_g$ such that $m_1 = \mathbb{1}$, and we set

$$d_{m_g} := \dim(m_g) \in \Bbbk^\times \quad \text{for all } g \in G. \tag{139}$$

We furthermore pick a square root $d_{m_g}^{1/2}$. It is straightforward to verify that

$$A_g := m_g^* \otimes m_g, \tag{140}$$

is a Δ-separable symmetric Frobenius algebra in $\mathcal{B}$ for all $g \in G$. Moreover we have A_{gh}-$(A_g \otimes A_h)$-bimodules $T_{g,h} \in \mathcal{B}$ given by

$$T_{g,h} := m_{gh}^* \otimes m_g \otimes m_h \quad \text{with actions}$$

$$\text{(diagram)} \qquad \overset{\text{def}}{=} \qquad \text{(diagram)} \qquad . \tag{141}$$

with labels $T_{g,h}\ A_g$ on the left and $m_{gh}^*\ m_g\ m_h\quad m_g^*\ m_g$ on the right.

Above we use string diagram notation for morphisms in the G-crossed category $\widehat{\mathcal{B}}$. By (138), the object label attached to a string changes at crossings. For example, the first crossing in the action of A_g is the inverse braiding $m_g \otimes m_h \to \varphi(g^{-1})(m_h) \otimes m_g$, and the second crossing is the braiding $\varphi(g^{-1})(m_h) \otimes m_g \to m_g \otimes m_h$ (composed with a coherence isomorphism for φ). Thus the string labelled m_h at the bottom is labelled by $\varphi(g^{-1})(m_h) \in \mathcal{B}_{ghg^{-1}}$ between the crossings and again by m_h at the top.

One checks that indeed (cf. (33))

$$\text{(diagram)} \quad = \quad \text{(diagram)} \quad . \tag{142}$$

with labels $T_{g,h}\ A_g\ A_h$ below each diagram.

Setting $A := \bigoplus_{g \in G} A_g$, it follows that $T := \bigoplus_{g,h \in G} T_{g,h}$ is an A-$(A \otimes A)$-bimodule.

Now we define component maps

$$\alpha_{g,h,k} \colon T_{g,hk} \otimes T_{h,k} \longrightarrow T_{gh,k} \otimes T_{g,h} , \quad \bar{\alpha}_{g,h,k} \colon T_{gh,k} \otimes T_{g,h} \longrightarrow T_{g,hk} \otimes T_{h,k} \tag{143}$$

by

$$\alpha_{g,h,k} \overset{\text{def}}{=} \text{(diagram)} , \quad \bar{\alpha}_{g,h,k} \overset{\text{def}}{=} \text{(diagram)} . \tag{144}$$

Here and below we use the following shorthand notation in labelling string diagrams. A label g on a string indicates that its source and target object is m_g (or m_g^*, depending on orientation). We stress that this is independent of the position of the label g along the string. For example, passing along the string labelled k in the diagram for $\bar{\alpha}_{g,h,k}$, the components of the string in the complement of the crossings should be labelled by the objects m_k, $\varphi(h^{-1}g^{-1})(m_k)$, $\varphi(h^{-1})(m_k)$ and m_k, in this order.

The components $\alpha_{g,h,k}$ and $\bar{\alpha}_{g,h,k}$ assemble into module maps $\alpha := \sum_{g,h,k \in G} \alpha_{g,h,k}$ and $\bar{\alpha} := \sum_{g,h,k \in G} \bar{\alpha}_{g,h,k}$, as can be checked by verifying identities as in (36)–(37).

