COCOMPLETION OF RESTRICTION CATEGORIES

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ABSTRACT.   Restriction categories were introduced as a way of generalising the notion of partial map category. In this paper, we define a notion of cocompleteness for restriction categories, and describe the free cocompletion of a small restriction category as a suitably defined category of restriction presheaves. We also consider free cocompletions in the case where our restriction category is only locally small.

1. Introduction

The notion of a partial function is ubiquitous in many areas of mathematics, including computability theory, complexity theory, algebraic geometry, algebraic topology and analysis. It is thus unsurprising that over the years, many attempts have been made to abstract the notion of partiality. One historical thread, starting with work of Ehresmann on pseudogroups of partial transformations [Ehresmann, 1957], can be traced directly to modern inverse semigroup theory [Lawson, 1998]

Another line of development is one motivated by theoretical computer science. This began with [Heller, 1983]'s introduction of *dominical categories* as an axiomatisation of a general notion of partiality. No doubt this work had an influence on [Longo & Moggi, 1984], where the authors introduced the notion of *concrete category with partial morphisms*; while [Rosolini, 1986] extends the notion of dominical category to that of a *p-category*, the main difference being the exclusion of a zero map from the axioms. At around the same time, [Carboni, 1987] studied bicategories with a partial map structure; while dominical categories and their applications to recursion theory, were considered in greater detail in [Di Paola & Heller, 1987].

In both dominical categories and *p*-categories, the domain of definition of a map $\varphi\colon X \to Y$ is not expressed in terms of a subobject of X, but rather via an idempotent $\operatorname{dom}\varphi\colon X \to X$, to be thought of as the partial identity map with the same degree of definition as φ. This establishes the connection between partiality and idempotents, which is also crucial in the context of semigroup theory. However, the step of presenting a category of partial maps *purely* in terms of a "restriction structure", assigning to each map $f\colon X \to Y$ a suitable idempotent $\bar{f}\colon X \to X$ of its domain, was not made until

[Grandis, 1990], wherein is introduced the notion of *e-cohesive category*. These were later rediscovered by Cockett and Lack, who termed them *restriction categories* and investigated their properties in the series of paper [Cockett & Lack, 2002, Cockett & Lack, 2003, Cockett & Lack, 2007].

Since restriction categories are categories with extra structure, it would not be unreasonable to think that one could give a notion of colimit in the restriction setting; for example, [Cockett & Lack, 2007, Lemma 2.1] already describes a notion of *restriction coproduct*. In this paper, as a first step towards understanding more general restriction colimits, we introduce a notion of *cocomplete restriction category*, and describe the free completion of a restriction category under colimits. In the follow-up paper [Lin, 2019], the second author extends this notion of restriction cocompletion to join restriction categories, and uses it to characterise the *manifold completion* of a join restriction category as described in [Grandis, 1990]. Possible future work would be to extend our notions to categories endowed with a restriction *tangent* structure [Cockett & Cruttwell, 2014], and thereby show that the free cocompletion of a restriction tangent category also has a restriction tangent structure.

The starting point for our work, in Section 2, is a revision of background material from [Cockett & Lack, 2002]. In particular, we recall the notions of restriction category and of $\mathcal{M}$-category, and the relation between the two. An $\mathcal{M}$-*category* is a category $\mathcal{C}$ endowed with a class $\mathcal{M}$ of monomorphisms which is closed under composition and pullback-stable; each such gives rise to a restriction category $\mathbf{Par}(\mathcal{C}, \mathcal{M})$ of "$\mathcal{M}$-partial maps", and a restriction category arises in this way precisely when it is *split*—meaning that every idempotent $\overline{f}$ has a splitting.

In Section 3, we introduce cocomplete restriction categories, and free restriction cocompletion. It will be convenient first to study the analogous notions for $\mathcal{M}$-categories: we recall from [Cockett & Lack, 2002] the *presheaf $\mathcal{M}$-category* of an $\mathcal{M}$-category, then introduce notions of cocomplete $\mathcal{M}$-category and cocontinuous $\mathcal{M}$-functor, and prove that the category of $\mathcal{M}$-presheaves is the free $\mathcal{M}$-cocompletion. Then, using the fact that $\mathcal{M}$-categories are the same as split restriction categories, can read off a definition of cocomplete restriction category and cocontinuous restriction functor, and a construction of the free restriction cocompletion, which we are able to identify as being exactly the restriction category described in [Cockett & Lack, 2002, Theorem 3.8].

In Section 4, we give a second description of the free restriction cocompletion in terms of *restriction presheaves*. We begin by introducing the notion of restriction presheaf on a restriction category $\mathbb{X}$, and form them into a split restriction category $\mathsf{PSh}_r(\mathbb{X})$. We then show that this restriction category is equivalent to the one exhibited in Section 3 as the restriction cocompletion, and so is itself a description of the restriction cocompletion.

Finally, in Section 5, we consider how restriction cocompletion can be extended to the case of restriction categories which are not small, but only *locally small*. We begin again with the case of $\mathcal{M}$-categories, by associating to any locally small $\mathcal{M}$-category an $\mathcal{M}$-category $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ of *small* presheaves. We show that this is not only locally small and cocomplete, but also the free cocompletion of $\mathbf{C}$. Then, like before, by transporting across the equivalence with restriction categories, we are able to characterise the restriction

cocompletion of a locally small restriction categories.

