FIBRATIONS OF AU-CONTEXTS BEGET FIBRATIONS OF TOPOSES

SINA HAZRATPOUR AND STEVEN VICKERS

ABSTRACT. Suppose an extension map $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ in the 2-category $\mathfrak{Con}$ of contexts for arithmetic universes satisfies a Chevalley criterion for being an (op)fibration in $\mathfrak{Con}$. If M is a model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$ with nno, then the classifier $p\colon \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ satisfies the representable definition of being an (op)fibration in the 2-category $\mathcal{E}\mathfrak{Top}$ of elementary toposes (with nno) and geometric morphisms.

Contents

1. Introduction

For many special constructions of topological spaces (which for us will be point-free, and generalized in the sense of Grothendieck), a structure-preserving morphism between the presenting structures gives a map between the corresponding spaces. Two very simple examples are: a function $f\colon X \to Y$ between sets already is a map between the corresponding discrete spaces; and a homomorphism $f\colon K \to L$ between two distributive lattices gives a map *in the opposite direction* between their spectra. The covariance or contravariance of this correspondence is a fundamental property of the construction.

In topos theory we can relativize this process. A presenting structure in an elementary topos $\mathcal{E}$ will give rise to a bounded geometric morphism $p\colon \mathcal{F} \to \mathcal{E}$, where $\mathcal{F}$ is the topos

of sheaves over $\mathcal{E}$ for the space presented by the structure. Then we commonly find that the covariant or contravariant correspondence mentioned above makes every such p an opfibration or fibration in the 2-category of toposes and geometric morphisms.

If toposes are taken as bounded over some fixed base $\mathcal{S}$, in the 2-category $\mathcal{B}\mathfrak{Top}/\mathcal{S}$, then there are often easy proofs using the Chevalley criterion to show that the generic such p, taken over the classifying topos for the relevant presenting structures, is an (op)fibration. See [SVW12] for some simple examples of the idea, though there are still questions of strictness left unanswered there.

However, often there is no natural choice of base topos $\mathcal{S}$, and Johnstone [Joh02, B4.4] proves (op)fibrational results in $\mathcal{B}\mathfrak{Top}$. These are harder both to state (the Chevalley criterion is not available) and to prove, but stronger because slicing over $\mathcal{S}$ restricts the 2-cells.

In this paper we show how to use the arithmetic universe (AU) techniques of [Vic17] to get simple proofs using the Chevalley criterion of the stronger, base-independent (op)fibration results in $\mathcal{E}\mathfrak{Top}$, the 2-category of elementary toposes with nno, and arbitrary geometric morphisms.

Our starting point is the following construction in [Vic17], using the 2-category $\mathfrak{Con}$ of AU-contexts in [Vic19]. Suppose $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ is an extension map in $\mathfrak{Con}$, and M is a model of $\mathbb{T}_0$ in $\mathcal{S}$, an elementary topos with nno. Then there is a geometric theory $\mathbb{T}_1/M$, of models of $\mathbb{T}_1$ whose $\mathbb{T}_0$-reduct is M, and so we get a classifying topos $p\colon \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$. As a generalized space (relative to base $\mathcal{S}$), we view it as the fibre of U over M. Our main result (Theorem 8.2) is that –

if U is an (op)fibration in $\mathfrak{Con}$, using the Chevalley criterion,
then p is an (op)fibration is $\mathcal{E}\mathfrak{Top}$, using the representable definition.

Throughout, we assume that *all our elementary toposes are equipped with natural numbers object (nno)*. Without an nno the ideas of generalized space do not go far (because it is needed in order to get an object classifier), and AU techniques don't apply.

2. Overview

In §3 we review relevant 2-categories of toposes, including our new 2-category $\mathcal{G}\mathfrak{Top}$ in which the objects are bounded geometric morphisms.

In §4 we quickly review the main aspects of the theory of AU-contexts, our AU analogue of geometric theories in which the need for infinitary disjunctions in many situations has been satisfied by a type-theoretic style of sort constructions that include list objects (and an nno). The contexts are "sketches for arithmetic universes" [Vic19], and we review the principal syntactic constructions on them that are used for continuous maps and 2-cells.

§5 reviews the connection between contexts and toposes as developed in [Vic17], along with some new results. A central construction shows how context extension maps $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ can be treated as *bundles* of generalized spaces: if M is a point of $\mathbb{T}_0$ (a model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$), then the fibre of U over M, as a generalized space over $\mathcal{S}$, is

a bounded geometric morphism $p\colon \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ that classifies the models of $\mathbb{T}_1$ whose U-reduct is M. Much of the discussion is about understanding the universal property of such a classifier in the setting of $\mathcal{G}\mathfrak{Top}$.

We then move in §6 and §7 to a review of two styles of definition for (op)fibrations in 2-categories, the *Chevalley* and *representable* criteria.

A functor $p\colon E \to B$ is, trivially, a fibration iff the functor $\mathfrak{Cat}(1,p)\colon \mathfrak{Cat}(1,E) \to \mathfrak{Cat}(1,B)$ is, and an obvious generalization is to replace $\mathfrak{Cat}$ by some other 2-category $\mathcal{K}$ to obtain a notion of fibration in $\mathcal{K}$. However, even when $\mathcal{K}$ has a terminal object, there may fail to be enough 1-cells from 1 to B to make a satisfactory definition this way. This is generally the case with 2-categories of toposes such as $\mathcal{E}\mathfrak{Top}$.

One therefore uses a more general *representable* definition, that for every A the functor $\mathcal{K}(A,p)\colon \mathcal{K}(A,E) \to \mathcal{K}(A,B)$ is a fibration, with the additional condition that the naturality squares are fibred functors.

If $\mathcal{K}$ has pullbacks of p, then these can be considered the fibres of p. Suppose we have $\alpha\colon g \to f$ between B' and B. Then by the representable definition $\alpha \centerdot f^*p$ has a cartesian lift $\alpha'\colon g' \to p^*f$:

$$
\begin{array}{c}
\xymatrix{
}
\end{array}
\qquad\qquad (1)
$$

g' now gives us a morphism from f^*E to g^*E, in other words a morphism between the fibres over f and g but in the opposite direction to that of α. This brings us closer to the "indexed category" view of fibrations, with 2-cells between base points (f and g) lifting to maps between the fibres (f^*E and g^*E), and to the examples mentioned at the outset.

Now suppose $\mathcal{K}$ has comma objects, which unfortunately $\mathcal{E}\mathfrak{Top}$ and $\mathcal{B}\mathfrak{Top}$ do not, so far as we know, although $\mathcal{B}\mathfrak{Top}/\mathcal{S}$ and our $\mathfrak{Con}$ do. Then we may capture the data of the above diagram in a generic way by taking B' to be the cotensor $\mathbf{2} \pitchfork B$ of B with the walking arrow $\mathbf{2}$. In such a $\mathcal{K}$, the fibration structure for arbitrary B' and α can be derived from generic structure for the generic α. The structure needs to be defined just once, instead of many times for all B'. We shall call this a *Chevalley criterion*. For ordinary fibrations the idea was attributed to Chevalley by Gray [Gray66], and subsequently referred to as the Chevalley criterion by Street [Str74].

As discussed in [Joh93], a shortcoming with all this is that examples such as $\mathcal{E}\mathfrak{Top}$ are really bicategories, and this calls for a relaxed notion of fibration for the functors $\mathcal{K}(A,p)$. We return to this in §7. Our main task in §6 is to clarify the 2-categorical structure needed, and the strictness issues, when we apply the Chevalley criterion in $\mathfrak{Con}$.

§8 then provides the main result, Theorem 8.2. Suppose $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ is a context extension map, and $p\colon \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ is a classifier constructed as in §5. Then if U is an (op)fibration, so is p.

3. Background: 2-categories of toposes

The setting for our main result is the 2-category $\mathcal{ETop}$ whose 0-cells are elementary toposes (equipped with nno), whose 1-cells are geometric morphisms, and whose 2-cells are geometric transformations.

However, our concern with generalized spaces means that we must also take care to deal with *bounded* geometric morphisms. Recall that a geometric morphism $p\colon \mathcal{E} \to \mathcal{S}$ is *bounded* whenever there exists an object $B \in \mathcal{E}$ (a *bound* for p) such that every $A \in \mathcal{E}$ is a subquotient of an object of the form $(p^*I) \times B$ for some $I \in \mathcal{S}$: that is one can form following span in $\mathcal{E}$, with the left leg a mono and the right leg an epi.

$$
\begin{array}{ccc}
 & E & \\
 & & \\
(p^*I) \times B & & A
\end{array}
$$
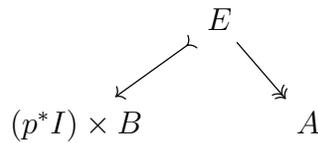
The significance of this notion can be seen in the relativized version of Giraud's Theorem (see [Joh02, B3.4.4]): p is bounded if and only if $\mathcal{E}$ is equivalent to the topos of sheaves over an internal site in $\mathcal{S}$. (In the original Giraud Theorem, relative to **Set**, the bound relates to the small set of generators.) It follows from this that the bounded geometric morphisms into $\mathcal{S}$ can be understood as the generalized spaces, the Grothendieck toposes, relative to $\mathcal{S}$.

Bounded geometric morphisms are closed under isomorphism and composition (see [Joh02, B3.1.10(i)]) and we get a 2-category $\mathcal{BTop}$ of elementary toposes, bounded geometric morphisms, and geometric transformations. It is a sub-2-category of $\mathcal{ETop}$, full on 2-cells.

Also [Joh02, B3.1.10(ii)], if a bounded geometric morphism q is isomorphic to pf, where p is also bounded, then so too is f. This means that if we are only interested in toposes bounded over $\mathcal{S}$, then we do not have to consider unbounded geometric morphisms between them. We can therefore take the "2-category of generalized spaces over $\mathcal{S}$" to be the slice 2-category $\mathcal{BTop}/\mathcal{S}$, where the 1-cells are triangles commuting up to an iso-2-cell. [Joh02, B4] examines $\mathcal{BTop}/\mathcal{S}$ in detail.

For the (op)fibrational results, [Joh02, B4] reverts to $\mathcal{BTop}$. This is appropriate, since the properties hold with respect to arbitrary geometric transformations, whereas working in $\mathcal{BTop}/\mathcal{S}$ limits the discussion to those that are isomorphisms over $\mathcal{S}$.

Unbounded geometric morphisms are rarely encountered in practice, and so it might appear reasonable to stay in $\mathcal{BTop}$ or $\mathcal{BTop}/\mathcal{S}$ [Joh02, B3.1.14]. However, one notable property of bounded geometric morphisms is that their bipullbacks along *arbitrary* geometric morphisms exist in $\mathcal{ETop}$ and are still bounded [Joh02, B3.3.6]. (Note that where [Joh02] says pullback in a 2-category, it actually means bipullback – this is explained there in section B1.1.) Thus for any geometric morphism of base toposes $f\colon \mathcal{S}' \to \mathcal{S}$, we have the change of base pseudo-2-functor $f\colon \mathcal{BTop}/\mathcal{S} \to \mathcal{BTop}/\mathcal{S}'$. One might say the "2-category of Grothendieck toposes" is indexed over $\mathcal{ETop}_\cong$ (where the 2-cells in

$\mathcal{ETop}_\cong$ are restricted to isos). [Vic17] develops this in its use of AU techniques to obtain base-independent topos results, and there is little additional effort in allowing change of base along arbitrary geometric morphisms. To avoid confronting the coherence issues of indexed 2-categories it takes a fibrational approach, with a 2-category $\mathcal{GTop}$ "of Grothendieck toposes" fibred (in a bicategorical sense) over $\mathcal{ETop}_\cong$.

We shall take a similar approach, but note that our 2-category $\mathcal{GTop}$, which we are about to define, is *not* the same as that of [Vic17] – we allow arbitrary geometric transformations "downstairs". We shall write $\mathcal{GTop}_\cong$ when we wish to refer to the $\mathcal{GTop}$ of [Vic17].

3.1. DEFINITION. *The 2-category $\mathcal{GTop}$ is defined as follows. We use a systematic "upstairs-downstairs" notation with overbars and underbars to help navigate diagrams.*
- *0-cells are bounded geometric morphisms $x\colon \overline{x} \to \underline{x}$.*
- *For any 0-cells x and y, the 1-cells from y to x are given by $f = \langle \overline{f}, \overset{\blacktriangledown}{f}, \underline{f}\rangle$ where $\underline{f}\colon \underline{y} \to \underline{x}$ and $\overline{f}\colon \overline{y} \to \overline{x}$ are geometric morphisms, and the geometric transformation $\overset{\blacktriangledown}{f}\colon x\overline{f} \Rightarrow \underline{f}y$ is an isomorphism.*
- *If f and g are 1-cells from y to x, then 2-cells from f to g are of the form $\alpha = \langle \overline{\alpha}, \underline{\alpha}\rangle$ where $\overline{\alpha}\colon \overline{f} \Rightarrow \overline{g}$ and $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$ are geometric transformations so that the obvious diagram of 2-cells commutes.*
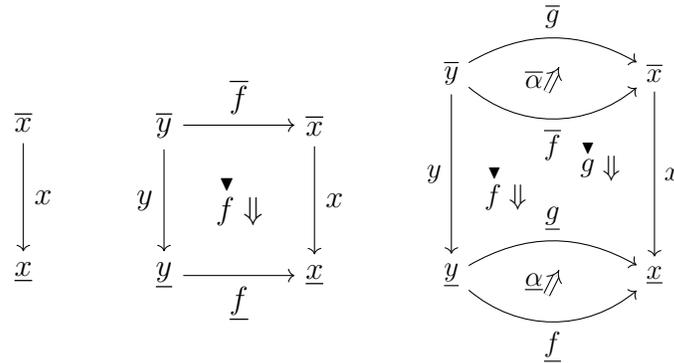
Composition of 1-cells $k\colon z \to y$ and $f\colon y \to x$ is given by pasting them together, more explicitly it is given by $fk := \langle \overline{f} \circ \overline{k}, \overset{\blacktriangledown}{f} \odot \overset{\blacktriangledown}{k}, \underline{f} \circ \underline{k}\rangle$ where $\overset{\blacktriangledown}{f} \odot \overset{\blacktriangledown}{k} := (\underline{f} \centerdot \overset{\blacktriangledown}{k}) \circ (\overset{\blacktriangledown}{f} \centerdot \overline{k})$. Vertical composition of 2-cells consists of vertical composition of upper and lower 2-cells. Similarly, horizontal composition of 2-cells consists of horizontal composition of upper and lower 2-cells. Identity 1-cells and 2-cells are defined trivially.