Finally we define $\psi \in \text{End}_{AA}(A)$ and $\phi \in \text{End}(\mathbb{1}) = \Bbbk$ by

$$\psi\big|_{A_g} \overset{\text{def}}{=} d_{m_g}^{-1/2} \cdot \text{id}_{A_g} , \quad \phi^2 := \frac{1}{|G|} . \tag{145}$$

5.1. THEOREM. *Let $\mathcal{B} = \mathcal{B}_1$ be the neutral component of a ribbon crossed G-category $\widehat{\mathcal{B}}$ as above. Then for every choice of simple objects $\{m_g \in \mathcal{B}_g\}_{g \in G}$ the tuple $\mathcal{A}^m := (A, T, \alpha, \bar{\alpha}, \psi, \phi)$ defined in (140)–(145) is a special orbifold datum for $\mathcal{B}$.*

PROOF. We have to show that $\mathcal{A}^m$ satisfies the conditions (41)–(45). Once the latter are written out in terms of the algebra actions (141), the component maps (144) and the definition (145) of ψ and ϕ, the verification becomes a straightforward exercise in the graphical calculus for ribbon crossed G-categories. Here we provide details for only two conditions; the remainder is checked analogously.

One of the more involved conditions is the second identity of (43). In components, its right-hand side is

while for the left-hand side we compute:

By the ribbon property, this expression is indeed equal to (146).

We also show that the left- and right-hand side of (45) agree:

5.2. REMARK. Note that in $\widehat{\mathcal{B}}$, all the A_g, $g \in G$, are Morita equivalent to the algebra $\mathbb{1}$. In $\mathcal{B}_1$ typically only A_1 is Morita equivalent to $\mathbb{1}$, but the bimodules $T_{g,h}$ still exhibit A_{gh} as Morita equivalent to $A_g \otimes A_h$ in $\mathcal{B}_1$. Thus $A_g \otimes A_{g^{-1}}$ is Morita equivalent to $\mathbb{1}$ in $\mathcal{B}_1$, that is, all A_g are necessarily Azumaya algebras in $\mathcal{B}_1$, see [VOZ] and e. g. [FRS1, Sect. 10]. Commutative separable algebras in braided tensor categories are Azumaya iff they are isomorphic to the tensor unit [VOZ, Thm. 4.9], and so in this sense the construction in Section 3.18 is complementary to the one described here.

It is natural to ask to what extent Theorem 5.1 depends on the choice of simple objects $m_g \in \mathcal{B}_g$. To answer this question, let $\{\widetilde{m}_g \in \mathcal{B}_g\}_{g \in G}$ be another choice of simple objects. Hence by setting

$$\widetilde{A}_g := \widetilde{m}_g^* \otimes \widetilde{m}_g, \quad \widetilde{A} := \bigoplus_{g \in G} \widetilde{A}_g \tag{149}$$

and similarly defining $\widetilde{T}, \widetilde{\alpha}, \widetilde{\widetilde{\alpha}}, \widetilde{\psi}$ in terms of $\{\widetilde{m}_g\}$ along the lines of (141)–(145), we have a second special orbifold datum

$$\mathcal{A}^{\widetilde{m}} := \left(\widetilde{A}, \widetilde{T}, \widetilde{\alpha}, \widetilde{\widetilde{\alpha}}, \widetilde{\psi}, \phi \right) \tag{150}$$

for $\mathcal{B}$.

To relate $\mathcal{A}^{\widetilde{m}}$ to $\mathcal{A}^m$, first note that $\widetilde{A}_g$ and A_g are Morita equivalent (in $\mathcal{B} = \mathcal{B}_1$) for all $g \in G$: for $X_g := \widetilde{m}_g^* \otimes m_g$ we have $X_g^* \otimes_{\widetilde{A}_g} X_g \cong A_g$ and $X_g \otimes_{A_g} X_g^* \cong \widetilde{A}_g$. Hence

$$X := \bigoplus_{g \in G} X_g \tag{151}$$

is an $\widetilde{A}$-A-bimodule exhibiting a Morita equivalence between A and $\widetilde{A}$, and Definition 3.10 and Proposition 3.11 give us another special orbifold datum for $\mathcal{B}$, the Morita transport of $\mathcal{A}^m$ along X:

$$X(\mathcal{A}^m) = \left(\widetilde{A}, T^X, \alpha^X, \bar{\alpha}^X, \psi^X, \phi \right). \tag{152}$$