2. Restriction category preliminaries

Throughout the paper, we will use boldface $\mathbf{C}$ and sometimes calligraphic $\mathcal{E}$ to denote ordinary categories and 2-categories, and reserve blackboard bold $\mathbb{X}$ for restriction categories. Later on, when we consider presheaves $P\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, we will often write the action of a map $f\colon B \to A$ in $\mathbf{C}$ on an element $x \in PA$ as $x \cdot f \in PB$.

2.1. RESTRICTION CATEGORIES. In this section, we recall the definition of a restriction category and basic lemmas from [Cockett & Lack, 2002].

2.2. DEFINITION. *A* restriction category *is a category* $\mathbb{X}$ *together with, for each pair of objects* $A, B \in \mathbb{X}$, *a function* $\mathbb{X}(A, B) \to \mathbb{X}(A, A)$, *whose action we notate by* $f \mapsto \bar{f}$, *all subject to the following conditions:*

(R1) $f \circ \bar{f} = f$;

(R2) $\bar{g} \circ \bar{f} = \bar{f} \circ \bar{g}$ *for* $f\colon A \to B$, $g\colon A \to C$;

(R3) $\overline{g \circ \bar{f}} = \bar{g} \circ \bar{f}$ *for* $f\colon A \to B$, $g\colon A \to C$;

(R4) $\bar{h} \circ f = f \circ \overline{h \circ f}$ *for* $f\colon A \to B$, $h\colon B \to C$.

The assigments $f \mapsto \bar{f}$ *collectively are called the* restriction structure *on* $\mathbb{X}$, *and we call* $\bar{f}$ *the* restriction *of* f.

(Note that (R2) corresponds with property (v) in [Robinson & Rosolini, 1988, Proposition 1.4], and (R4) with property (iii) of the same proposition).

2.3. EXAMPLES.

(1) The category of sets and partial functions $\mathbf{Set}_p$ is a restriction category, where the restriction on each partial function $f\colon A \to B$ is given by

$$\bar{f}(a) = \begin{cases} a & \text{if } f \text{ is defined at } a \in A; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

(2) In a similar way, we have a restriction category $\mathbf{Top}_p$ of topological spaces and partial functions defined on an *open* subset of the domain.

(3) If $\mathcal{E}$ is any Grothendieck topos, then there is a restriction category $\mathcal{E}_p$ whose objects are those of $\mathcal{E}$, and whose maps from X to Y are *partial maps* in $\mathcal{E}$: equivalence classes of spans $X \hookleftarrow R \to Y$ with left leg a monomorphism. We will see a more general version of this construction in Section 2.7 below.

The restriction $\bar{f}$ of any map f in a restriction category satisfies the following basic properties (see [Cockett & Lack, 2002, pp. 227, 230] for details).

2.4. LEMMA. *Let $\mathbb{X}$ be a restriction category, and let $f\colon A \to B$ and $g\colon B \to C$ be morphisms in $\mathbb{X}$. Then*

(1) $\bar{f}$ is idempotent;

(2) $\bar{f} \circ \overline{gf} = \overline{gf}$;

(3) $\overline{\bar{g}f} = \overline{gf}$;

(4) $\bar{f} = \bar{\bar{f}}$;

(5) $\bar{f} = 1$ if f is a monomorphism;

(6) $\mathbb{X}(A, B)$ has a partial order given by $f \leq f'$ if and only if $f = f' \circ \bar{f}$.

(Note the fact that $\bar{f}$ is idempotent corresponds with property (vi) in [Robinson & Rosolini, 1988, Proposition 1.4], and that $\overline{\bar{g}f} = \overline{gf}$ is property (ii)).

A map $f \in \mathbb{X}$ is called a *restriction idempotent* if $f = \bar{f}$, and is *total* if $\bar{f} = 1$. If $f\colon A \to B$ and $g\colon B \to C$ are total maps in a restriction category, then gf is also total since $\overline{gf} = \overline{\bar{g}f} = \bar{f} = 1$. Therefore, as identities are total, the objects and total maps of any restriction category $\mathbb{X}$ form a subcategory $\mathsf{Total}(\mathbb{X})$.

2.5. DEFINITION. *A functor $F\colon \mathbb{X} \to \mathbb{Y}$ between restriction categories is called a* restriction functor *if $F(\bar{f}) = \overline{F(f)}$ for all maps $f \in \mathbb{X}$, and a natural transformation $\alpha\colon F \Rightarrow G$ is a* restriction transformation *if its components are total. We denote by* **rCat** *the 2-category of restriction categories, restriction functors and restriction transformations.*

2.6. SPLIT RESTRICTION CATEGORIES. There is an important full sub-2-category $\mathbf{rCat}_s$ of **rCat**, the objects of which are restriction categories whose restriction idempotents split. Recall that a restriction idempotent $\bar{f}$ *splits* if there exist maps m and r such that $mr = \bar{f}$ and $rm = 1$. We call the maps m arising in this manner *restriction monics*.