This is a particular case of our more general Construction 7.4. $\mathcal{GTop}$ is $\mathcal{K}_\mathcal{D}$ when $\mathcal{K}$ is $\mathcal{ETop}$ and $\mathcal{D}$ is the class of bounded geometric morphisms.

Much of our development will turn on the codomain 2-functor

$$\mathbb{Cod}\colon \mathcal{GTop} \to \mathcal{ETop}.$$

It is important to note that this codomain functor is *not* a fibration in any 2-categorical sense, as it is not well behaved with respect to arbitrary 2-cells in $\mathcal{ETop}$. This is easy to see if one takes the point of view of indexed 2-categories, and considers the corresponding

change-of-base functors. (It becomes a fibration if one restricts the downstairs 2-cells to be isos, as in [Vic17].) However, it will still be interesting to consider its cartesian 1-cells and 2-cells, which we do in section 7.

4. Background: The 2-category **$\mathfrak{Con}$** of contexts

The observation underlying [Vic19] is that important geometric theories can be expressed in coherent logic (no infinite disjunctions), provided that new sorts can be constructed in a type-theoretic style that includes free algebra constructions. Models can then be sought in any arithmetic universe (list-arithmetic pretopos), and that includes any elementary topos with nno; moreover, the inverse image functors of geometric morphisms are AU-functors.

In the following table we illustrates some of the differences between the AU approach and toposes. More details about the expressive power of AUs can be found in [MV12].

	AUs	Grothendieck toposes
Classifying category	$\mathbf{AU}\langle\mathbb{T}\rangle$	$\mathcal{S}[\mathbb{T}]$
$\mathbb{T}_1 \to \mathbb{T}_2$	$\mathbf{AU}\langle\mathbb{T}_2\rangle \to \mathbf{AU}\langle\mathbb{T}_1\rangle$	$\mathcal{S}[\mathbb{T}_1] \to \mathcal{S}[\mathbb{T}_2]$
Base	Base independent	Base $\mathcal{S}$
Infinities	Intrinsic; provided by List e.g. $N = \mathsf{List}(1)$	Extrinsic; provided by $\mathcal{S}$ e.g. infinite coproducts
Results	A single result in AUs	A family of results by varying $\mathcal{S}$

The system developed in [Vic19] expresses those geometric theories using *sketches*. They are, first of all, finite-limit-finite-colimit sketches. Each has an underlying directed graph of *nodes* and *edges*, reflexive to show the identity $s(X)$ for each node X, and with some triangles specified as commutative. On top of that, certain nodes are specified as being terminal or initial, and certain cones and cocones are specified as being for pullbacks or pushouts. In addition, there is a new notion of *list universal* to specify parameterized list objects, together with their empty list and cons operations. From these we can also construct, for example, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$. ($\mathbb{R}$, however, cannot be constructed as an object, a "set". It is a point-free space of Dedekind sections, presented by a geometric theory.)

A homomorphism of AU-sketches preserves all the structure: it maps nodes to nodes, edges to edges, commutativities to commutativities and universals to universals.

We shall need to restrict the sketches we deal with, to our *contexts*. These are built up as *extensions* of the empty sketch $\mathbb{1}$, each extension a finite sequence of *simple extension* steps of the following types: adding a new primitive node, adding a new edge, adding a commutativity, adding a terminal, adding an initial, adding a pullback universal, adding a pushout, and adding a list object.

The following is an example of simple extension by adding a pullback universal.

4.1. EXAMPLE. Suppose $\mathbb{T}$ is a sketch that already contains data in the form of a cospan of edges: $\xrightarrow{\;u_1\;} \xleftarrow{\;u_2\;}$. Then we can make a simple extension of $\mathbb{T}$ to $\mathbb{T}'$ by adding a

pullback universal for that cospan, a cone in the form

$$\begin{array}{ccc} \mathsf{P} & \xrightarrow{\;\mathsf{p}^2\;} & \\ & \bullet & \\ \mathsf{p}^1 \downarrow & \;\mathsf{p} & \downarrow u_2 \\ & \bullet & \\ & \xrightarrow[u_1]{} & \end{array}$$
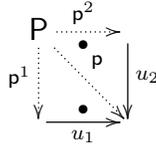
Along with the new universal itself, we also add a new node P, the pullback; four new edges (the projections $\mathsf{p}^1, \mathsf{p}^2, \mathsf{p}$ and the identity for P) and two commutativities $u_1\mathsf{p}^1 \sim \mathsf{p}$ and $u_2\mathsf{p}^2 \sim \mathsf{p}$.

An important feature of extensions is that the subjects of the universals (for instance, P and the projections in the above example) must be *fresh* – not already in the unextended sketch. This avoids the possibility of giving a single node two different universal properties, and allows the property that every non-strict model has a canonical strict isomorph.

The next fundamental concept is the notion of *equivalence extension*. This is an extension that can be expressed in a sequence of steps for which each introduces structure that must be present, and uniquely, given the structure in the unextended sketch. Unlike an ordinary extension, we cannot arbitrarily add nodes, edges or commutativities – they must be justified. Examples of equivalence extensions are to add composite edges; commutativities that follow from the rules of category theory; pullbacks, fillins and uniqueness of fillins, and similarly for terminals, initials, pushouts and list objects; and inverses of edges that must be isomorphisms by the rules of pretoposes. Thus the presented AUs for the two contexts are isomorphic.

The previous example, of adding a pullback universal, is already an equivalence extension. Another is that of adding a fillin edge. Suppose we have a pullback universal as above, and we also have a commutative square

$$\begin{array}{ccc} & \xrightarrow{\;v_2\;} & \\ & \bullet & \\ v_1 \downarrow & \;v & \downarrow u_2 \\ & \bullet & \\ & \xrightarrow[u_1]{} & \end{array}$$
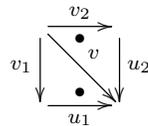
Then as an equivalence extension we can add a fillin edge $\xrightarrow{\;w\;} \mathsf{P}$, with commutativities $\mathsf{p}^1 w \sim v_1$ and $\mathsf{p}^2 w \sim v_2$.

Similarly, if we have two fillin candidates with the appropriate commutativities, then as an equivalence extension we can add a commutativity to say that the fillins are equal.

Any sketch homomorphism between contexts gives a model reduction map (in the reverse direction), but those are much too rigidly bound to the syntax to give us a good general notion of model map. We seek something closer to geometric morphisms, and in fact we shall find a notion of *context map* that captures exactly the strict AU-functors between the corresponding arithmetic universes $\mathbf{AU}\langle \mathbb{T}\rangle$. A *context map* $H\colon \mathbb{T}_0 \to \mathbb{T}_1$ is a sketch homomorphism from $\mathbb{T}_1$ to some equivalence extension $\mathbb{T}_0'$ of $\mathbb{T}_0$.

Each model M of $\mathbb{T}_0$ gives – by the properties of equivalence extensions – a model of $\mathbb{T}_0'$, and then by model reduction along the sketch homomorphism it gives a model $M \mathbin{.} H$ of $\mathbb{T}_1$.

Thus context maps embody a localization by which equivalence extensions become invertible. Of course, every sketch homomorphism is, trivially, a map in the reverse direction. Context extensions are sketch homomorphisms, and the corresponding maps backwards are *context extension maps*. They have some important properties, which we shall see in the next section.

At this point let us introduce the important example of the *hom context* $\mathbb{T}^{\rightarrow}$ of a context $\mathbb{T}$. We first take two disjoint copies of $\mathbb{T}$ distinguished by subscripts 0 and 1, giving two sketch homomorphisms $i_0, i_1 \colon \mathbb{T} \to \mathbb{T}^{\rightarrow}$. Second, for each node X of $\mathbb{T}$, we adjoin an edge $\theta_X \colon X_0 \to X_1$. Also, for each edge $u \colon X \to Y$ of $\mathbb{T}$, we adjoin a connecting edge $\theta_u \colon X_0 \to Y_1$ together with two commutativities:

$$
\begin{array}{ccc}
X_0 & \xrightarrow{\;\theta_X\;} & X_1 \\[-2pt]
& \bullet & \\
u_0 \downarrow & \searrow \theta_u & \downarrow u_1 \\
& & \\
Y_0 & \xrightarrow[\theta_Y]{\;\bullet\;} & Y_1
\end{array}
$$

A model of $\mathbb{T}^{\rightarrow}$ comprises a pair M_0, M_1 of models of $\mathbb{T}$, together with a homomorphism $\theta \colon M_0 \to M_1$.

We define a *2-cell* between maps $H_0, H_1 \colon \mathbb{T}_0 \to \mathbb{T}_1$ to be a map from $\mathbb{T}_0$ to $\mathbb{T}_1^{\rightarrow}$ that composes with the maps $i_0, i_1 \colon \mathbb{T}_1^{\rightarrow} \to \mathbb{T}_1$ to give H_0 and H_1

Finally, an *objective equality* between context maps H_0 and H_1 is a 2-cell for which the homomorphism between strict models must always be an identity. This typically arises when a context introduces the same universal construction twice on the same data.

From these definitions we obtain a 2-category $\mathfrak{Con}$ whose 0-cells are contexts, 1-cells are context maps modulo objective equality, and 2-cells are 2-cells. It has all finite PIE-limits (limits constructible from finite products, inserters, equifiers). Although it does not possess all (strict) pullbacks of arbitrary maps, it has all (strict) pullbacks of context extension maps along any other map.

We now list some of the most useful example of contexts. For more examples see [Vic19, §3.2].

4.2. EXAMPLE. The context $\mathbb{O}$ has nothing but a single node, X, and an identity edge $s(X)$ on X. A model of $\mathbb{O}$ in an AU (or topos) $\mathcal{A}$ is a "set" in the broad sense of an object of $\mathcal{A}$, and so $\mathbb{O}$ plays the role of the object classifier in topos theory. There is also a context $\mathbb{O}_\bullet$ which in addition to the generic node X has another node 1 declared as terminal, and moreover, it has an edge $x \colon 1 \to X$. (This is the effect of adding a generic point to the context $\mathbb{O}$.) Its models are the pointed sets. This time we must distinguish between strict and non-strict models. In a strict model, 1 is interpreted as *the canonical* terminal object.

There is a context extension map $U \colon \mathbb{O}_\bullet \to \mathbb{O}$ which corresponds to the sketch inclusion in the opposite direction, sending the generic node in $\mathbb{O}$ to the generic node in $\mathbb{O}_\bullet$. As a model reduction, U simply forgets the point.

The context $\mathbb{O}^{\rightarrow}$ comprises two nodes X_0 and X_1 and their identities, and an edge $\theta_X \colon X_0 \to X_1$. A model of $\mathbb{O}^{\rightarrow}$ in an AU (or topos) $\mathcal{A}$ is exactly a morphism in $\mathcal{A}$.

We can define the diagonal context map $\pi_{\mathbb{T}}\colon \mathbb{T} \to \mathbb{T}^{\to}$ by the opspan (id, F) of sketch morphisms where F sends edges θ_X to $s(X)$, θ_u to u and commutativities to degenerate commutativities of the form $us(X) \sim u$ and $s(Y)u \sim u$.

One central issue for models of sketches is that of *strictness*. The standard sketch-theoretic notion of models is non-strict: for a universal, such as a pullback of some given opspan, the pullback cone can be interpreted as any pullback of the opspan. Contexts give us good handle over strictness. The following result appears in [Vic17, Proposition 1]:

4.3. REMARK. Let $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ be a context extension map, for an extension $\mathbb{T}_1$ of $\mathbb{T}_0$. Suppose in some AU $\mathcal{A}$ we have a model M_1 of $\mathbb{T}_1$, a strict model M_0' of $\mathbb{T}_0$, and an isomorphism $\phi\colon M_0' \cong M_1U$.

$$
\begin{array}{ccc}
\mathbb{T}_1 & \qquad M_1' \overset{\tilde{\phi}}{\underset{\cong}{\dashrightarrow}} M_1 \\
{\scriptstyle U}\downarrow & \qquad \vdots\;\,\Big\downarrow \qquad\qquad \Big\downarrow \\
\mathbb{T}_0 & \qquad M_0' \underset{\cong}{\overset{\phi}{\longrightarrow}} M_1U
\end{array}
$$

Then there is a unique model M_1' of $\mathbb{T}_1$ and isomorphism $\tilde{\phi}\colon M_1' \cong M_1$ such that
  (i) M_1' is strict,
 (ii) $M_1'U = M_0'$,
(iii) $\tilde{\phi}U = \phi$, and
 (iv) $\tilde{\phi}$ is equality on all the primitive nodes used in extending $\mathbb{T}_0$ to $\mathbb{T}_1$.
We call M_1' the *canonical strict isomorph* of M_1 along ϕ.

The fact that we can uniquely lift strict models to strict models as in the remark above will be crucial in §5 and §8.

5. Background: Classifying toposes of contexts in $\mathcal{G}\mathfrak{Top}$

In this part, we shall review how [Vic17] exploits the fact that, for any geometric morphism $f\colon \mathcal{E} \to \mathcal{F}$ between elementary toposes with nno, the inverse image functor f^* is an AU-functor. It preserves the finite colimits and finite limits immediately from the definition, and the preservation of list objects follows quickly from their universal property and the adjunction of f.

By straightforwardly applying f^* we transform a model of M of a context $\mathbb{T}$ in $\mathcal{F}$ to a model in $\mathcal{E}$. However, we shall be interested in *strict* models, and f^* is in general non-strict as an AU-functor. For this reason we reserve the notation $f^\star M$ for the canonical strict isomorph of the straightforward application. By this means, the 1-cells of $\mathcal{E}\mathfrak{Top}$ act strictly on the categories of strict $\mathbb{T}$-models. This extends to 2-cells. If we have geometric morphisms $f, g\colon \mathcal{E} \rightrightarrows \mathcal{F}$ and a geometric transformation $\alpha\colon f \Rightarrow g$, then we get a model homomorphism $\alpha^\star M\colon f^\star M \to g^\star M$.