5.3. LEMMA. *There is a T-compatible isomorphism from $X(\mathcal{A}^m)$ to $\mathcal{A}^{\widetilde{m}}$.*

PROOF. The T-compatible isomorphism $\rho: T^X \to \widetilde{T}$ can be assembled from component maps $\rho_{g,h}: (T^X)_{g,h} = X_{gh}^* \otimes_{A_{gh}} T_{g,h} \otimes_{A_g \otimes A_h} (X_g \otimes X_h) \to \widetilde{T}_{g,h}$ obtained via the universal property from the map $X_{gh}^* \otimes T_{g,h} \otimes X_g \otimes X_h \to \widetilde{T}_{g,h}$ given by

$$\frac{1}{\sqrt{d_{m_{gh}} d_{m_g} d_{m_h}}} \cdot \mathrm{id}_{\widetilde{m}_{gh}} \otimes \mathrm{ev}_{m_{gh}} \otimes \mathrm{ev}_{m_g} \otimes \mathrm{id}_{\widetilde{m}_g} \otimes \mathrm{ev}_{m_h} \otimes \mathrm{id}_{\widetilde{m}_h} \tag{153}$$

for all $g, h \in G$. Checking the defining condition (64) of T-compatibility for (the components of) ρ is a straightforward string-diagram manipulation of the same type as in the proof of Theorem 5.1. ∎

As one would expect, if one maps $\mathcal{B}$ into a modular tensor category via a ribbon functor, all the above orbifold data lead to isomorphic orbifold TQFTs, and in this sense the construction does not depend on the choice of simple objects:

5.4. COROLLARY. *Let $\mathcal{B}, \mathcal{C}$ be ribbon fusion categories, and let $\mathcal{C}$ be modular. Let $F \colon \mathcal{B} \to \mathcal{C}$ be a ribbon functor and let $\{m_g \in \mathcal{B}_g\}_{g \in G}$ and $\{\widetilde{m}_g \in \mathcal{B}_g\}_{g \in G}$ be two choices of simple objects as above. Then*

$$(\mathcal{Z}^{\mathcal{C}})_{F(\mathcal{A}^m)} \cong (\mathcal{Z}^{\mathcal{C}})_{F(X(\mathcal{A}^m))} \cong (\mathcal{Z}^{\mathcal{C}})_{F(\mathcal{A}^{\widetilde{m}})} . \tag{154}$$

PROOF. The first isomorphism in (154) is Corollary 3.17. The second isomorphism follows from Lemma 3.16 and Lemma 5.3, and from the fact that F maps T-compatible isomorphisms to T-compatible isomorphisms. ∎

5.5. REMARK. Recall the notions of G-crossed extension and G-equivariantisation, e.g. from [ENOM, Tu2]. By Theorem 5.1 every G-crossed extension $\mathcal{C}_G^\times$ of a modular tensor category $\mathcal{C}$ gives rise to an orbifold datum for $\mathcal{Z}^{\mathcal{C}}$. We expect that the associated orbifold TQFT is isomorphic to the Reshetikhin–Turaev theory $\mathcal{Z}^{\mathrm{RT}, (\mathcal{C}_G^\times)^G}$ corresponding to the G-equivariantisation $(\mathcal{C}_G^\times)^G$ of $\mathcal{C}_G^\times$; this is in line with previous work on gauging global symmetry groups [BBCW, CGPW] and on geometric group orbifolds of 3-2-1-extended TQFTs [SW].

5.6. REMARK. There is a generalisation of Theorem 5.1 which does not need the strong assumptions of semisimplicity and finiteness inherent to fusion categories: Suppose that $\widehat{\mathcal{B}}$ is a $\Bbbk$-linear ribbon crossed G-category such that in every graded component $\mathcal{B}_g$ there exists a simple object m_g with invertible quantum dimension $d_{m_g} \in \mathrm{End}(\mathbb{1})$ which in turn has a square root. Then the proof of Theorem 5.1 still goes through to show that $\mathcal{A}^m = (A, T, \alpha, \bar{\alpha}, \psi, \phi)$ defined as in (140)–(145) is a special orbifold datum for $\mathcal{B} = \mathcal{B}_1$.