The inclusion $\mathbf{rCat}_s \hookrightarrow \mathbf{rCat}$ has a left biadjoint K_r [Cockett & Lack, 2002, p. 242], which on objects takes a restriction category $\mathbb{X}$ to the split restriction category $\mathsf{K}_r(\mathbb{X})$ with the following data (note the construction is Freyd's splitting of idempotents [Freyd, 1964]):

Objects are pairs (A, e), where A is an object of $\mathbb{X}$ and $e\colon A \to A$ is a restriction idempotent on A;

Morphisms $f\colon (A, e) \to (A', e')$ are morphisms $f\colon A \to A'$ in $\mathbb{X}$ satisfying the condition $e'fe = f$;

Restriction of $f\colon (A, e) \to (A, e')$ is given by $\bar{f}\colon (A, e) \to (A, e)$.

The unit at $\mathbb{X}$ of this biadjunction is the restriction functor $J\colon \mathbb{X} \to \mathsf{K}_r(\mathbb{X})$ which takes an object A to $(A, 1_A)$ and a map $f\colon A \to A'$ to $f\colon (A, 1_A) \to (A', 1_{A'})$. In fact, J is a full embedding.

2.7. $\mathcal{M}$-CATEGORIES AND PARTIAL MAP CATEGORIES. A *stable system of monics* $\mathbf{C}_{\mathcal{M}}$ in a category $\mathbf{C}$ is a collection of monics in $\mathbf{C}$ which includes all isomorphisms, is closed under composition, and for which the pullback of any map in $\mathbf{C}_{\mathcal{M}}$ along any map of $\mathbf{C}$ exists and is in $\mathbf{C}_{\mathcal{M}}$. An $\mathcal{M}$-*category* [Cockett & Lack, 2002, p. 245] is a category $\mathbf{C}$ together with a stable system of monics $\mathbf{C}_{\mathcal{M}}$. We usually write this as a pair $(\mathbf{C}, \mathbf{C}_{\mathcal{M}})$, or sometimes, where the meaning is clear, simply as $\mathbf{C}$.

If $\mathbf{C}$ and $\mathbf{D}$ are $\mathcal{M}$-categories, a functor F between them is called an $\mathcal{M}$-*functor* if $m \in \mathbf{C}_{\mathcal{M}}$ implies $Fm \in \mathbf{D}_{\mathcal{M}}$ and moreover F preserves pullbacks along maps in $\mathbf{C}_{\mathcal{M}}$. If $F, G\colon \mathbf{C} \to \mathbf{D}$ are $\mathcal{M}$-functors, then a natural transformation between them is called $\mathcal{M}$-*cartesian* if the naturality square of α at each $m \in \mathbf{C}_{\mathcal{M}}$ is a pullback (cf. [Cockett & Lack, 2002, p. 247]). We denote by $\mathcal{M}\mathbf{Cat}$ the 2-category of $\mathcal{M}$-categories, $\mathcal{M}$-functors and $\mathcal{M}$-cartesian natural transformations.

Associated with any $\mathcal{M}$-category $\mathbf{C}$ is a split restriction category $\mathsf{Par}(\mathbf{C})$ called the *category of partial maps in* $\mathbf{C}$. It has the same objects as $\mathbf{C}$, while morphisms from X to Y are equivalence classes of spans $m\colon X \leftarrow Z \to Y\colon f$ with $m \in \mathbf{C}_{\mathcal{M}}$. Here, the equivalence relation is that $(m, f) \sim (n, g)$ if and only if there exists an isomorphism φ with $m\varphi = n$ and $f\varphi = g$. Composition in this category is by pullback, identities are of the form $(1, 1)$ and the restriction of (m, f) is (m, m) (cf. [Cockett & Lack, 2002, pp. 246, 247]).

The assignment $\mathbf{C} \mapsto \mathsf{Par}(\mathbf{C})$ is the action on objects of a 2-functor $\mathsf{Par}\colon \mathcal{M}\mathbf{Cat} \to \mathbf{rCat}_s$. On 1–cells, if $F\colon \mathbf{C} \to \mathbf{D}$ is an $\mathcal{M}$-functor, then $\mathsf{Par}(F)$ acts as F does on objects, and on morphisms sends (m, f) to (Fm, Ff). On 2-cells, if $\alpha\colon F \Rightarrow G$ is $\mathcal{M}$-cartesian, then $\mathsf{Par}(\alpha)$ is defined componentwise by $\mathsf{Par}(\alpha)_A = (1_{FA}, \alpha_A)$.

2.8. THEOREM. *The* 2-*functor* $\mathsf{Par}\colon \mathcal{M}\mathbf{Cat} \to \mathbf{rCat}_s$ *is an equivalence of* 2-*categories.*

PROOF. This is [Cockett & Lack, 2002, Theorem 3.4]. ∎

3. Cocompletion of restriction categories

For any small category $\mathbf{C}$, the category of presheaves $\mathsf{PSh}(\mathbf{C})$ is the *free cocompletion* of $\mathbf{C}$. That is, for any small-cocomplete category $\mathcal{E}$, the following is an equivalence of categories:

$$(-) \circ \mathbf{y}\colon \mathbf{Cocomp}(\mathsf{PSh}(\mathbf{C}), \mathcal{E}) \to \mathbf{Cat}(\mathbf{C}, \mathcal{E})$$

where $\mathbf{y}$ is the Yoneda embedding, $\mathbf{Cat}$ is the 2-category of small categories and $\mathbf{Cocomp}$ is the 2-category of small-cocomplete categories and cocontinuous functors. (For the rest of this paper, we shall take *colimits* to mean *small colimits*, and *cocomplete* to mean *small-cocomplete* unless otherwise indicated).