It will later be crucial to know how $(-)^\star$ interacts with transformation of models by context maps. Given a context map $H\colon \mathbb{T}_1 \to \mathbb{T}_0$, the models $f^\star(M{\cdot}H)$ and $(f^\star M){\cdot}H$ are isomorphic but not always equal. For instance, take $H\colon \mathbb{O}_\bullet \to \mathbb{O}$ to be the non-extension context map that sends the generic node of $\mathbb{O}$ to the terminal node in $\mathbb{O}_\bullet$, and M a strict model of $\mathbb{O}_\bullet$. However, [Vic17, Lemma 9] demonstrates that if H is an extension map, then they are indeed equal.

We take one step further to investigate the action of 1-cells and 2-cells in $\mathcal{G}\mathfrak{Top}$ on strict models of context extensions.

5.1. DEFINITION. *Let $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ be a context extension map and $p\colon \overline{p} \to \underline{p}$ a geometric morphism.*

*Then a **strict model** of U in p is a pair $M = (\overline{M}, \underline{M})$ where $\underline{M}$ is a strict $\mathbb{T}_0$-model in $\underline{p}$ and $\overline{M}$ is a strict $\mathbb{T}_1$-model in $\overline{p}$ such that $\overline{M}{\cdot}U = p^\star\underline{M}$.*

*A U-**morphism of models** $\varphi\colon M \to M'$ is a pair $(\overline{\varphi}, \underline{\varphi})$ where $\overline{\varphi}\colon \overline{M} \to \overline{M}'$ and $\underline{\varphi}\colon \underline{M} \to \underline{M}'$ are homomorphisms of $\mathbb{T}_1$- and $\mathbb{T}_0$-models such that $\overline{\varphi}{\cdot}U = p^\star\underline{\varphi}$.*

Strict U-models and U-morphisms in p form a category p-$\mathbf{Mod}$-U.

5.2. CONSTRUCTION. Suppose $f\colon q \to p$ is a 1-cell in $\mathcal{G}\mathfrak{Top}$ and let M be a model of U in p. We define a model $f^\star M$ of U in q, with downstairs part $\underline{f}^\star\underline{M}$, as follows.

Note that $\overset{\blacktriangledown}{f}{}^\star\underline{M}$ is an isomorphism of $\mathbb{T}_0$-models in $\overline{q}$ between $\overline{f}^\star p^\star\underline{M}$ and $q^\star\underline{f}^\star\underline{M}$. Also, $(\overline{f}^\star\overline{M}){\cdot}U = \overline{f}^\star(\overline{M}{\cdot}U) = \overline{f}^\star p^\star\underline{M}$. We define the isomorphism $\overset{\blacktriangledown}{f}{}^\star\overline{M}\colon \overline{f}^\star\overline{M} \to f^\star\overline{M}$ to be the canonical strict isomorph of $\overline{f}^\star\overline{M}$ along $\overset{\blacktriangledown}{f}{}^\star\underline{M}$, and then $f^\star M := (f^\star\overline{M}, \underline{f}^\star\underline{M})$ is a strict model of U in q.

The construction extends to U-model homomorphisms $\varphi\colon M \to M'$, as in the diagram on the left.

This can be encapsulated in the functor

$$f\text{-}\mathbf{Mod}\text{-}U\colon p\text{-}\mathbf{Mod}\text{-}U \to q\text{-}\mathbf{Mod}\text{-}U,\ M \mapsto f^\star M.$$

By the properties of the canonical strict isomorph, it is strictly functorial with respect to f. Furthermore, if $\alpha\colon f \Rightarrow g$ is a 2-cell in $\mathcal{G}\mathfrak{Top}$, then the bottom square in the above

right-hand diagram commutes and we define $\alpha^\star \overline{M}$ to be the unique $\mathbb{T}_1$-model morphism which completes the top face to a commutative square. We may also write $f^\star \underline{M}$ and $\alpha^\star \underline{M}$ for $\underline{f^\star M}$ and $\underline{\alpha^\star M}$.

The upshot is that each 2-cell $\alpha \colon f \Rightarrow g$ in $\mathcal{GT}\mathfrak{op}$ gives rise to a natural transformation α-**Mod**-U between functors f-**Mod**-U and g-**Mod**-U and $(\alpha$-**Mod**-$U)(M) = \alpha^\star M$. Hence ()-**Mod**-U is actually a 2-functor.

5.3. PROPOSITION. ()-**Mod**-$U \colon \mathcal{GT}\mathfrak{op}^{\mathrm{op}} \to \mathfrak{Cat}$ *is a strict 2-functor.*

A main purpose of [Vic17] is to explain how a context extension map $U \colon \mathbb{T}_1 \to \mathbb{T}_0$ may be thought of as a *bundle*, each point of the base giving rise to a space, its fibre. In terms of toposes, a point of the base $\mathbb{T}_0$ is a model M of $\mathbb{T}_0$ in some elementary topos $\mathcal{S}$. Then the space is a Grothendieck topos over $\mathcal{S}$, in other words a bounded geometric morphism. It should be the classifying topos for a theory $\mathbb{T}_1/M$ of models of $\mathbb{T}_1$ that reduce to M.

[Vic17] describes $\mathbb{T}_1/M$ using the approach it calls "elephant theories", namely that set out in [Joh02, B4.2.1]. An elephant theory over $\mathcal{S}$ specifies the category of models of the theory in every bounded $\mathcal{S}$-topos $q \colon \mathcal{E} \to \mathcal{S}$, together with the reindexing along geometric morphisms. Then $\mathbb{T}_1/M$ is defined by letting $\mathcal{E}$-**Mod**-$\mathbb{T}_1/M$ be the category of strict models of $\mathbb{T}_1$ in $\mathcal{E}$ that reduce by U to $q^\star M$.

The extension by which $\mathbb{T}_1$ was built out of $\mathbb{T}_0$ shows that the elephant theory $\mathbb{T}_1/M$, while not itself a context, is geometric over $\mathcal{S}$ in the sense of [Joh02, B4.2.7], and hence has a classifying topos $p \colon \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$, with generic model G, say. Its classifying property is that for each bounded $\mathcal{S}$-topos $\mathcal{E}$ we have an equivalence of categories

$$\Phi \colon {}^{\mathcal{BT}\mathfrak{op}}/_{\mathcal{S}} \; (\mathcal{E}, \mathcal{S}[\mathbb{T}_1/M]) \simeq \mathcal{E}\text{-}\mathbf{Mod}\text{-}\mathbb{T}_1/M$$

defined as $\Phi(f) := f^\star G$. (We shall take *equivalence* of categories to mean a functor that is full, faithful and essentially surjective.)

5.4. EXAMPLE. Consider the (unique) context map ! from $\mathbb{O}$ to the empty context $\mathbb{1}$. In any elementary topos $\mathcal{S}$ there is a unique model ! of $\mathbb{1}$, and the classifier for $\mathbb{O}/!$ is the object classifier over $\mathcal{S}$, the geometric morphism $[\mathbf{FinSet}, \mathcal{S}] \to \mathcal{S}$ where $\mathbf{FinSet}$ here denotes the category of finite sets as an internal category in $\mathcal{S}$, its object of objects being the nno N. The generic model of $\mathbb{O}$ in $[\mathbf{FinSet}, \mathcal{S}]$ is the inclusion functor $\mathrm{Inc} \colon \mathbf{FinSet} \hookrightarrow \mathbf{Set}$. As an internal diagram it is given by the second projection of the order $<$ on N, since $\{m \mid m < n\}$ has cardinality n. Given an object M of $\mathcal{S}$, the classifying topos for $\mathbb{O}_\bullet/M$ is the slice topos $\mathcal{S}/M$. Hence the classifying topos of $\mathbb{O}_\bullet$ is the slice topos $[\mathbf{FinSet}, \mathcal{S}]/\mathrm{Inc}$. The generic model of $\mathbb{O}_\bullet$ in $[\mathbf{FinSet}, \mathcal{S}]/\mathrm{Inc}$ is the pair $(\mathrm{Inc}, \pi \colon \mathrm{Inc} \to \mathrm{Inc} \times \mathrm{Inc})$ where Δ is the diagonal transformation which renders the diagram below commutative:

$$\mathrm{Inc} \xrightarrow{\;\;\Delta\;\;} \mathrm{Inc} \times \mathrm{Inc}$$

with maps id and π_2 to Inc.

So far the discussion of p as classifier has been firmly anchored to $\mathcal{S}$ and M, but notice that (G, M) is a model of U in p. We now turn to discussing how it fits in more generally with ()-**Mod**-U by spelling out the properties of p as classifying topos that are shown in [Vic17]. The main result there, Theorem 5.12, says that P is "locally representable" over Q in the following fibration tower.

$$(\mathcal{G}\mathfrak{Top}_{\cong}\text{-}U)^{\mathrm{co}} \xrightarrow{\ P\ } (\mathcal{G}\mathfrak{Top}_{\cong}\text{-}(\mathbb{T}_0 \subset \mathbb{T}_0))^{\mathrm{co}} \xrightarrow{\ Q\ } (\mathcal{E}\mathfrak{Top}_{\cong}\text{-}\mathbb{T}_0)^{\mathrm{co}}$$

There is a slight change of notation from [Vic17]. $\mathcal{G}\mathfrak{Top}$ there, unlike ours, restricts the 2-cells to be isomorphisms downstairs. This is needed to make P and Q fibrations. To emphasize the distinction we have written $\mathcal{G}\mathfrak{Top}_{\cong}$ above.

The objects of $\mathcal{G}\mathfrak{Top}_{\cong}\text{-}U$ are pairs (q, N) where $q \colon \overline{q} \to \underline{q}$ is a bounded geometric morphism and $N = (\overline{N}, \underline{N})$ is a model of U in q. A 1-cell from (q_0, N_0) to (q_1, N_1) is a triple (f, f^-, f_-) such that $f \colon q_0 \to q_1$ in $\mathcal{G}\mathfrak{Top}$, $(f^-, f_-) \colon N_0 \to f^\star N_1$ is a homomorphism of U-models, *and f_- is an isomorphism.* It is P-cartesian iff f^- too is an isomorphism. A 2-cell is a 2-cell $\alpha \colon f \Rightarrow g$ in $\mathcal{G}\mathfrak{Top}_{\cong}$ (α an iso) such that $\alpha^\star N_1 \circ (f^-, f_-) = (g^-, g_-)$.

$\mathcal{G}\mathfrak{Top}_{\cong}\text{-}(\mathbb{T}_0 \subset \mathbb{T}_0)$ is similar, but without the $\overline{N}$s and f^-s.

Let us now unravel the local representability. It says that for each $(\mathcal{S}, M)$ in $\mathcal{E}\mathfrak{Top}_{\cong}\text{-}\mathbb{T}_0$ there is a classifier $(p \colon \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}, (G, M))$ in $\mathcal{G}\mathfrak{Top}_{\cong}\text{-}U$, where G is the generic model of $\mathbb{T}_1/M$.

5.5. PROPOSITION. *[Vic17, Proposition 19] The properties that characterize p as classifier are equivalent to the following.*
  (i) *For every object (q, N) of $\mathcal{G}\mathfrak{Top}_{\cong}\text{-}U$, 1-cell $\underline{f} \colon \underline{q} \to \underline{p}$ in $\mathcal{E}\mathfrak{Top}$ and isomorphism $f_- \colon \underline{N} \to \underline{f}^\star M$, there is a P-cartesian 1-cell $\overline{(f, f^-, f_-)} \colon (q, N) \to (p, (G, M))$ over $(\underline{f}, f_-)$. In other words, there is f over $\underline{f}$ and an isomorphism (P-cartesianness) $f^- \colon \overline{N} \cong f^\star G$ over f_-.*
  (ii) *Suppose $(f, f^-, f_-), (g, g^-, g_-) \colon (q, N) \to (p, (G, M))$ in $\mathcal{G}\mathfrak{Top}_{\cong}\text{-}U$, with (g, g^-, g_-) being P-cartesian (g^- is an iso). Suppose also we have $\underline{\alpha} \colon \underline{g} \Rightarrow \underline{f}$ so that $\underline{\alpha}^\star M$ commutes with f_- and g_-. (Note the reversal of 2-cells compared with [Vic17, Proposition 19]. This is because the fibration tower uses the 2-cell duals $(\mathcal{G}\mathfrak{Top}_{\cong}\text{-}U)^{\mathrm{co}}$ etc.) Then $\underline{\alpha}$ has a unique lift $\alpha \colon g \Rightarrow f$ such that $(\alpha^\star G)g^- = f^-$.*

In the case where we have identity 1-cells and 2-cells downstairs, it can be seen that this matches the usual characterization of classifier for $\mathbb{T}_1/M$ in $\mathcal{B}\mathfrak{Top}/\mathcal{S}$.

Although the properties described above insist on the 2-cells $\underline{\alpha}$ and model homomorphisms f_- downstairs being isomorphisms, we shall generalize this in a new result, Proposition 5.7.

We first remark on the construction of finite lax colimits in the 2-category $\mathcal{E}\mathfrak{Top}$ and more specifically cocomma objects which will be used in our proof. There is a forgetful 2-functor $\mathcal{U}$ from $\mathcal{E}\mathfrak{Top}^{\mathrm{op}}$ to the 2-category of categories which sends a topos $\mathcal{E}$ to its underlying category $\mathcal{E}$, a geometric morphism $f \colon \mathcal{E} \to \mathcal{F}$ to its inverse image part $f^* \colon \mathcal{F} \to \mathcal{E}$ and a geometric transformation $\theta \colon f \Rightarrow g$ to the natural transformation $\theta^* \colon f^* \Rightarrow g^*$.

The 2-functor $\mathcal{U}$ transforms colimits in $\mathcal{E}\mathfrak{Top}$ to limits in $\mathfrak{Cat}$. This in particular means that the underlying category of a coproduct of toposes, for instance, is the product of their underlying categories. The same is true for cocomma objects. More specifically, for any topos $\mathcal{E}$, with cocomma topos $(\mathrm{id}_{\mathcal{E}} \uparrow \mathrm{id}_{\mathcal{E}})$ equipped with geometric morphisms $i_0, i_1 \colon \mathcal{E} \rightrightarrows (\mathrm{id}_{\mathcal{E}} \uparrow \mathrm{id}_{\mathcal{E}})$ and 2-cell θ between them, the data $\langle i_0^\star, i_1^\star, \theta^\star \rangle$ specifies the corresponding comma category $\mathrm{id}_{\mathcal{U}(\mathcal{E})} \downarrow \mathrm{id}_{\mathcal{U}(\mathcal{E})}$. For more details on the construction of cocomma toposes see [Joh02, B3.4.2]. Another useful remark is about the relation of topos models of $\mathbb{T}^{\rightarrow}$ and models of $\mathbb{T}$.