5.7. EXAMPLE. As a class of concrete examples of G-crossed extensions and their associated orbifold data, we consider *Tambara–Yamagami categories* $\mathcal{TY}_{H,\chi,\tau}$. Recall that Tambara and Yamagami [TY] classified $\mathbb{Z}_2$-extensions of pointed categories, i.e. of fusion categories where all objects are invertible. Such extensions are constructed from tuples (H, χ, τ), where H is a finite abelian group, $\chi \colon H \times H \to \Bbbk^\times$ is a nondegenerate symmetric bicharacter, and $\tau \in \Bbbk$ is a square root of $|H|^{-1}$. Writing $\mathbb{Z}_2 = \{\pm 1\}$, the graded components of $\mathcal{TY}_{H,\chi,\tau}$ are the category of H-graded vector spaces and vector spaces, respectively: $(\mathcal{TY}_{H,\chi,\tau})_{+1} = \mathrm{vect}_H$ and $(\mathcal{TY}_{H,\chi,\tau})_{-1} = \mathrm{vect}$. The fusion rules for the $+1$-component are as in vect_H, the single simple object in the -1-component is noninvertible (unless $|H| = 1$), and the category $\mathcal{TY}_{H,\chi,\tau}$ has a canonical spherical structure such that the quantum dimensions of all objects are positive, see e.g. [GNN] for details.

  (i) Consider the case of a Tambara–Yamagami category where the bicharacter χ comes from a quadratic form $q \colon H \to \Bbbk^\times$, i.e. it satisfies $\chi(a, b) = \frac{q(a \cdot b)}{q(a)q(b)}$ for all $a, b \in H$. Then the category $\mathcal{TY}_{H,\chi,\tau}$ is a $\mathbb{Z}_2$-crossed extension of $\mathrm{vect}_{H,\chi}$ with the braiding

on simple objects $a, b \in H$ given by (where we use $a \otimes b = ab = ba = b \otimes a$ in vect_H)

$$c_{a,b} = \chi(a, b) \cdot \mathrm{id}_{a \cdot b} : a \otimes b \longrightarrow b \otimes a, \tag{155}$$

see [GNN, Prop. 5.1]. From Theorem 5.1 we obtain orbifold data $\mathcal{A}_\tau$ in $(\mathcal{TY}_{H,\chi,\tau})_{+1} = \mathrm{vect}_{H,\chi}$ for each choice of square root τ of $|H|^{-1}$. Following the construction in the proof, we see that the algebra in the orbifold datum is $A = \mathbb{1} \oplus A_H$, where $A_H := \bigoplus_{h \in H} h$ corresponds to the nontrivial element $-1 \in \mathbb{Z}_2$. By inspecting the fusion rules of $\mathcal{TY}_{H,\chi,\tau}$, one finds that the bimodules $T_{g,h}$ are given by $T_{1,1} = \mathbb{1}$ for $1 \in \mathbb{Z}_2$ and $T_{g,h} = A_H$ in all other cases.

(ii) For $H = \mathbb{Z}_2$ the corresponding Tambara–Yamagami categories reduce to the familiar Ising categories [EGNO]. Consider, for example, the quadratic form q such that $q(+1) = 1$ and $q(-1) = \mathrm{i}$, and $\tau = \pm\frac{1}{\sqrt{2}}$. Then $\mathcal{TY}_{\mathbb{Z}_2,\chi,\tau}$ are $\mathbb{Z}_2$-extensions of $\mathrm{vect}_{\mathbb{Z}_2,\chi}$ with a symmetric braiding coming from χ: The simple object in degree -1 has a self-braiding which is -1 times the identity. As in the general case in part (i) above we obtain orbifold data $\mathcal{A}_\tau$ in the ribbon category $\mathrm{vect}_{\mathbb{Z}_2,\chi}$ for both choices of τ.