Our objective in this section is to show that there is an analogous notion of cocompletion for small restriction categories $\mathbb{X}$. To do so, we will exploit the 2-equivalence between $\mathcal{M}\mathbf{Cat}$ and $\mathbf{rCat}_s$, by first defining and constructing the free cocompletion of a small $\mathcal{M}$-category, and then transferring across the 2-equivalence to obtain corresponding notions for restriction categories.

3.1. AN $\mathcal{M}$-CATEGORY OF PRESHEAVES. If $\mathbf{C}$ is a small $\mathcal{M}$-category, then there are various ways of making the category $\mathsf{PSh}(\mathbf{C})$ of presheaves on the underlying category of $\mathbf{C}$ into an $\mathcal{M}$-category. For our purposes, we will be interested in the following one. We say a map $\mu\colon P \to Q$ is in $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$ if for all $D \in \mathbf{C}$ and all presheaf maps $\gamma\colon \mathbf{y}D \to Q$, there is a map $m\colon C \to D$ in $\mathbf{C}_{\mathcal{M}}$ making the following a pullback square:

$$
\begin{array}{ccc}
\mathbf{y}C & \longrightarrow & P \\
{\scriptstyle \mathbf{y}m}\downarrow & & \downarrow{\scriptstyle \mu} \\
\mathbf{y}D & \xrightarrow{\ \gamma\ } & Q
\end{array}
$$

where $\mathbf{y}\colon \mathbf{C} \to \mathsf{PSh}(\mathbf{C})$ is the usual Yoneda embedding. We denote the $\mathcal{M}$-category $(\mathsf{PSh}(\mathbf{C}), \mathsf{PSh}(\mathbf{C})_{\mathcal{M}})$ arising in this way by $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$. It is easy to see that $\mathbf{y}m \in \mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$ whenever $m \in \mathbf{C}_{\mathcal{M}}$; since the Yoneda embedding also preserves all pullbacks, it is thus an $\mathcal{M}$-functor $\mathbf{y}\colon \mathbf{C} \to \mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$.

3.2. COCOMPLETE $\mathcal{M}$-CATEGORIES. It is well known that for any small category $\mathbf{C}$, the Yoneda embedding $\mathbf{y}\colon \mathbf{C} \to \mathsf{PSh}(\mathbf{C})$ exhibits $\mathsf{PSh}(\mathbf{C})$ as the free cocompletion of $\mathbf{C}$. It is therefore natural to ask whether there is a sense in which, for a small $\mathcal{M}$-category $(\mathbf{C}, \mathbf{C}_{\mathcal{M}})$, the Yoneda embedding $\mathbf{y}\colon (\mathbf{C}, \mathbf{C}_{\mathcal{M}}) \to \mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$ exhibits $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$ as a free cocompletion of $(\mathbf{C}, \mathbf{C}_{\mathcal{M}})$. We now give appropriate definitions of *cocomplete $\mathcal{M}$-category* and *cocontinuous $\mathcal{M}$-functor* which will make this true.

3.3. DEFINITION. *An $\mathcal{M}$-category $(\mathbf{C}, \mathbf{C}_{\mathcal{M}})$ is said to be* cocomplete *if the underlying category $\mathbf{C}$ is cocomplete and the inclusion $\mathbf{C} \to \mathsf{Par}(\mathbf{C})$ preserves colimits. An $\mathcal{M}$-functor $F\colon (\mathbf{C}, \mathbf{C}_{\mathcal{M}}) \to (\mathbf{D}, \mathbf{D}_{\mathcal{M}})$ is called* cocontinuous *if the underlying functor $F\colon \mathbf{C} \to \mathbf{D}$ is cocontinuous. We denote by $\mathcal{M}\mathbf{Cocomp}$ the 2-category of cocomplete $\mathcal{M}$-categories, cocontinuous $\mathcal{M}$-functors and $\mathcal{M}$-cartesian natural transformations.*

3.4. EXAMPLE. Let $\mathbf{Set}$ denote the category of all small sets, and consider the $\mathcal{M}$-category $(\mathbf{Set}, \mathbf{Inj})$, where $\mathbf{Inj}$ are the injective functions. In this case, $\mathbf{Set}$ is cocomplete and the inclusion $\mathbf{Set} \hookrightarrow \mathsf{Par}(\mathbf{Set}, \mathbf{Inj}) \cong \mathbf{Set}_p$ has a right adjoint $X \mapsto X + \{*\}$. It follows that $(\mathbf{Set}, \mathbf{Inj})$ is a cocomplete $\mathcal{M}$-category.

In fact, as we now explain, this example is a particular instance of a much wider class of cocomplete $\mathcal{M}$-categories.

3.5. DEFINITION. *Let $\mathbf{C}$ be an $\mathcal{M}$-category. An $\mathcal{M}$-subobject of $D \in \mathbf{C}$ is an isomorphism class of $\mathbf{C}_{\mathcal{M}}$-maps with codomain D. We write $\mathbf{Sub}_{\mathbf{C}_{\mathcal{M}}}(D)$ for the set of $\mathcal{M}$-subobjects of D.*

We noted above that the Yoneda embedding sends $\mathbf{C}_{\mathcal{M}}$-maps to $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$-maps; in fact, slightly more is true. The following result is [Rosolini, 1986, Proposition 3.1.1].