5.6. LEMMA. *Models of $\mathbb{T}^{\rightarrow}$ in a topos $\mathcal{E}$ are equivalent to models of $\mathbb{T}$ in the cocomma topos $(\mathrm{id}_{\mathcal{E}} \uparrow \mathrm{id}_{\mathcal{E}})$.*

5.7. PROPOSITION. *Let $U \colon \mathbb{T}_1 \to \mathbb{T}_0$ be a context extension map, M a strict model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$, and $p \colon \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ the corresponding classifying topos with generic model G.*

Let $q \colon \overline{q} \to \underline{q}$ be a bounded geometric morphism, and let $(f_i, f_i^-, f_{i-}) \colon (q, N_i) \to (p, (G, M))$ $(i = 0, 1)$ be two P-cartesian 1-cells in $\mathcal{G}\mathfrak{Top}_{\cong}\text{-}U$.

Suppose $\varphi \colon N_0 \to N_1$ is a homomorphism of U-models and $\underline{\alpha} \colon \underline{f}_0 \Rightarrow \underline{f}_1$ is such that the left hand diagram in below commutes. Then there exists a unique 2-cell $\alpha \colon f_0 \Rightarrow f_1$ over $\underline{\alpha}$ such that the right hand diagram commutes.

$$
\begin{array}{ccc}
\underline{N}_0 \xrightarrow{f_{0-}} \underline{f}_0^{\star}M & \qquad & \overline{N}_0 \xrightarrow{f_0^-} f_0^{\star}G \\
\Big\downarrow{\underline{\varphi}} \qquad \Big\downarrow{\underline{\alpha}^{\star}M} & & \Big\downarrow{\overline{\varphi}} \qquad \Big\downarrow{\alpha^{\star}G} \\
\underline{N}_1 \xrightarrow{f_{1-}} \underline{f}_1^{\star}M & & \overline{N}_1 \xrightarrow{f_1^-} f_1^{\star}G
\end{array}
$$

PROOF. Note that we do not assume that $\underline{\alpha}$ and φ are isomorphisms, so φ need not be a 1-cell in $\mathcal{G}\mathfrak{Top}_{\cong}$. To get round this, we use cocomma toposes.

Let $\underline{q}' = \underline{q} \uparrow \underline{q}$ and $\overline{q}' = \overline{q} \uparrow \overline{q}$ be the two cocomma toposes, with bounded geometric morphism $q' \colon \overline{q}' \to \underline{q}'$. We now have two 1-cells $i_0, i_1 \colon q \to q'$ in $\mathcal{G}\mathfrak{Top}$, equipped with identities for $\overset{\blacktriangledown}{i}_0$ and $\overset{\blacktriangledown}{i}_1$, and a 2-cell $\theta \colon i_0 \Rightarrow i_1$. The pair $\varphi = (\overline{\varphi}, \underline{\varphi})$ is a model of U in q'.

The geometric transformation $\underline{\alpha}$ gives us a geometric morphism $\underline{a} \colon \underline{q}' \to \mathcal{S}$, with an isomorphism $a_- \colon \underline{\varphi} \cong \underline{a}^{\star}M$, so a 1-cell in $\mathcal{E}\mathfrak{Top}_{\cong}\text{-}\mathbb{T}_0$. This lifts to a P-cartesian 1-cell $(a, a^-, a_-) \colon (q', \varphi) \to (p, (G, M))$ in $\mathcal{G}\mathfrak{Top}_{\cong}\text{-}U$. We now have the following diagrams in $\mathcal{G}\mathfrak{Top}$ and $\mathcal{G}\mathfrak{Top}_{\cong}\text{-}U$.
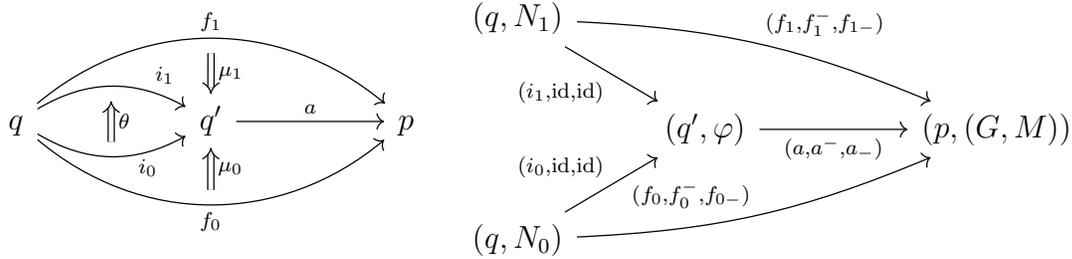
In the right hand diagram all the 1-cells are P-cartesian, and it follows there are unique iso-2-cells $\mu_i\colon (f_1, f_1^-, f_{1-}) \to (a, a^-, a_-)(i_1, \mathrm{id}, \mathrm{id})$ lifting the identity 2-cells downstairs. Now by composing μ_0, $a \centerdot \theta$ and μ_1^{-1} we get the required α.

To show uniqueness of the geometric transformation α, suppose we have another, β, with the same properties. In other words, $\underline{\alpha} = \underline{\beta}$ and $\alpha^\star(G, M) = \beta^\star(G, M)$. We thus get two 1-cells $a, b\colon q' \Rightarrow p$, $a = (f_0, \alpha, f_1)$ and $b = (f_0, \beta, f_1)$. We have $\underline{a} = \underline{b}$ and $a^\star(G, M) = b^\star(G, M)$ and it follows that there is a unique vertical 2-cell $\iota\colon a \Rightarrow b$ such that $\iota^\star(G, M)$ is the identity.

By composing horizontally with θ, we can analyse ι as a pair of 2-cells $\iota_\lambda\colon f_\lambda \Rightarrow f_\lambda$ ($\lambda = 0, 1$) such that the following diagram commutes.

$$\begin{array}{ccc} f_0 & \xrightarrow{\ \iota_0\ } & f_0 \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ f_1 & \xrightarrow{\ \iota_1\ } & f_1 \end{array}$$

Now we see that each ι_λ is the unique vertical 2-cell such that $\iota_\lambda{}^\star(G, M)$ is the identity, so ι_λ is the identity on f_λ and $\alpha = \beta$.

$\blacksquare$

6. The Chevalley criterion in $\mathfrak{Con}$

In [Str74], and later in [Str80], Ross Street develops an elegant algebraic approach to study fibrations, opfibrations, and two-sided fibrations in 2-categories and bicategories. In the case of (op)fibrations the 2-category is required to be *finitely complete,* with strict finite conical limits[1] and cotensors with the (free) walking arrow category $\mathbb{2}$. Given those, it also has strict comma objects. Then he defined a fibration (opfibration) as a pseudo-algebra of a certain right (resp. left) slicing 2-monad. In the case of bicategories, they are defined via "hyperdoctrines" on bicategories.

For (op)fibrations internal to 2-categories, he showed [Str74, Proposition 9] that his definition was equivalent to a Chevalley criterion. However, for our purposes we prefer

---

[1] i.e. weighted limits with set-valued weight functors. They are ordinary limits as opposed to a more general weighted limit.

to start from the Chevalley criterion and bypass Street's characterization via pseudo-algebras.

Note also that Street weakened the original Chevalley criterion of Gray, by allowing the adjunction to have isomorphism counit. We shall revert to the original and more strict requirement of having the identity as the counit.

We do not wish to assume existence of all pullbacks since our main 2-category $\mathfrak{Con}$ does not have them. Instead, we assume our 2-categories in this section to have all finite strict PIE-limits [PR91], in other words those reducible to Products, Inserters and Equifiers. This is enough to guarantee existence of all strict comma objects since for any opspan $A \xrightarrow{f} B \xleftarrow{g} C$ in a 2-category $\mathscr{K}$ with (strict) finite PIE-limits, the comma object $f{\downarrow}g$ can be constructed as an inserter of $f\pi_A, g\pi_C \colon A \times C \rightrightarrows B$. Moreover, it is a result of [Vic19, Lemma 44] that $\mathfrak{Con}$ has finite PIE-limits.

Pullbacks are not PIE-limits, so sometimes we shall be interested in whether they exist.

6.1. DEFINITION. *A 1-cell $p \colon E \to B$ in a 2-category $\mathscr{K}$ is **carrable** whenever a strict pullback of p along any other 1-cell $f \colon B' \to B$ exists in $\mathscr{K}$. As usual, we write $f^*p \colon f^*E \to B'$ for a chosen pullback of p along f.*

[Vic19] proves that all context extension maps are carrable.

From now on in this section, we assume that $\mathscr{K}$ is a 2-category with all finite PIE-limits. Note that for AUs and elementary toposes, we assume that the structure is given *canonically* – this is essential if we are to consider strict models. For our $\mathscr{K}$ here we do not assume there are canonical PIE-limits or pullbacks. Indeed, in $\mathfrak{Con}$ (so far as we know) they do not exist. 1-cells are defined only modulo objective equality, and the construction of those limits depends on the choice of representatives of 1-cells.

We first describe the Chevalley criterion in the style of [Str74]. Suppose B is an object of $\mathscr{K}$, and p is a 0-cell in the strict slice 2-category $\mathscr{K}/B$. By the universal property of (strict) comma object $B{\downarrow}p$, there is a unique 1-cell $\Gamma_1 \colon E{\downarrow}E \to B{\downarrow}p$ with properties $R(p)\Gamma_1 = d_0(p{\downarrow}p)$, $\hat{d}_1\Gamma_1 = e_1$, and $\phi_p \cdot \Gamma_1 = p \cdot \phi_E$.

$$
\begin{array}{c}
\begin{array}{ccc}
E{\downarrow}E & \xrightarrow{\quad e_1 \quad} & \\
\Big\downarrow{\scriptstyle p{\downarrow}p} \quad {\scriptstyle \Gamma_1} & & \\
B{\downarrow}B & B{\downarrow}p \xrightarrow{\ \hat{d}_1\ } E & \\
& R(p)\Big\downarrow \quad {\scriptstyle \phi_p\Uparrow} \quad \Big\downarrow{\scriptstyle p} & \\
& B \xrightarrow{\ 1\ } B &
\end{array}
\end{array}
\qquad (2)
$$

6.2. DEFINITION. *Consider p as above. We call p a **(Chevalley) fibration** if the 1-cell Γ_1 has a right adjoint Λ_1 with counit ε an identity in the 2-category $\mathscr{K}/B$.*

Dually one defines (Chevalley) **opfibrations** *as 1-cells* $p\colon E \to B$ *for which the morphism* $\Gamma_0\colon E{\downarrow}E \to p{\downarrow}B$ *has a left adjoint* Λ_0 *with unit* η *an identity.*

Street [Str74], but using isomorphisms for the counits ε instead of identities, showed that the Chevalley criterion is equivalent to a certain pseudoalgebra structure on p. Gray [Gray66] showed that Chevalley fibrations in the 2-category $\mathfrak{Cat}$ of (small) categories correspond to well-known (cloven) Grothendieck fibrations.

Although pullbacks are not assumed to exist (they are not PIE-limits), the comma objects $p{\downarrow}B$ and $B{\downarrow}p$ can be expressed as pullbacks along the two projections from $B{\downarrow}B$ to B. Let us at this point reformulate the fibration property using the notation as it will appear in $\mathfrak{Con}$ when p is an extension map $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ (and hence carrable).

Let $\mathrm{dom}, \mathrm{cod}\colon \overrightarrow{\mathbb{T}_0} \to \mathbb{T}_0$ be the domain and codomain context maps corresponding to sketch homomorphisms $i_0, i_1\colon \mathbb{T}_0 \to \overrightarrow{\mathbb{T}_0}$. We define the context extension maps $\mathrm{dom}^*\mathbb{T}_1 \to \overrightarrow{\mathbb{T}_0}$ and $\mathrm{cod}^*\mathbb{T}_1 \to \overrightarrow{\mathbb{T}_0}$ as the pullbacks of U along dom and cod. A model of $\mathrm{dom}^*(\mathbb{T}_1)$ is a pair $(N, f\colon M_0 \to M_1)$ where f is a homomorphism of models of $\mathbb{T}_0$ and N is a model of $\mathbb{T}_1$ such that $N \boldsymbol{.} U = M_0$. Models of $\mathrm{cod}^*(\mathbb{T}_1)$ are similar, except that $N \boldsymbol{.} U = M_1$. There are induced context maps $\Gamma_0\colon \overrightarrow{\mathbb{T}_1} \to \mathrm{dom}^*(\mathbb{T}_1)$ and $\Gamma_1\colon \overrightarrow{\mathbb{T}_1} \to \mathrm{cod}^*(\mathbb{T}_1)$. Given a model $f\colon N_0 \to N_1$ of $\overrightarrow{\mathbb{T}_1}$, Γ_i sends it to $(N_i, f\boldsymbol{.}\overrightarrow{U}\colon N_0\boldsymbol{.}U \to N_1\boldsymbol{.}U)$.

$$(3)$$

6.3. REMARK. A consequence of the counit of the adjunction $\Gamma_1 \dashv \Lambda_1$ being the identity is that the adjunction triangle equations are expressed in simpler forms; we have $\Gamma_1\boldsymbol{.}\eta_1 = \mathrm{id}_{\Gamma_1}$ and $\eta_1 \boldsymbol{.} \Lambda_1 = \mathrm{id}_{\Lambda_1}$.

6.4. REMARK. The composite $\Gamma_0\Lambda_1$ is a 1-cell from $\mathrm{cod}^*(\mathbb{T}_1)$ to $\mathrm{dom}^*(\mathbb{T}_1)$. Moreover, there is a 2-cell from $\pi_0\Gamma_0\Lambda_1$ to π_1 constructed as $\pi_0\Gamma_0\Lambda_1 \xRightarrow{\theta_{\mathbb{T}_1}\Lambda_1} \pi_1\Gamma_1\Lambda_1 = \pi_1$. These two, the 1-cell and the 2-cell, will appear again as the structure discussed in Remark 7.10.