We end with an example which relates the orbifold data of Example 5.7(ii) to our constructions in Section 3:

5.8. EXAMPLE. Let $\mathcal{C}_k$ be the modular tensor category associated to the affine Lie algebra $\widehat{\mathfrak{sl}}(2)_k$ at a positive integer level k. The category $\mathcal{C}_k$ has $k + 1$ simple objects $U_0, U_1, \ldots, U_k$, all of which are self-dual. The object U_k is invertible and has ribbon-twist $\theta_{U_k} = \mathrm{i}^k \cdot \mathrm{id}_{U_k}$. The simple Δ-separable symmetric Frobenius algebras in $\mathcal{C}_k$ are known up to Morita equivalence from the classification of $\mathcal{C}_k$-module categories [Os] and follow an ADE pattern. Depending on the level k, there are one, two or three such Morita classes:

- all k (case A): For every value of k one has the Morita class $[A_\mathrm{A}]$ of the simple Δ-separable symmetric Frobenius algebra $A_\mathrm{A} := \mathbb{1}$. For $k = 1 \bmod 2$ this is furthermore the only such Morita class, so these values of k do not provide interesting examples of the constructions in Section 3.18 or in this section.

- $k = 0 \bmod 4$ (case D_even): There is an up-to-isomorphism unique structure of a Δ-separable symmetric Frobenius algebra on $A_\mathrm{D} := U_0 \oplus U_k$. Its Morita class $[A_\mathrm{D}]$ is different from $[A_\mathrm{A}]$. The algebra A_D is commutative and one can thus apply the construction in Section 3.18. The algebra A_D is not Azumaya (and hence no algebra in $[A_\mathrm{D}]$ is), and so it cannot appear as part of a G-extension as discussed in this section.

- $k = 2 \bmod 4$ (case D_odd): As above, $A_\mathrm{D} := U_0 \oplus U_k$ is a Δ-separable symmetric Frobenius algebra in an up-to-isomorphism unique way, and its Morita class $[A_\mathrm{D}]$ is distinct from $[A_\mathrm{A}]$. But this time, A_D is noncommutative and in fact Azumaya. The full subcategory spanned by U_0 and U_k is ribbon-equivalent to $\mathrm{vect}_{\mathbb{Z}_2,\chi}$ with the

symmetric braiding from χ as in (ii) above. Put differently, there is a fully faithful ribbon functor $F\colon \mathrm{vect}_{\mathbb{Z}_2,\chi} \to \mathcal{C}_k$. The two orbifold data $\mathcal{A}_\tau$ for $\tau = \pm\frac{1}{\sqrt{2}}$ in $\mathrm{vect}_{\mathbb{Z}_2,\chi}$ give by Proposition 3.7 orbifold data $F(\mathcal{A}_\tau)$ in $\mathcal{C}_k$.

- $k \in \{10, 28\}$ (cases E_6, E_8): There are commutative simple Δ-separable symmetric Frobenius algebras A_{E_6}, A_{E_8}, which provide a third Morita class $[A_{\mathrm{E}_6}]$, resp. $[A_{\mathrm{E}_8}]$, in addition to $[A_\mathrm{A}]$ and $[A_\mathrm{D}]$ at these levels. The corresponding categories of local modules are equivalent to the modular tensor categories obtained from $\widehat{\mathfrak{sp}}(4)_1$ and $\widehat{\mathfrak{g}}(2)_1$, respectively, see [Os]. The construction in Section 3.18 applies, and as mentioned there, we expect the orbifolds corresponding to A_{E_6} and A_{E_8} to be equivalent to the Reshetikhin–Turaev TQFTs obtained from these two modular tensor categories.

- $k = 16$ (case E_7): There is a simple Δ-separable symmetric Frobenius algebra A_{E_7} which generates a third Morita class in addition to $[A_\mathrm{A}]$ and $[A_\mathrm{D}]$ at this level. The Morita class $[A_{E_7}]$ does not contain a commutative representative, and the algebra A_{E_7} is not Azumaya. We do not know if A_{E_7} can form part of an orbifold datum.

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