3.6. LEMMA. *Let $\mathbf{C}$ be an $\mathcal{M}$-category. Then there exists an isomorphism as follows:*

$$
\mathbf{Sub}_{\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}}(\mathbf{y}C) \cong \mathbf{Sub}_{\mathbf{C}_{\mathcal{M}}}(C).
$$

Now consider an $\mathcal{M}$-category $(\mathcal{E}, \mathcal{E}_{\mathcal{M}})$ with a terminal object for which there exists a *generic $\mathcal{M}$-subobject* $\tau\colon 1 \to \Sigma$. By this, we mean an $\mathcal{E}_{\mathcal{M}}$-map $\tau\colon 1 \to \Sigma$ with the property that, for any map $m\colon A \to B$ in $\mathcal{E}_{\mathcal{M}}$, there is a unique map $\tilde{m}\colon B \to \Sigma$ making the following square a pullback:

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \tau} \\ B & \xrightarrow{\;\tilde{m}\;} & \Sigma \;. \end{array}$$

Now, suppose that the pullback functor $\tau^*\colon \mathcal{E}/\Sigma \to \mathcal{E}$ has a right adjoint Π_τ. Then by a familiar argument—see, for example, [Johnstone, 2002, Proposition 2.4.7]—$\mathcal{E}$ has a *partial map classifier* for every object $C \in \mathcal{E}$ given by the domain of $\Pi_\tau(C \times \Sigma \to \Sigma)$ and this in turn implies that the inclusion $\mathcal{E} \hookrightarrow \mathsf{Par}(\mathcal{E}, \mathcal{E}_{\mathcal{M}})$ has a right adjoint. In fact, the partial map category $\mathsf{Par}(\mathcal{E}, \mathcal{E}_{\mathcal{M}})$ is equivalent to the Kleisli category of the monad induced by the adjunction $\mathcal{E} \leftrightarrows \mathsf{Par}(\mathcal{E}, \mathcal{E}_{\mathcal{M}})$; see [Mulry, 1994, Lemma 2.10].

In this situation, $\mathcal{E} \to \mathsf{Par}(\mathcal{E}, \mathcal{E}_{\mathcal{M}})$, being a left adjoint, will necessarily preserve all colimits. Thus $(\mathcal{E}, \mathcal{E}_{\mathcal{M}})$ will be a cocomplete $\mathcal{M}$-category so long as $\mathcal{E}$ itself is cocomplete.

3.7. EXAMPLES.

(1) Let $\mathcal{E}$ be a cocomplete elementary topos and $\mathcal{M}$ the class of all monics in $\mathcal{E}$. By definition of topos, there is a generic $\mathcal{M}$-subobject, and every pullback functor has a right adjoint as $\mathcal{E}$ is locally cartesian closed. So $(\mathcal{E}, \mathcal{M})$ is a cocomplete $\mathcal{M}$-category.

(2) Similarly, if $\mathcal{E}$ is a cocomplete quasitopos and $\mathcal{M}$ the class of all regular monics in $\mathcal{E}$, then $(\mathcal{E}, \mathcal{M})$ is a cocomplete $\mathcal{M}$-category.

(3) For any small $\mathcal{M}$-category $(\mathbf{C}, \mathbf{C}_{\mathcal{M}})$, the $\mathcal{M}$-category $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$ admits a generic $\mathcal{M}$-subobject $\tau\colon 1 \to \Sigma$, where $\Sigma(C) = \mathbf{Sub}_{\mathbf{C}_{\mathcal{M}}}(C)$; see [Rosolini, 1986, Proposition 3.1.1]. Like before, $\mathsf{PSh}(\mathbf{C})$ is a cocomplete, locally cartesian closed category and so we conclude that $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$ is a cocomplete $\mathcal{M}$-category.
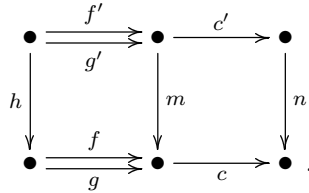
The following result gives a characterisation of cocomplete $\mathcal{M}$-categories.

3.8. PROPOSITION. *Suppose $(\mathbf{C}, \mathbf{C}_{\mathcal{M}})$ is an $\mathcal{M}$-category, and $\mathbf{C}$ is cocomplete. Then the following statements are equivalent:*

(1) $(\mathbf{C}, \mathbf{C}_{\mathcal{M}})$ is a cocomplete $\mathcal{M}$-category, i.e., $\mathbf{C} \hookrightarrow \mathsf{Par}(\mathbf{C})$ preserves colimits;

(2) The following conditions hold:

    (a) If $\{m_i\colon A_i \to B_i\}_{i\in\mathcal{I}}$ is a small family of maps in $\mathbf{C}_{\mathcal{M}}$, then the coproduct $\sum_{i\in\mathcal{I}} m_i$ is in $\mathbf{C}_{\mathcal{M}}$ and the coproduct coprojection squares below are pullbacks for every $i \in \mathcal{I}$:

$$\begin{array}{ccc} A_i & \xrightarrow{\;\iota_{A_i}\;} & \sum_{i\in\mathcal{I}} A_i \\ {\scriptstyle m_i}\downarrow & & \downarrow{\scriptstyle \sum_{i\in\mathcal{I}} m_i} \\ B_i & \xrightarrow{\;\iota_{B_i}\;} & \sum_{i\in\mathcal{I}} B_i \;. \end{array}$$