7. The representable definition in $\mathcal{ETop}$

In this section we turn to the representable definition of fibration, particularly in $\mathcal{ETop}$. The only fibrations there that we shall be interested in are bounded as geometric

morphisms, and we find it convenient to consider them as objects in $\mathcal{G}\mathfrak{Top}$. Part of our analysis will consider cartesian properties with respect to the codomain functor $\mathbb{C}\mathrm{od}\colon \mathcal{G}\mathfrak{Top} \to \mathcal{E}\mathfrak{Top}$, and so relates to work on 2-categorical or bicategorical fibrations.

[Her99] generalizes the notion of fibration to strict 2-functors between strict 2-categories. His archetypal example of strict 2-fibration is the 2-category **Fib** of Grothendieck fibrations, fibred over the 2-category of categories via the codomain functor $\mathbb{C}\mathrm{od}\colon \mathbf{Fib} \to \mathfrak{Cat}$. Much later [Bak12] in his talk, and [Buc14] in his paper develop these ideas to define fibrations of bicategories.

We shall examine the cartesian 1-cells and 2-cells for our codomain 2-functor $\mathbb{C}\mathrm{od}\colon \mathcal{G}\mathfrak{Top} \to \mathcal{E}\mathfrak{Top}$, but we might as well do this in the abstract. We assume for the rest of this section that $\mathcal{K}$ is a 2-category (abstracting $\mathcal{E}\mathfrak{Top}$).

7.1. REMARK. We recall that a **bipullback** of an opspan $A \xrightarrow{f} C \xleftarrow{g} B$ in a 2-category $\mathcal{K}$ is given by a 0-cell P together with 1-cells d_0, d_1 and an iso-2-cell $\pi\colon fd_0 \Rightarrow gd_1$ satisfying the following universal properties.

(BP1) Given any iso-cone $(l_0, l_1, \lambda\colon fl_0 \cong gl_1)$ over f, g (with vertex X), there exists a 1-cell u with two iso-2-cells γ_0 and γ_1 such that the pasting diagrams below are equal.
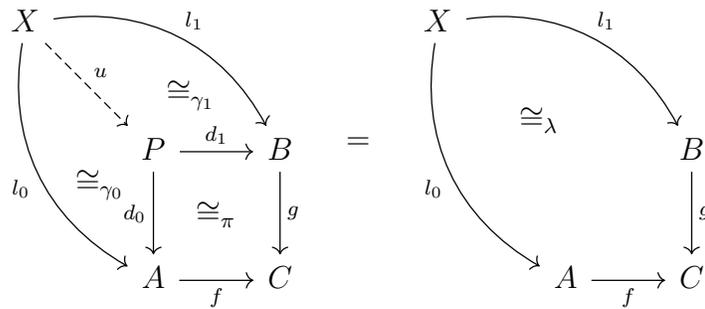
(BP2) Given 1-cells $u, v\colon X \rightrightarrows P$ and 2-cells $\alpha_i\colon d_i u \Rightarrow d_i v$ such that

$$
\begin{array}{ccc}
fd_0u & \xrightarrow{\;f\cdot\alpha_0\;} & fd_0v \\
{\scriptstyle \pi\cdot u}\downarrow & & \downarrow{\scriptstyle \pi\cdot v} \\
gd_1u & \xrightarrow[\;g\cdot\alpha_1\;]{} & gd_1v
\end{array} \; ,
$$

then there is a unique $\beta\colon u \Rightarrow v$ such that each $\alpha_i = d_i \cdot \beta$.

The two conditions (BP1) and (BP2) together are equivalent to saying that the functor

$$\mathcal{K}(X, P) \xrightarrow{\;\cong\;} \mathcal{K}(X, f) \downarrow_\cong \mathcal{K}(X, g),$$

obtained from post-composition by the pseudo-cone $\langle d_0, \pi, d_1 \rangle$, is an equivalence of categories. The right hand side here is an isocomma category.

Note the distinction from pseudopullbacks, for which the equivalence is an isomorphism of categories.

7.2. DEFINITION. *(cf. Definition 6.1.) A 1-cell* $x\colon \overline{x} \to \underline{x}$ *in* $\mathscr{K}$ *is **bicarrable** whenever a bipullback of p along any other 1-cell* $\underline{f}$ *exists in* $\mathscr{K}$*. We frequently use the diagram below to represent a chosen such bipullback:*

$$
\begin{array}{ccc}
\underline{f}^*\overline{x} & \xrightarrow{\ \overline{f}\ } & \overline{x} \\
{\scriptstyle \underline{f}^*x}\big\downarrow & {}^{\blacktriangledown}\ \overline{f}\ \Downarrow & \big\downarrow{\scriptstyle x} \\
\underline{y} & \xrightarrow[\ \underline{f}\ ]{} & \underline{x}
\end{array}
$$

where the 2-cell $\overset{\blacktriangledown}{\overline{f}}$ *is an iso-2-cell.*

   Similarly, we say x *is **pseudocarrable** if pseudopullbacks exist.*

   Of course, bipullbacks are defined up to equivalence and the class of bicarrable 1-cells is closed under bipullback.

   An important fact in $\mathcal{ETop}$ is that all bounded geometric morphisms are bicarrable [Joh02, B3.3.6].

   We now recall the representable definition of fibration from [Joh93, definition 3.1].

7.3. DEFINITION. *Let* $\mathscr{K}$ *be a 2-category and* $x\colon \overline{x} \to \underline{x}$ *a 1-cell in* $\mathscr{K}$*.*
   (a) A 2-cell $\overline{\alpha}\colon \overline{f} \Rightarrow \overline{e}\colon \overline{y} \to \overline{x}$ *is* cartesian *(with respect to x) if, for every 1-cell* $\overline{h}\colon \overline{w} \to \overline{y}$*, the whiskering* $\overline{\alpha}\boldsymbol{.}\overline{h}$ *is cartesian with respect to the functor* $\mathscr{K}(\overline{w},x)\colon \mathscr{K}(\overline{w},\overline{x}) \to \mathscr{K}(\overline{w},\underline{x})$*.*
   (b) x *is a* fibration *in* $\mathscr{K}$ *if, given any* $\overline{e}\colon \overline{y} \to \overline{x}$*,* $\underline{f}\colon \overline{y} \to \underline{x}$ *and* $\underline{\alpha}\colon \underline{f} \Rightarrow x\overline{e}$*, there exists a cartesian 1-cell* $\overline{\alpha}\colon \overline{f} \Rightarrow \overline{e}$ *and an iso-2-cell* $\overset{\blacktriangledown}{\overline{f}}\colon x\overline{f} \cong \underline{f}$ *such that* $x\boldsymbol{.}\overline{\alpha} = \underline{\alpha}\,\overset{\blacktriangledown}{\overline{f}}$*. We say* x *is a* strict fibration *if the above condition always holds with* $\overset{\blacktriangledown}{\overline{f}} = \mathrm{id}_{\underline{f}}$*.*

   Note that (b) is stronger than just saying $\mathscr{K}(\overline{y},x)$ is a fibration. The stronger definition of cartesian in (a) covers the additional stipulation that the naturality squares should be fibred functors, that is, for any 1-cell $\overline{h}\colon \overline{w} \to \overline{y}$, the top row functor in the commutative diagram below preserves cartesian morphisms.

$$
\begin{array}{ccc}
\mathscr{K}(\overline{y},\overline{x}) & \xrightarrow{\ \mathscr{K}(\overline{h},\overline{x})\ } & \mathscr{K}(\overline{w},\overline{x}) \\
{\scriptstyle \mathscr{K}(\overline{y},x)}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{K}(\overline{w},x)} \\
\mathscr{K}(\overline{y},\underline{x}) & \xrightarrow[\ \mathscr{K}(\overline{h},\underline{x})\ ]{} & \mathscr{K}(\overline{w},\underline{x})
\end{array}
$$

Now, x is a strict (resp. weak) fibration in $\mathscr{K}$ iff the natural transformation $\mathscr{K}(-,x)\colon \mathscr{K}(-,\overline{x}) \Rightarrow \mathscr{K}(-,\underline{x})$ is a (representable) presheaf over $\mathscr{K}$ of Grothendieck (resp. Street aka weak) fibrations. (We have ignored the fact that $\mathscr{K}$ is a 2-category, and typically large.)

7.4. CONSTRUCTION. Suppose $\mathscr{K}$ is a 2-category. Let $\mathcal{D}$ be a chosen class of bicarrable 1-cells in $\mathscr{K}$, which we shall call "display 1-cells", with the following properties.

- Every identity 1-cell is in $\mathcal{D}$.
- If $x\colon \overline{x} \to \underline{x}$ is in $\mathcal{D}$, and $\underline{f}\colon \underline{y} \to \underline{x}$ in $\mathscr{K}$, then there is some bipullback y of x along $\underline{f}$ such that $y \in \mathcal{D}$.

We form a 2-category $\mathscr{K}_{\mathcal{D}}$ whose 0-cells are the elements $x \in \mathcal{D}$, and whose 1-cells and 2-cells are taken in exactly the same manner as for $\mathcal{GTop}$ (Definition 3.1), using elements of $\mathcal{D}$ for bounded geometric morphisms and 1-cells and 2-cells in $\mathscr{K}$ for geometric morphisms and geometric transformations.

Notice that $\mathscr{K}_{\mathcal{D}}$ is a sub-2-category of the 2-category $\mathscr{K}^2 := \mathbf{Fun}_{ps}(\mathbb{2}, \mathscr{K})$, where the latter consists of (strict) 2-functors, pseudo-natural transformations and modifications from the interval category (aka free walking arrow category) $\mathbb{2}$. We write $\mathbb{C}\mathrm{od}, \mathbb{D}\mathrm{om}\colon \mathscr{K}^2 \to \mathscr{K}$ (and also for their restrictions to $\mathscr{K}_{\mathcal{D}}$) for the (strict) 2-functors that map everything to the downstairs or upstairs parts. Thus $\mathbb{C}\mathrm{od}$ maps $x \mapsto \underline{x}$, $f \mapsto \underline{f}$, and $\langle \overline{\alpha}, \underline{\alpha} \rangle \mapsto \underline{\alpha}$.

We now examine cartesian 1-cells and 2-cells of $\mathscr{K}_{\mathcal{D}}$ with respect to $\mathbb{C}\mathrm{od}\colon \mathscr{K}_{\mathcal{D}} \to \mathscr{K}$, following the definitions of [Buc14, 3.1]. Note that, although we deal only with 2-categories and 2-functors between them, we follow the bicategorical definitions, in which uniqueness appears only at the level of 2-cells.

7.5. PROPOSITION. *A 1-cell $f\colon y \to x$ in $\mathscr{K}_{\mathcal{D}}$ is cartesian with respect to $\mathbb{C}\mathrm{od}$ iff, as square in $\mathscr{K}$, it is a bipullback.*

PROOF. Consider the following diagram.

$$(4)$$

The top, bottom and back faces all commute up to identity. The left and right faces commute up to iso-2-cells, corresponding to those in the 1-cells of $\mathscr{K}_{\mathcal{D}}$. By definition, they are both iso-comma squares (pseudopullbacks). The front face has an iso-2-cell, by whiskering the iso-2-cell $\overset{\blacktriangledown}{f}$.

The definition of cartesianness for f is that, for all w, the back face is a bipullback. f itself is a bipullback (in $\mathscr{K}$) iff, for all objects $\overline{w}$ of $\mathscr{K}$, the front face is a bipullback of categories.

$\Rightarrow$: Given $\overline{w}$, let $w = \mathrm{id}_{\overline{w}} \colon \overline{w} \to \overline{w}$. The back face is a bipullback by hypothesis. Then the two edges $\mathbb{D}\mathbf{om}$ in the diagram are both equivalences and the edges $\mathscr{K}(w, \underline{y})$ and $\mathscr{K}(w, \underline{x})$ are identities. It follows that the front face is a bipullback.

$\Leftarrow$: Given w, by hypothesis the front face is a bipullback. The iso-comma for the back face has objects $(\underline{g} \colon \underline{w} \to \underline{y}, h \colon w \to x, \theta \colon \underline{fg} \cong \underline{h})$, and this is equivalent to the category of structures $(\underline{g}, u \colon \overline{w} \to \underline{y}, \phi \colon \underline{g}w \cong u, \overline{h}, \overset{\blacktriangledown}{h} \colon x\overline{h} \cong \underline{f}u)$. The result now follows from the fact that the front and left faces are bipullback and iso-comma. ∎

7.6. REMARK. A 2-cell $\alpha \colon f \Rightarrow g \colon y \to x$ in $\mathscr{K}_{\mathcal{D}}$ is cartesian for Buckley's definition if it is cartesian as a 1-cell with respect to the functor $\mathbb{C}\mathbf{od}_{yx} \colon \mathscr{K}_{\mathcal{D}}(y, x) \to \mathscr{K}(\underline{y}, \underline{x})$.

We now define a notion that, on the one hand, conveniently leads to a characterization of when P is a fibration; but, on the other hand, turns out in the next section to be useful even when P is not a fibration.

7.7. DEFINITION. *Let $P \colon \mathscr{E} \to \mathscr{B}$ be a 2-functor. We call an object e of $\mathscr{E}$ **fibrational** with respect to P iff*
(B1) every $f \colon b' \to b = P(e)$ has a cartesian lift,
(B2) for every 0-cell e' in $\mathscr{E}$, the functor

$$P_{e',e} \colon \mathscr{E}(e', e) \to \mathscr{B}(P(e'), P(e))$$

is a Grothendieck fibration of categories, and
(B3) whiskering on the left preserves cartesianness of 2-cells in $\mathscr{E}$ between 1-cells with codomain e.

Clearly, every object of $\mathscr{E}$ is fibrational iff P is a fibration in the sense of [Her99, Theorem 2.8]. Buckley's definition requires a strengthening of $(B3)$ in which cartesianness is also preserved by whiskering on the right – see [Buc14, Remark 2.1.9]. It is also noteworthy that conditions $(B2)$ and $(B3)$ together make the 2-functor $P_{-,e} \colon \mathscr{E}^{\mathrm{op}} \to \mathfrak{Cat} \!\downarrow\! \mathfrak{Cat}$ lift to $P_{-,e} \colon \mathscr{E}^{\mathrm{op}} \to \mathbf{Fib}$ for every $e \in \mathscr{E}$.

7.8. LEMMA. *Let x be an object of $\mathscr{K}_{\mathcal{D}}$ and let $\alpha \colon g \Rightarrow f \colon y \to x$ be a 2-cell.*
(a) If $\overline{\alpha}$ is cartesian with respect to x (Definition 7.3), then α is cartesian with respect to $\mathbb{C}\mathbf{od}$.
(b) If x satisfies condition $(B3)$ of Definition 7.7 then the converse also holds.