(b) *Suppose $m \in \mathbf{C}_{\mathcal{M}}$ admits the same pullback h along parallel maps f, g of $\mathbf{C}$. If f', g' are the pullbacks of f, g along m, and c, c' are the coequalisers of f, g and f', g' respectively, then the unique induced map n making the right square below commute is in $\mathbf{C}_{\mathcal{M}}$ and also makes the right square a pullback:*

$$
\begin{array}{ccccc}
\bullet & \underset{g'}{\overset{f'}{\rightrightarrows}} & \bullet & \overset{c'}{\longrightarrow} & \bullet \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} \\
\bullet & \underset{g}{\overset{f}{\rightrightarrows}} & \bullet & \underset{c}{\longrightarrow} & \bullet \;.
\end{array}
$$

(c) *Colimits are stable under pullback along $\mathbf{C}_{\mathcal{M}}$-maps.*

PROOF. For the proof of $(1) \implies (2)$, we will be using Lemma 3.14 and Corollary 3.16 (both to be proven later).
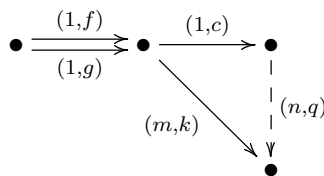
$(1) \implies (2a)$ Let $\mathcal{I}$ be a small set considered as a discrete category, and $H, K \colon \mathcal{I} \to \mathbf{C}$ be functors taking objects $i \in \mathcal{I}$ to A_i and B_i respectively. Let $\alpha \colon H \Rightarrow K$ be the natural transformation whose component at i is given by $m_i \colon A_i \to B_i$, and observe that all naturality squares are trivially pullbacks. Then by Lemma 3.14, the sum $\sum_{i \in \mathcal{I}} m_i$ is in $\mathbf{C}_{\mathcal{M}}$ and for every $i \in \mathcal{I}$, the coproduct coprojection squares are pullbacks.

$(1) \implies (2b)$ In a similar way, take $\mathbf{I}$ to be the category with two objects and a pair of parallel maps between them and apply Lemma 3.14.
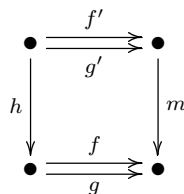
$(1) \implies (2c)$ See Corollary 3.16.

$(2) \implies (1)$ To show that the inclusion $\mathbf{C} \hookrightarrow \mathsf{Par}(\mathbf{C})$ is cocontinuous, it is enough to show that it preserves all small coproducts and coequalisers.
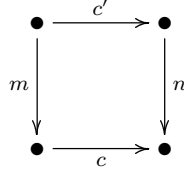
So suppose c is a coequaliser of f and g in $\mathbf{C}$. To show the inclusion preserves this coequaliser, we need to show that for any map (m, k) such that $(m, k)(1, f) = (m, k)(1, g)$, there is a unique map (n, q) making the following diagram commute:
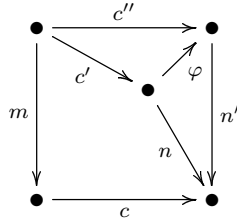
$$
\begin{array}{ccc}
\bullet \underset{(1,g)}{\overset{(1,f)}{\rightrightarrows}} \bullet & \overset{(1,c)}{\longrightarrow} & \bullet \\
& {\scriptstyle (m,k)} \searrow & \big\downarrow {\scriptstyle (n,q)} \\
& & \bullet
\end{array}
$$

Now the condition $(m, k)(1, f) = (m, k)(1, g)$ is precisely the condition that the pullbacks of m along f and g can be chosen to be the same map h, as displayed in:

$$
\begin{array}{ccc}
\bullet & \underset{g'}{\overset{f'}{\rightrightarrows}} & \bullet \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle m} \\
\bullet & \underset{g}{\overset{f}{\rightrightarrows}} & \bullet
\end{array}
$$

and that moreover $kf' = kg'$. Taking c' to be the coequaliser of f' and g', our assumption then implies there is a unique map $n \in \mathbf{C}_\mathcal{M}$ making the following diagram a pullback:
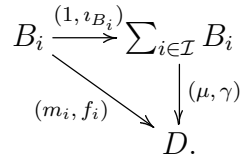
$$\begin{array}{ccc} \bullet & \xrightarrow{c'} & \bullet \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ \bullet & \xrightarrow{c} & \bullet \end{array}$$

Since c' is the coequaliser of f' and g' and $kf' = kg'$, there exists a unique map q such that $c'q = k$. This gives a map $(n, q) \in \mathsf{Par}(\mathbf{C})$ such that $(n, q)(1, c) = (m, k)$. To see it must be unique, suppose (n', q') also satisfies the condition $(n', q')(1, c) = (m, k)$. Since, by assumption, colimits are stable under pullback along $\mathbf{C}_\mathcal{M}$-maps, the pullback of c along n' must be a coequaliser of f' and g', say c''.

$$\begin{array}{ccc} \bullet & \xrightarrow{c''} & \bullet \\ {\scriptstyle m}\downarrow \; {\scriptstyle c'}\searrow & {\scriptstyle\varphi}\nearrow & \downarrow{\scriptstyle n'} \\ & \bullet & \\ & {\scriptstyle n}\searrow & \\ \bullet & \xrightarrow{c} & \bullet \end{array}$$

Now as coequalisers are unique up to isomorphism, there is an isomorphism φ such that $c'' = \varphi c'$. Now the calculation $n'\varphi c' = n'c'' = cm = nc'$ implies $n'\varphi = n$ as c' is an epimorphism, so that n and n' represent the same $\mathcal{M}$-subobject. Similarly, $q = q'\varphi$, and so we have $(n, q) = (n', q')$.

Next, suppose $\sum_{i \in \mathcal{I}} B_i$ is a small coproduct in $\mathbf{C}$, with coproduct coprojections $(\imath_{B_i} \colon B_i \to \sum_{i \in \mathcal{I}} B_i)_{i \in \mathcal{I}}$. This coproduct will be preserved by $\mathbf{C} \to \mathsf{Par}(\mathbf{C})$ if for any object $D \in \mathsf{Par}(\mathbf{C})$ and family of maps $\big((m_i, f_i) \colon B_i \to D\big)_{i \in \mathcal{I}}$, there exists a unique map $(\mu, \gamma) \colon \sum_{i \in \mathcal{I}} B_i \to D$ making the following diagram commute for every $i \in \mathcal{I}$:

$$\begin{array}{ccc} B_i & \xrightarrow{(1, \imath_{B_i})} & \sum_{i \in \mathcal{I}} B_i \\ & {\scriptstyle (m_i, f_i)}\searrow & \downarrow{\scriptstyle (\mu, \gamma)} \\ & & D. \end{array}$$

By assumption, $\sum_{i \in \mathcal{I}} m_i$ is in $\mathbf{C}_\mathcal{M}$, and so the map $\big(\sum_{i \in \mathcal{I}} m_i, f\big) \colon \sum_{i \in \mathcal{I}} B_i \to D$ is well-defined, where f is the unique map $\sum_{i \in \mathcal{I}} \mathrm{dom}(f_i) \to D$ induced from the family of maps $\{f_i\}_{i \in \mathcal{I}}$ using the universal property of coproduct. Since the coproduct coprojection squares are pullbacks, taking $\mu = \sum_{i \in \mathcal{I}} m_i$ and $\gamma = f$ certainly makes the above diagram commute. The uniqueness of (μ, γ) now follows by the stability of colimits under pullback using an analogous argument to the case of coequalisers.

Hence, as the inclusion $\mathbf{C} \hookrightarrow \mathsf{Par}(\mathbf{C})$ preserves all small coproducts and all coequalisers, it preserves all small colimits. ∎

3.9. REMARK. There is yet another formulation for the condition that the inclusion $\mathbf{C} \hookrightarrow \mathsf{Par}(\mathbf{C})$ preserves all small colimits. Namely, the inclusion is cocontinuous if and only if the presheaf $\mathbf{Sub}_{\mathbf{C}_{\mathcal{M}}} \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$, which on objects takes C to the set of $\mathcal{M}$-subobjects of C, is continuous, and moreover, colimits are stable under pullback along maps in $\mathbf{C}_{\mathcal{M}}$. The proof of this result is similar to the proof of Lemma 3.8.

Also, by conditions (2a) and (2c), observe that cocomplete $\mathcal{M}$-categories must be $\mathcal{M}$-*extensive*, meaning that for every $i \in \mathcal{I}$ (with $\mathcal{I}$ a small set), if the following square commutes with the bottom row being coproduct injections and $m, m_i \in \mathcal{M}$ (for all $i \in \mathcal{I}$), then the top row must be a coproduct diagram if and only if each square is a pullback:

$$
\begin{array}{ccc}
A_i & \longrightarrow & Z \\
{\scriptstyle m_i}\downarrow & & \downarrow{\scriptstyle m} \\
B_i & \xrightarrow[\iota_{B_i}]{} & \sum_{i \in \mathcal{I}} B_i.
\end{array}
$$

Finally, compare this characterisation of cocomplete $\mathcal{M}$-categories to the characterisation of p-categories with coproducts in [Rosolini, 1988], where it was noted that these coproducts were in fact preserved by embeddings into certain partial map categories.

We now use the previous result to give an example of an $\mathcal{M}$-category with cocomplete underlying category which is not itself cocomplete.

3.10. EXAMPLE. Consider the category $\mathbf{Ab}$ of small abelian groups, made into an $\mathcal{M}$-category by equipping it with the class of all monomorphisms. Denote the trivial group by 0 and the group of integers by $\mathbb{Z}$. The coproduct of $\mathbb{Z}$ with itself is just the direct sum $\mathbb{Z} \oplus \mathbb{Z}$ with coprojections $\iota_1 \colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ and $\iota_2 \colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ sending n to $(n, 0)$ and $(0, n)$ respectively. Let $\Delta \colon \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ denote the diagonal map, which is clearly a monomorphism, and observe that both squares in the following diagram are pullbacks:

$$
\begin{array}{ccccc}
0 & \longrightarrow & \mathbb{Z} & \longleftarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \Delta} & & \downarrow \\
\mathbb{Z} & \xrightarrow[\iota_1]{} & \mathbb{Z} \oplus \mathbb{Z} & \xleftarrow[\iota_2]{} & \mathbb{Z}.
\end{array}
$$

However, the top row is certainly not a coproduct diagram in $\mathbf{Ab}$. Therefore $(\mathbf{Ab}, \text{monos})$ is not $\mathcal{M}$-extensive, and hence by Proposition 3.8, is not a cocomplete $\mathcal{M}$-category.