PROOF. (a) Suppose we have $h \colon y \to x$, $\gamma \colon h \Rightarrow f$ and $\underline{\beta} \colon \underline{h} \Rightarrow \underline{f}$ with $\underline{\gamma} = \underline{\alpha} \circ \underline{\beta}$. By cartesianness with respect to x, we can find unique $\overline{\beta} \colon \overline{h} \Rightarrow \overline{g}$ over $\overset{\blacktriangledown}{g}{}^{-1} \circ (\underline{\beta}y) \circ \overset{\blacktriangledown}{h}$ such that $\overline{\gamma} = \overline{\alpha} \circ \overline{\beta}$. Then $\langle \overline{\beta}, \underline{\beta} \rangle \colon h \Rightarrow g$ is the required 2-cell in $\mathscr{K}_{\mathcal{D}}$.

(b) Let α' be got by whiskering α with the 1-cell $\langle 1_{\overline{y}}, \mathrm{id}_y, y \rangle \colon \overline{y} \to y$, where $\overline{y}$, as object of $\mathscr{K}_{\mathcal{D}}$, denotes $1_{\overline{y}}$. The 2-cell α' is cartesian with respect to $\mathbb{C}\mathbf{od} \colon \mathscr{K}_{\mathcal{D}}(\overline{y}, x) \to \mathscr{K}(\overline{y}, \underline{x})$. $\mathscr{K}_{\mathcal{D}}(\overline{y}, x)$ is equivalent to $\mathscr{K}(\overline{y}, \overline{x})$, and it follows that $\overline{\alpha} = \overline{\alpha}'$ is cartesian for $\mathscr{K}(\overline{y}, x)$. The rest of Definition 7.3 (a) follows by whiskering α with $\langle \overline{h}, \mathrm{id}, y\overline{h} \rangle \colon \overline{w} \to y$. ∎

7.9. PROPOSITION. *Let x be an object of $\mathscr{K}_{\mathcal{D}}$. Then x is a fibrational object with respect to $\mathbb{C}\mathbf{od}$ iff it is a fibration (Definition 7.3 (b)) in $\mathscr{K}$.*

PROOF. $\Rightarrow$: Suppose we have $\overline{e}$, $\underline{f}$ and $\underline{\alpha}$ as in Definition 7.3 (b). Any cartesian lift $\alpha\colon f \Rightarrow e$ of $\underline{\alpha}$ provides us with the required $\overline{\alpha}$, $\overline{f}$ and $\overset{\blacktriangledown}{f}$; then by Lemma 7.8, $\overline{\alpha}$ is cartesian.

$\Leftarrow$: Condition $(B1)$ of Definition 7.7 follows from Proposition 7.5, as every 1-cell in $\mathcal{D}$ is bicarrable in $\mathscr{K}$.

For condition $(B2)$, suppose we have $e\colon y \to x$ and $\underline{\alpha} : \underline{f} \Rightarrow \underline{e}\colon \underline{y} \to \underline{x}$. Let $\overline{\alpha}\colon \overline{f} \Rightarrow \overline{e}$, with $\overset{\blacktriangledown}{f}$, be a cartesian lift of $\overset{\blacktriangledown}{e}{}^{-1} \circ (\underline{\alpha} \cdot y)\colon \underline{f}y \to x\overline{e}$. These provide us with $\alpha\colon f \Rightarrow e$ as required, and it is cartesian by Lemma 7.8.

For condition $(B3)$, let $\alpha\colon g \Rightarrow f\colon y \to x$ be cartesian for $\mathbb{C}\mathbf{od}$. The above construction applied to $\underline{\alpha}$, together with uniqueness of cartesian lifts up to isomorphism, shows that $\overline{\alpha}$ is cartesian for x. If $h\colon w \to y$ then it is clear from the definition that $\overline{\alpha} \cdot \hbar$ is also cartesian for x, and now by Lemma 7.8 $\alpha \cdot h$ is cartesian for $\mathbb{C}\mathbf{od}$. ∎

Note that the weak nature of Definition 7.3 (b), with the appearance of $\overset{\blacktriangledown}{f}$, is dealt with in the definition of $\mathscr{K}_{\mathcal{D}}$, where 1-cells are squares commuting only up to isomorphism. Definition 7.7 $(B2)$ always uses strict fibrations.

7.10. REMARK. Suppose x is a fibrational object in $\mathscr{K}_{\mathcal{D}}$, and $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}\colon \underline{y} \to \underline{x}$ is a 2-cell. Let $f\colon x_f \to x$ and $g\colon x_g \to x$ be cartesian lifts of $\underline{f}$ and $\underline{g}$ (obtained as bipullbacks), so $\underline{x}_f = \underline{x}_g = \underline{y}$. By $(B2)$ $\underline{\alpha}$ has a cartesian lift $\alpha'\colon f' \Rightarrow g$. We can then factor f', up to an iso-2-cell γ, as $f \circ \ell_\alpha$ where ℓ_α is vertical ($\underline{\ell}_\alpha = \mathrm{id}_{\underline{y}}$) and $f\colon x_f \to x$. From α' and γ we obtain a cartesian 2-cell $\alpha\colon f \circ \ell_\alpha \Rightarrow g$ (diagram on the left below).

After appropriate changes of notation, this translates into the diagram on the right, which summarizes the structure used in [Joh02, B4.4.1].
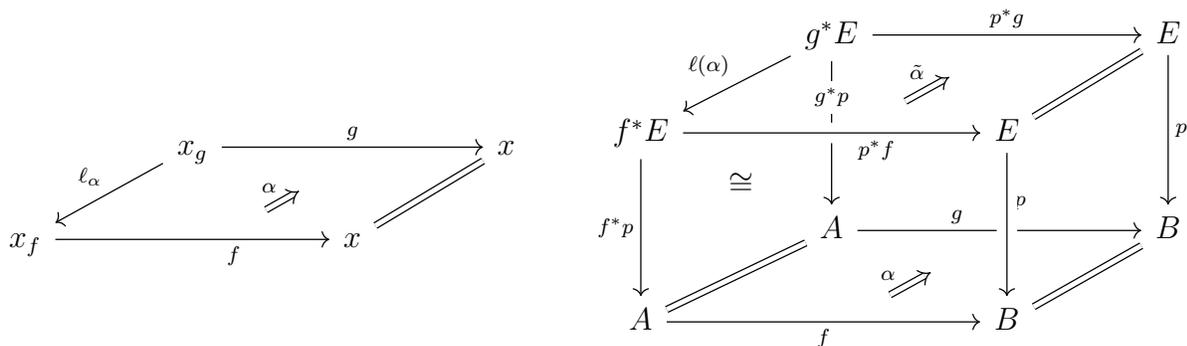
That structure, together with various coherence isomorphisms to deal with the fact that x_f and x_g are defined only up to equivalence, and ℓ_α only up to isomorphism, is the elementary unravelling of a definition shown in [Joh93, Proposition 3.3] to be equivalent to the representable definition of fibration. (Note how our consistent notation allows the diagrams of [Joh93] to be collapsed in the vertical dimension.)

Despite being quite complex, this description is closer to underlying intuitions of fibration as indexed category, and to some of our motivating examples. The 1-cells $\underline{f}$ and $\underline{g}$ may be thought of as "generalized objects" of the base $\underline{x}$, and then x_f and x_g are the generalized fibres over them. Then for a generalized morphism $\underline{\alpha}$, ℓ_α gives (contravariantly) the corresponding map between the fibres. $\overline{\alpha}$ shows how morphisms in $\overline{x}$ work across fibres.

8. Main results

We are now at a stage that we can state our main theorem.

8.1. LEMMA. *Let $U \colon \mathbb{T}_1 \to \mathbb{T}_0$ be a context extension map with the fibration property in the Chevalley style (Definition 6.2), let M be a model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$, and let $p \colon \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ be the classifier for $\mathbb{T}_1/M$ with generic model G. Suppose $f, g \colon q \rightrightarrows p$ are 1-cells in $\mathcal{G}\mathfrak{Top}$ and $\alpha \colon f \Rightarrow g$ a 2-cell. We write $\varphi := \alpha^\star(G, M)$, so that $\overline{\varphi} = \alpha^\star G$ is a model of $\mathbb{T}_1^{\rightarrow}$ in $\overline{q}$.*

Then α is a cartesian 2-cell (in $\mathcal{G}\mathfrak{Top}$ over $\mathcal{E}\mathfrak{Top}$) iff $\eta_{\overline{\varphi}}$ is an isomorphism, where $(\eta_{\overline{\varphi}}, \mathrm{id})$ is the unit for $\overline{q}$-$\mathbf{Mod}$-$\Gamma_1 \dashv \overline{q}$-$\mathbf{Mod}$-$\Lambda_1$.

PROOF. ($\Rightarrow$): Let $\overline{N}$ be the domain of $\overline{\varphi}.\Gamma_1.\Lambda_1$, and let $\underline{N} := \underline{f}^\star M$. Then (see diagram (3))

$$\overline{N}.U = \overline{\varphi}.\Gamma_1.\Lambda_1.\Gamma_0.\pi_0.U = \overline{\varphi}.\Gamma_1.\Lambda_1.U^{\rightarrow}.\mathrm{dom} = \overline{\varphi}.\Gamma_1.\Lambda_1.\Gamma_1.U_1.\mathrm{dom}$$
$$= \overline{\varphi}.\Gamma_1.U_1.\mathrm{dom} = \overline{\varphi}.U^{\rightarrow}.\mathrm{dom} = \overline{\varphi}.\mathrm{dom}.U = (f^\star G).U = q^\star \underline{N},$$

and so $N := (\overline{N}, \underline{N})$ is a model of U in q.

By the classifier property of p (Proposition 5.5), and taking $\underline{e} := \underline{f}$ and $e_- := \mathrm{id} \colon \underline{N} = \underline{f}^\star M$, we obtain $e \colon q \to p$ and $(e^-, \mathrm{id}) \colon N \cong e^\star(G, M)$. Now by Proposition 5.7 we get a unique $\gamma \colon e \Rightarrow g$ over $\underline{\gamma} := \underline{\alpha}$ such that $\overline{\varphi}.\Gamma_1.\Lambda_1 = (\gamma^\star G)e^-$. Again by Proposition 5.7 we get a unique $\beta' \colon f \Rightarrow e$ over $\mathrm{id}_{\underline{f}}$ such that $e^- \eta_{\overline{\varphi}} = \beta'^\star G$, and since $(\gamma^\star G)(\beta'^\star G) = \alpha^\star G$ it follows that $\gamma\beta' = \alpha$.

By cartesianness of α we also have a unique $\beta \colon e \Rightarrow f$ over $\mathrm{id}_{\underline{f}}$ such that $\gamma = \alpha\beta$, and since $\alpha\beta\beta' = \gamma\beta' = \alpha$ it follows that $\beta\beta' = \mathrm{id}_f$. We deduce that $(\beta^\star G)e^- \eta_{\overline{\varphi}} = \mathrm{id}_{f^\star G}$.

Finally $\eta_{\overline{\varphi}}(\beta^\star G)e^- = \mathrm{id}_{\overline{N}}$ follows from the adjunction $\Gamma_1 \dashv \Lambda_1$, because both sides reduce by Γ_1 to the identity. Hence $\eta_{\overline{\varphi}}$ is an isomorphism, with inverse $(\beta^\star G)e^-$.

($\Leftarrow$): Let $e\colon q \to p$ with $\gamma\colon e \Rightarrow f$ such that $\underline{\gamma} = \underline{\alpha}\underline{\beta}$.

By the adjunction $\Gamma_1 \dashv \Lambda_1$ there is a unique $\mathbb{T}_1$-morphism $\overline{\psi}\colon e^\star G \to \overline{N}$ over $\underline{\beta}^\star M$ such that $(\overline{\varphi}\bullet\Gamma_1\bullet\Lambda_1)\overline{\psi} = \gamma^\star G$. Because $\eta_{\overline{\varphi}}$ is an isomorphism this corresponds to a unique $\overline{\psi}'\colon e^\star G \to f^\star G$ over $\underline{\beta}^\star M$ such that $\overline{\varphi}\overline{\psi}' = \gamma^\star G$. By Proposition 5.7 this corresponds to a unique $\beta\colon e \Rightarrow f$ over $\underline{\beta}$ such that $(\alpha^\star G)(\beta^\star G) = \gamma^\star G$, i.e. unique such that $\alpha\beta = \gamma$. This proves that α is cartesian. ∎

8.2. THEOREM. *If $U\colon \mathbb{T}_1 \to \mathbb{T}_0$ is a context extension map with (op)fibration property in the Chevalley style (Definition 6.2), and M a model of $\mathbb{T}_0$ in an elementary topos $\mathcal{S}$, then $p\colon \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ is an (op)fibration in the 2-category $\mathcal{ETop}$ by the representable definition.*

PROOF. Here we only prove the theorem for the case of fibrations. A proof for the opfibration case is similarly constructed. According to Proposition 7.9, in order to establish that p is a fibration in the 2-category $\mathcal{ETop}$, we have to verify that conditions (B1)-(B3) in Definition 7.7 hold for $P = \mathbb{C}\mathrm{od}\colon \mathcal{K}_{\mathcal{D}} \to \mathcal{K}$, where $\mathcal{K} = \mathcal{ETop}$, $\mathcal{D}$ is the class of bounded geometric morphisms, and so $\mathcal{K}_{\mathcal{D}}$ is $\mathcal{GTop}$.

By Proposition 7.5, condition (B1) follows from the fact that p is bicarrable.

To prove condition (B2), let $q\colon \overline{q} \to q$ be a bounded geometric morphism, let $g\colon q \to p$ be a 1-cell in $\mathcal{K}_{\mathcal{D}}$, let $\underline{f}\colon \underline{q} \to \mathcal{S}$ be geometric morphism and $\underline{\alpha}\colon \underline{f} \Rightarrow \underline{g}$ a geometric transformation.

We seek f over $\underline{f}$ with a cartesian lift $\alpha\colon f \Rightarrow g$ of $\underline{\alpha}$. Notice that for the given model M of $\mathbb{T}_0$ in $\mathcal{S}$, the component M of the natural transformation $\underline{\alpha}$ gives us a morphism

$\underline{\alpha}^\star M \colon \underline{f}^\star M \to \underline{g}^\star M$ of $\mathbb{T}_0$-models in $\underline{q}$, hence a $\mathbb{T}_0^{\to}$-model in $\underline{q}$. Let us write it as $\underline{\varphi} \colon \underline{N}_f \to \underline{N}_g$. Then $q^\star \underline{\varphi}$ is a model of $\mathbb{T}_0^{\to}$ in $\overline{q}$.

Let G be the generic model of $\mathbb{T}_1/M$ in $\mathcal{S}[\mathbb{T}_1/M]$, so that (G, M) is a model of U in p. Hence we get $(\overline{N}_g, \underline{N}_g) := g^\star(G, M)$ a model of U in q, and

$$\mathfrak{g} := (\overline{N}_g, q^\star \underline{\varphi}) \in \overline{q}\text{-}\mathbf{Mod}\text{-}\mathrm{cod}^\ast(\mathbb{T}_1).$$

Then $\mathfrak{g} \centerdot \Lambda_1$ (see diagram (3)) is a model $\overline{\varphi} \colon \overline{N}_f \to \overline{N}_g$ of $\mathbb{T}_1^{\to}$ in $\overline{q}$, with $\overline{N}_f = \mathfrak{g} \centerdot (\Lambda_1; \Gamma_0; \pi_0)$. We also see that $\overline{\varphi} \centerdot U^{\to} = \mathfrak{g} \centerdot (\Lambda_1; U^{\to}) = q^\star \underline{\varphi}$, so $\varphi := (\overline{\varphi}, \underline{\varphi}) \colon N_f \to N_g$ is a homomorphism of U-models in q.

$$
\begin{array}{ccc}
\overline{N}_f & \xrightarrow{\ \overline{\varphi}\ } & \overline{N}_g \\
\downarrow & & \downarrow \\
q^\star \underline{N}_f & \xrightarrow{\ q^\star \underline{\varphi}\ } & q^\star \underline{N}_g
\end{array}
$$

Thus we get two objects (q, N_f) and (q, N_g) of $\mathcal{GTop}_{\cong}\text{-}U$ together with φ as in Proposition 5.7. In addition we have $(p, (G, M))$, and a P-cartesian 1-cell

$$(g, (\mathrm{id} \colon \overline{N}_g = g^\star G, \mathrm{id} \colon \underline{N} = \underline{g}^\star M)) \colon (q, N_g) \to (p, (G, M)).$$

By the classifier property we can also find a P-cartesian 1-cell

$$(f, (f^-, f_-)) \colon (q, N_f) \to (p, (G, M)).$$

We can now apply Proposition 5.7 to find a 2-cell $\alpha \colon f \Rightarrow g$ over $\underline{\alpha}$ that gives us $\overline{\varphi}$.

Since $\overline{\varphi}$ is defined to be of the form $\mathfrak{g} \centerdot \Lambda_1$, so $\overline{\varphi} \centerdot \Gamma_1 \centerdot \Lambda_1 = \overline{\varphi}$, we find that $\eta_{\overline{\varphi}}$ is the identity and $\eta_{\alpha^\star G}$ is an isomorphism. It follows from Lemma 8.1 that α is cartesian.

For proving (B3), suppose we have $f, g \colon q \rightrightarrows p$ and a cartesian 2-cell $\alpha \colon f \Rightarrow g$. By Lemma 8.1, $\eta_{\alpha^\star G}$ is an isomorphism. Take any 1-cell $k \colon q' \to q$ in $\mathcal{GTop}$ where $q' \colon \overline{q}' \to \underline{q}'$. Relative to the isomorphism of models $k^\star(\mathfrak{g} \centerdot \Lambda_1) \cong (k^\star \mathfrak{g}) \centerdot \Lambda_1$, $k^\star$ preserves the unit η, and so $\eta_{k^\star \alpha^\star G}$ is an isomorphism and, by Lemma 8.1, $\alpha \centerdot k$ is cartesian. ∎

8.3. REMARK. There is a shorter proof[2] of Theorem 8.2 which circumvents the 2-functor $\mathbb{Cod}$ and directly uses Proposition 5.7. It goes as follows: for an (op)fibration $U \colon \mathbb{T}_1 \to \mathbb{T}_0$ of AU-contexts, and any AU $\mathcal{A}$, we get a Grothendieck (op)fibration $\mathcal{A}\text{-}\mathbf{Mod}\text{-}U \colon \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathbb{T}_1 \to \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathbb{T}_0$ of categories (of strict models), and moreover, given an (strict) AU-functor $F \colon \mathcal{A} \to \mathcal{B}$, we get a morphism of Grothendieck (op)fibrations (i.e. a strictly commuting square where the top functor preserves cartesian arrows):

$$
\begin{array}{ccc}
\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathbb{T}_1 & \xrightarrow{F\text{-}\mathbf{Mod}\text{-}\mathbb{T}_1} & \mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathbb{T}_1 \\
{\scriptstyle \mathcal{A}\text{-}\mathbf{Mod}\text{-}U}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathcal{B}\text{-}\mathbf{Mod}\text{-}U} \\
\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathbb{T}_0 & \xrightarrow[F\text{-}\mathbf{Mod}\text{-}\mathbb{T}_0]{} & \mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathbb{T}_0
\end{array}
$$

---

[2]We are indebted to the anonymous referee for this elegant proof.

Therefore the right vertical functor in the diagram below is a Grothendieck fibration of presheaves of categories (or equivalently a presheaf of Grothendieck fibrations and fibred functors), where $J \colon \mathcal{ETop} \hookrightarrow \mathcal{A}\mathrm{U}^{\mathrm{op}}$ is the inclusion of the 2-category of elementary toposes into the opposite of 2-category of AUs.

$$
\begin{array}{ccc}
\mathcal{ETop}(-, \mathcal{S}[\mathbb{T}_1/M]) & \xrightarrow{\widehat{G}} & J(-)\text{-}\mathbf{Mod}\text{-}\mathbb{T}_1 \\
{\scriptstyle p_*}\Big\downarrow & & \Big\downarrow{\scriptstyle J(-)\text{-}\mathbf{Mod}\text{-}U} \\
\mathcal{ETop}(-, \mathcal{S}) & \xrightarrow[\widehat{M}]{} & J(-)\text{-}\mathbf{Mod}\text{-}\mathbb{T}_0
\end{array}
\tag{5}
$$

In above $\widehat{M}$ and $\widehat{G}$ are respectively the natural transformations induced by strictifying pullbacks of models M and G.

Now, we claim that Proposition 5.7 implies that the diagram above is a bipullback of presheaves of categories. To witness this pointwise, for any elementary topos $\mathcal{E}$, having a geometric morphism $\underline{f} \colon \mathcal{E} \to \mathcal{S}$ of elementary toposes together with a strict model N of $\mathbb{T}_1$ in $\mathcal{E}$ which reduces, via U, to $\widehat{M}(\underline{f}) := \underline{f}^\star M$ is equivalent to a 1-cell $f \colon 1_E \to p$ in $\mathcal{GTop}$ which is the same as a geometric morphism $\overline{f} \colon \mathcal{E} \to \mathcal{S}[\mathbb{T}_1/M]$ satisfying $p \circ \overline{f} \cong \underline{f}$ and $\overline{f}^\star G = N$. It is easy to see that these pointwise bipullbacks are preserved under the base change along arbitrary geometric morphisms $\mathcal{E}' \to \mathcal{E}$, and therefore, the diagram (5) is a bipullback of presheaves of categories.

It is a classical fact that a bipullback of a Grothendieck fibration is a Street (aka weak) fibration, and this fact implies that p_* is a Street fibration, and therefore by [Joh93, Proposition 3.3], the geometric morphism $p \colon \mathcal{S}[\mathbb{T}_1/M] \to \mathcal{S}$ is a fibration in the 2-category $\mathcal{ETop}$.

Theorem 8.2 can now be applied to the examples in section 6.

9. Examples and applications

We begin this section by the detailing one of the simplest non-trivial opfibrations of contexts, which classifies the base-independent local homeomorphisms of toposes via Theorem 8.2. We then outline two other important examples of context extensions, the first an opfibration and the second a fibration. We do not have space here to give full details as sketches. Rather, our aim is to explain why the known geometric theories can be expressed as contexts. At the end of this section we conjecture several other examples. Further details are in [Haz19].

9.1. EXAMPLE. *Local homeomorphisms are opfibrations.* The context extension $U \colon \mathbb{O}_\bullet \to \mathbb{O}$ (Example 4.2) is an extension map with the opfibration property. First we form the pullbacks of the context extension U along the two context maps dom and cod. U_0 and U_1 are U *reindexed* along dom and cod: the same simple extension steps, but with the

data for each transformed by dom or cod.

$$
\begin{array}{ccc}
\mathrm{dom}^*(\mathbb{O}_\bullet) \xrightarrow{\pi_0} \mathbb{O}_\bullet & \qquad & \mathrm{cod}^*(\mathbb{O}_\bullet) \xrightarrow{\pi_1} \mathbb{O}_\bullet \\
\downarrow{\scriptstyle U_0} \qquad \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U_1} \qquad \downarrow{\scriptstyle U} \\
\mathbb{O}^{\to} \xrightarrow{\mathrm{dom}} \mathbb{O} & & \mathbb{O}^{\to} \xrightarrow{\mathrm{cod}} \mathbb{O}
\end{array}
$$

The context $\mathrm{dom}^*(\mathbb{O}_\bullet)$ has three nodes, a terminal 1, primitive nodes X_0 and X_1, with edges $x_0 \colon 1 \to X_0$, $\theta_X \colon X_0 \to X_1$, and identities on the three nodes. $\mathrm{cod}^*(\mathbb{O}_\bullet)$ is similar, but with $x_1 \colon 1 \to X_1$ instead of x_0.

There is, in addition, the arrow context $\mathbb{O}_\bullet^{\to}$ which consists of all the nodes, edges, and two commutativities $\theta_X x_0 \sim \theta_x$, $x_1 \theta_1 \sim \theta_x$ (marked by bullet points) as presented in the following diagram plus identity edges.

$$
\begin{array}{ccc}
1_1 & \xrightarrow[\bullet]{x_1} & X_1 \\
{\scriptstyle \theta_1}\uparrow & {\scriptstyle \theta_x}\nearrow & \uparrow{\scriptstyle \theta_X} \\
1_0 & \xrightarrow[x_0]{\bullet} & X_0
\end{array}
$$

There are context maps Γ_0 and Γ_1 which make the following diagram commute:

Γ_0 is the dual to the sketch morphism $\mathrm{dom}^* \mathbb{O}_\bullet \to \mathbb{O}_\bullet^{\to}$ that takes 1 to 1_0 and otherwise preserves notation. Γ_1 is similar.

More interestingly, Γ_0 has a left adjoint $\Lambda_0 \colon \mathrm{dom}^*(\mathbb{O}_\bullet) \to \mathbb{O}_\bullet^{\to}$. For this, X_0, θ_X, X_1 and x_0 in $\mathbb{O}_\bullet^{\to}$ are interpreted in $\mathrm{dom}^* \mathbb{O}_\bullet$ by the ingredients with the same name, and 1_0, 1_1 by 1 and θ_1 by the identity on 1. For θ_x and x_1 we need an equivalence extension of $\mathrm{dom}^* \mathbb{O}_\bullet$ got by adjoining the composite $\theta_X x_0$, and a commutativity for one of the unit laws of composition.

It is now obvious that $\Gamma_0 \Lambda_0 = \mathrm{id} \colon \mathrm{dom}^*(\mathbb{O}_\bullet) \to \mathrm{dom}^*(\mathbb{O}_\bullet)$. Less obvious, but true in this example, is that $\Lambda_0 \Gamma_0$ is the identity on $\mathbb{O}_\bullet^{\to}$. This follows from the rules for objective equality, and is essentially because in any strict model 1_0 and 1_1 are both interpreted as the canonical terminal object, and θ_1 as the identity on that.

Now, let $\mathcal{B}$ be a bounded topos over a base topos $\mathcal{S}$, and $f \colon M \to N$ a morphism in $\mathcal{B}$. The geometric morphism $p_U \colon \mathcal{S}[\mathbb{O}_\bullet] \to \mathcal{S}[\mathbb{O}]$ induced by the context extension map U

is an opfibration and a local homeomorphism. Using Remark 7.10, we get the 1-cells and 2-cells in the left diagram below where the inverse image of the geometric morphism ℓ_f is the base change f^* and the direct image is formed by the dependent product Π_f along f.
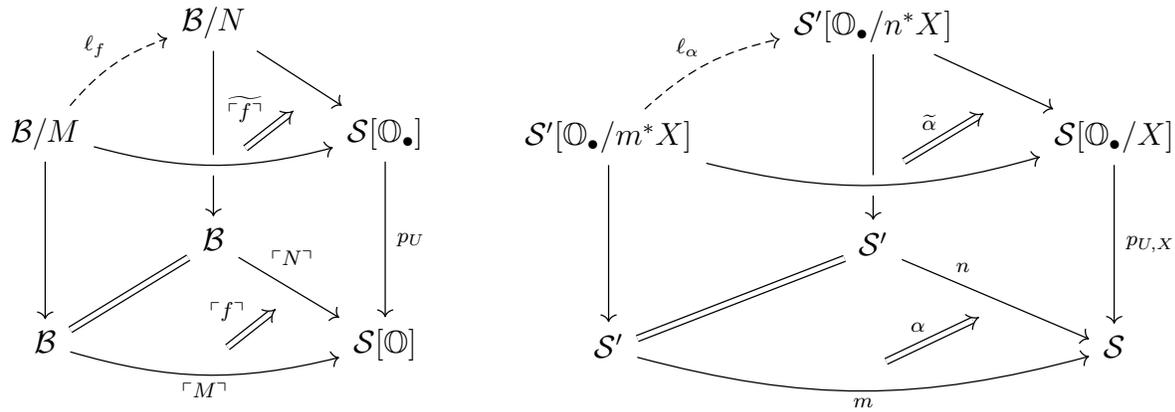
The diagram on the right is one that arises out of Theorem 8.2, and the two have the same formal shape. Note however some crucial differences. The left diagram works only for bounded $\mathcal{S}$-toposes $\mathcal{B}$ (i.e. the opfibration property of p_U is limited to the 2-category $\mathcal{BZop}/\mathcal{S}$), and the geometric transformations are restricted to being over $\mathcal{S}$. In the right diagram, $\mathcal{S}$ is a base only for the purposes of constructing $\mathcal{S}[\mathbb{O}_\bullet/X]$. The opfibration property works for arbitrary elementary topos $\mathcal{S}'$, geometric morphisms m and n, and geometric transformation α.

The classifiers for this example are the local homeomorphisms. Their opfibrational character follows simply from our results, though note that it can also be deduced as a special case of the torsor result, Example 9.2.

9.2. EXAMPLE. *Presheaf toposes are opfibrations.* Let $\mathbb{T}_0 = [C\colon \mathrm{Cat}]$ be the theory of categories. It includes nodes C_0 and C_1, primitive nodes introduced for the objects of objects and of morphisms; edges $d_0, d_1\colon C_1 \to C_0$ for domain and codomain and an edge for identity morphisms; another node C_2 for the object of composable pairs and introduced as a pullback; an edge $c\colon C_2 \to C_1$ for composition; and various commutativities for the axioms of category theory. The technique is general and would apply to any finite cartesian theory – this should be clear from the account in [PV07].

Now let us define the extension $\mathbb{T}_1 = [C\colon \mathrm{Cat}][T\colon \mathrm{Tor}(C)]$, where $\mathrm{Tor}(C)$ denotes the theory of torsors (flat presheaves) over C. The presheaf part is expressed by the usual procedure for internal presheaves. We declare a node T_0 with an edge $p\colon T_0 \to C_0$, and let T_1 be the pullback along d_1. Then the morphism part of the presheaf defines $xu = F(u)(x)$ if $d_1(u) = p(x)$, and this is expressed by an edge from T_1 to T_0 over d_1 satisfying various conditions which gives a right action by C on T. In fact this is another cartesian theory.

The flatness conditions are not cartesian, but are still expressible using contexts. First we must say that T_0 is non-empty: the unique morphism $T_0 \to 1$ is epi, in other words the cokernel pair has equal injections. Second, if $x, y \in T_0$ then there are u, v, z such that

$x = zu$ and $y = zv$. Third, if $xu = xv$ then there are w, z such that $x = zw$ and $wu = wv$. Again, these can be expressed by saying that certain morphisms are epi.

Now we have a context extension map $U \colon \mathbb{T}_1 \to \mathbb{T}_0$, which forgets the torsor.

Contexts $\mathbb{T}_0$ and $\mathbb{T}_1$, like all contexts, are finite. In section 5 we saw how for an infinite category C we can still access the infinite theory $\mathrm{Tor}(C)$ (infinitely many sorts and axioms, infinitary disjunctions) as $\mathbb{T}_1/C$, the "fibre of U over C".

We now demonstrate that U is indeed an opfibration. Consider a functor $F \colon C \to D$. If T is a torsor over C, we must define a torsor $T' = \mathrm{Tor}(F)(T)$ over D. Analogously let us write D as a C-D-bimodule, with a right action by D by composition, and a left action by C by composition after applying F. We define $\mathrm{Tor}(F)(T)$, a D-torsor, as the tensor $T \otimes_C D$. Its elements are pairs (x, f) with $x \in T$, $f \in D_1$ and $p(x) = d_1(f)$, modulo the equivalence relation generated by $(x, uf) \sim (xu, f)$. This can be defined using AU structure. Let us analyse an equation $(x, f) = (x', f')$ in more detail. It can be expressed as a chain of equations

$$(yu, k) \sim^{-1} (y, uk) = (y, u'k') \sim (yu', k'),$$

each for a quintuple (k, u, y, u', k') with $uk = u'k'$. Hence the overall equation $(x, f) = (x', f')$ derives from sequences (k_i) $(0 \le i \le n)$ and $(u_i), (y_i), (u'_i)$ $(0 \le i < n)$ such that $u_i k_i = u'_i k_{i+1}$, $y_i u'_i = y_{i+1} u_{i+1}$, $f = k_0$, $x = y_0 u_0$, $f' = k_n$ and $x' = y_{n-1} u'_{n-1}$. (We are thinking of k'_i as k_{i+1}.) By flatness of T we can replace the y_is by elements yv_i with $v_i u'_i = v_{i+1} u_{i+1}$, $x = yv_0 u_0$ and $x' = yv_{n-1} u'_{n-1}$.

We outline why $\mathrm{Tor}(F)(T)$ is flat (over D). First, it is non-empty, because T is. If $x \in T$ then $(x, \mathrm{id}_{F(p(x))}) \in \mathrm{Tor}(F)(T)$. Next, suppose $(x, f), (x', f') \in \mathrm{Tor}(F)(T)$. We can find y, u, u' with $x = yu$ and $x' = yu'$, and then $(x, f) = (yu, f) = (y, uf) = (y, \mathrm{id})uf$ and $(x', f') = (\mathrm{id}, y)u'f'$.

Finally, suppose $(x, g)f = (x, g)f'$. We must find h, g', y such that $hf = hf'$ and $(x, g) = (y, g')h$. Composing g' and h, we can instead look for $(y, h) = (x, g)$ such that $hf = hf'$. In fact, we can reduce to the case where $g = \mathrm{id}$. Suppose, then that we have $(x, f) = (x, f')$. By the analysis above, we get y and sequences $(k_i), (u_i), (v_i), (u'_i)$ such that $u_i k_i = u'_i k_{i+1}$, $v_i u'_i = v_{i+1} u_{i+1}$, $f = k_0$, $x = yv_0 u_0$, $f' = k_n$ and $x = yv_{n-1} u'_{n-1}$. Using flatness of T again, we can assume $v_0 u_0 = v_{n-1} u'_{n-1}$. Now put $h := v_0 u_0$, so $(y, h) = (y, v_0 u_0) = (yv_0 u_0, \mathrm{id}) = (x, \mathrm{id})$. Then, as required,

$$hf = v_0 u_0 k_0 = v_0 u'_0 k_1 = v_1 u_1 k_1 = \cdots = v_{n-1} u'_{n-1} k_n = hf'.$$

Although this reasoning is informal, its ingredients – and in particular the reasoning with finite sequences – are all present in AU structure.

Once we have $\mathrm{Tor}(F)(T)$ it is straightforward to define to define the function $T \to \mathrm{Tor}(F)(T)$, $x \mapsto (x, \mathrm{id})$ that makes a homomorphism of $\mathbb{T}_1$-models. Note in particular that the action is preserved: $xu \mapsto (xu, \mathrm{id}) = (x, u) = (x, \mathrm{id})u$. This gives us our Λ_0, and $\Gamma_0 \Lambda_0 = \mathrm{id}$. For the counit of the adjunction, let $(F, \theta) \colon (C, T) \to (D, T')$ be a $\mathbb{T}_1$-homomorphism. Then θ factors via $\mathrm{Tor}(F)(T)$ using $(x, f) \mapsto \theta(x)f$. This respects the equivalence, as $\theta(xu)f = \theta(x)F(u)f$ is a condition of $\mathbb{T}_1$-homomorphisms.

Note that the context extension map of Example 9.1 can be got from U above as a pullback. This is because there is a context map $\mathbb{O} \to [C \colon \mathrm{Cat}]$ taking a set X to the discrete category over it. A torsor over the discrete category is equivalent to an element of X.

The classifier for $\mathbb{T}_1/C$ is, by Diaconescu's theorem, the bounded geometric morphism $[C, \mathcal{S}] \to \mathcal{S}$ for C an internal category in $\mathcal{S}$, and hence is a typical presheaf topos. We now know that they are opfibrations in $\mathcal{E}\mathfrak{Top}$. This is already known, of course, and appears in [Joh02, B4.4.9]. Note, however, that our calculation to prove the opfibration property in $\mathfrak{Con}$ is elementary in nature. The proof of [Joh02] verifies that the class of all such geometric morphisms satisfies the "covariant tensor condition", and such a technique cannot work for AUs as it uses the direct image parts of geometric morphisms.

9.3. EXAMPLE. *Spectra of distributive lattices are fibrations.* Let $\mathbb{T}_0 = [L \colon \mathrm{DL}]$ be the finite algebraic theory of distributive lattices, a context. Now let $\mathbb{T}_1 = [L \colon \mathrm{DL}][F \colon \mathrm{Filt}(L)]$ be the theory of distributive lattices L equipped with prime filters F, and let $U \colon \mathbb{T}_1 \to \mathbb{T}_0$ be the corresponding extension map. $\mathbb{T}_1$ is built over $\mathbb{T}_0$ by adjoining a node F with a monic edge $F \to L$, and conditions to say that it is a filter (contains top and is closed under meet) and prime (inaccessible by bottom and join). For example, to say that bottom is not in F, we say that the pullback of F along bottom as edge $1 \to L$ is isomorphic to the initial object.

Given a model L of $\mathbb{T}_0$, the fibre of U over L is its spectrum $\mathrm{Spec}(L)$.

To show that U is a fibration, consider a distributive lattice homomorphism $f \colon L_0 \to L_1$. The map $\mathrm{Spec}(f) \colon \mathrm{Spec}(L_1) \to \mathrm{Spec}(L_0)$ can be expressed using contexts. It takes a prime filter F_1 of L_1 to its inverse image F_0 under f which is a prime filter of L_0. f restricts (uniquely) to a function from F_1 to F_0, and so we get a $\mathbb{T}_1$-homomorphism $f' \colon (L_1, F_1) \to (L_0, F_0)$. The construction so far can all be expressed using AU-structure, and so gives our $\Lambda_1 \colon \mathrm{cod}^*(\mathbb{T}_1) \to \mathbb{T}_1^{\to}$.

$$
\begin{array}{ccc}
(L_0, F_0 = f^{-1}(F_1)) & \xdashrightarrow{\ f'\ } & (L_1, F_1) \\
{\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\
L_0 & \xrightarrow[\ f\ ]{} & L_1
\end{array}
$$

Aided by the fact that $\Gamma_1 \colon \mathbb{T}_1^{\to} \to \mathrm{cod}^*(\mathbb{T}_1)$ is given by a sketch homomorphism (no equivalence extension of $\mathbb{T}_1^{\to}$ needed), we find that $\Gamma_1 \Lambda_1$ is the identity on $\mathrm{cod}^{-1}(\mathbb{T}_1)$. The unit $\eta \colon \mathrm{id} \Rightarrow \Lambda_1 \Gamma_1$ of the adjunction is given as follows. In $\mathbb{T}_1^{\to}$ we have a generic $f \colon (L_0, F_0) \to (L_1, F_1)$, and clearly f restricted to F_0 factors via $f^{-1}(F_1)$. Taking this with the identity on L_1 gives a $\mathbb{T}_1^{\to}$-homomorphism from $(L_0, F_0) \to (L_1, F_1)$ to $(L_0, f^{-1}(F_1) \to (L_1, F_1)$, and hence our η. The diagonal equations for the adjunction hold.

It follows that the classifier for $\mathbb{T}_1/L$, which is the spectrum of L, is a fibration. Since the spectra of distributive lattices correspond to propositional coherent theories, this fibrational nature is already known from [Joh02, B4.4.11], which says that any coherent

topos is a fibration. It will be interesting to see how far our methods can cover this general result.

We conjecture that further examples can be found as follows, from the basic idea that, given a style of presentation of spaces, homomorphisms between presentations can yield maps between the spaces.

- (Opfibration) Let $\mathbb{T}_0$ be the theory of sets equipped with an idempotent relation, and $\mathbb{T}_1$ extend it with a rounded ideal [Vic93]. Classically at least, the classifiers are the continuous dcpos with Scott topology.
- (Opfibration) Let $\mathbb{T}_0$ be the theory of generalized metric spaces, and $\mathbb{T}_1$ extend it with a Cauchy filter (point of the localic completion) [Vic05].
- (Fibration) Let $\mathbb{T}_0$ be the theory of normal distributive lattices, and $\mathbb{T}_1$ extend it with a rounded prime filter. This would be analogous to Example 9.3, and the classifiers are the compact regular spaces.
  [SVW12] discusses the consequences of the (op)fibrational characters on two topos approaches to quantum foundations. One uses point-set ideas, leading to local homeomorphisms, opfibrations. The other uses Gelfand duality, hence compact regular spaces, fibrations. That simple distinction leads to opposite ways of constructing their toposes, one with presheaves and the other with covariant functors.
- (Bifibration) Let $\mathbb{T}_0$ be the theory of strongly algebraic information systems, and let $\mathbb{T}_1$ extend it with an ideal [Vic99]. This is a special case of Example 9.2 – when the category C is a poset, then a torsor is just an ideal – and hence would be an opfibration. However, "strongly algebraic" includes extra geometric structure that restricts the homomorphisms so that they correspond to adjunctions between the corresponding domains. This would lead to an additional fibrational nature.

10. Concluding thoughts

What we have shown in this paper is that an important and extensive class of fibrations/opfibrations in the 2-category $\mathcal{ETop}$ of toposes arises from strict fibrations/opfibrations in the 2-category $\mathfrak{Con}$ of contexts. There are several advantages: first, the structure of strict fibrations/opfibrations in $\mathfrak{Con}$ is much easier to study because of explicit and combinatorial description of $\mathfrak{Con}$ and in particular due to existence of comma objects in there. Second, proofs concerning properties of based-toposes arising from $\mathfrak{Con}$ are very economical since one only needs to work with strict models of contexts. Not only does this approach help us to avoid taking the pain of working with limits and colimits in $\mathcal{ETop}/\mathcal{S}$ and bookkeeping of coherence issues arising in this way, but it also gives us insights in inner working of 2-categorical aspects of toposes via more concrete and constructive approach of contexts and context extensions.

There is also an advantage from foundational point of view; for any $\mathcal{S}$-topos $\mathcal{E}$, there are logical properties internal to $\mathcal{E}$ which are determined by internal logic of $\mathcal{S}$. A consequence of this work is that we can reason in 2-category of contexts to get uniform results about toposes independent of their base $\mathcal{S}$.

We hope that in the future work we can investigate the question that how much of 2-categorical structure of $\mathcal{E}\mathfrak{T}\mathfrak{op}$ can be presented by contexts, and more importantly whether we can find simpler proofs in $\mathfrak{Con}$ that can be transported to toposes.

11. Acknowledgements

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School of Computer Science, University of Birmingham, Birmingham, UK.

Email: `sinahazratpour@gmail.com`
         `s.j.vickers@cs.bham.ac.uk`