3.11. COCOMPLETION OF $\mathcal{M}$-CATEGORIES. Our goal now is to show for any small $\mathcal{M}$-category $\mathbf{C}$ and cocomplete $\mathcal{M}$-category $\mathbf{D}$, the following is an equivalence:

$$
(-) \circ \mathbf{y} \colon \mathcal{M}\mathbf{Cocomp}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}), \mathbf{D}) \to \mathcal{M}\mathbf{Cat}(\mathbf{C}, \mathbf{D}).
$$

We will do so by making use of the following four lemmas.

3.12. LEMMA. *Let* **C** *be an* $\mathcal{M}$-*category and* $m \in \mathbf{C}_{\mathcal{M}}$. *The following square is a pullback:*

$$
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
{\scriptstyle n}\downarrow & & \downarrow{\scriptstyle m} \\
C & \xrightarrow[f]{} & D
\end{array}
$$

if and only if the following diagram commutes in $\mathsf{Par}(\mathbf{C})$:

$$
\begin{array}{ccc}
C & \xrightarrow{(1,f)} & D \\
{\scriptstyle (n,1)}\downarrow & & \downarrow{\scriptstyle (m,1)} \\
A & \xrightarrow[(1,g)]{} & B.
\end{array}
$$

PROOF. This is an easy diagram chase. $\blacksquare$

3.13. LEMMA. *Let* $\mathbb{X}$ *be a restriction category,* **I** *any small category and* $L \colon \mathbf{I} \to \mathbb{X}$ *a functor. Suppose* $\operatorname{colim} L$ *exists and its colimiting coprojections* $(p_I \colon LI \to \operatorname{colim} L)_{I \in \mathbf{I}}$ *are total. If* $\varepsilon \colon L \Rightarrow L$ *is a natural transformation such that each component is a restriction idempotent, then* $\operatorname{colim} \varepsilon$ *is also a restriction idempotent:*

$$
\begin{array}{ccc}
LI & \xrightarrow{p_I} & \operatorname{colim} L \\
{\scriptstyle \varepsilon_I}\downarrow & & \downarrow{\scriptstyle \operatorname{colim} \varepsilon} \\
LI & \xrightarrow[p_I]{} & \operatorname{colim} L.
\end{array}
$$

PROOF. By the facts that $\overline{p_I} = 1$ and $\varepsilon_I = \overline{\varepsilon_I}$, we have

$$
\overline{\operatorname{colim} \varepsilon} \circ p_I = p_I \circ \overline{\operatorname{colim} \varepsilon \circ p_I} = p_I \circ \overline{p_I \circ \varepsilon_I} = p_I \circ \overline{\overline{p_I} \circ \varepsilon_I} = p_I \circ \overline{\varepsilon_I} = p_I \circ \varepsilon_I.
$$

Therefore, $\operatorname{colim} \varepsilon = \overline{\operatorname{colim} \varepsilon}$ by uniqueness. $\blacksquare$

3.14. LEMMA. *Let* **C** *be a cocomplete* $\mathcal{M}$-*category, and let* $H, K \colon \mathbf{I} \to \mathbf{C}$ *be functors (with* **I** *small). Suppose* $\alpha \colon H \Rightarrow K$ *is a natural transformation such that for each* $I \in \mathbf{I}$, α_I *is in* $\mathbf{C}_{\mathcal{M}}$ *and such that all naturality squares are pullbacks:*

$$
\begin{array}{ccc}
HI & \xrightarrow{Hf} & HJ \\
{\scriptstyle \alpha_I}\downarrow & & \downarrow{\scriptstyle \alpha_J} \\
KI & \xrightarrow[Kf]{} & KJ.
\end{array}
$$

Then $\operatorname{colim} \alpha$ *is in* $\mathbf{C}_{\mathcal{M}}$, *and the following is a pullback for every* $I \in \mathbf{I}$:

$$
\begin{array}{ccc}
HI & \xrightarrow{p_I} & \operatorname{colim} H \\
{\scriptstyle \alpha_I}\downarrow & & \downarrow{\scriptstyle \operatorname{colim} \alpha} \\
KI & \xrightarrow[q_I]{} & \operatorname{colim} K
\end{array}
$$

where p_I, q_I *are colimit coprojections.*

PROOF. Applying the inclusion $\imath\colon \mathbf{C} \to \mathsf{Par}(\mathbf{C})$ gives the following commutative diagram for each $I \in \mathbf{I}$:

$$
\begin{array}{ccc}
HI & \xrightarrow{(1,p_I)} & \mathrm{colim}\,H \\
{\scriptstyle(1,\alpha_I)}\big\downarrow & & \big\downarrow{\scriptstyle(1,\mathrm{colim}\,\alpha)} \\
KI & \xrightarrow[(1,q_I)]{} & \mathrm{colim}\,K.
\end{array}
$$

Observe that there is a natural transformation $\beta\colon \imath K \Rightarrow \imath H$ whose components are given by $\beta_I = (\alpha_I, 1)$. Indeed, we may simply apply Lemma 3.12 to our assumption that α_I is a pullback of α_J along Kf.

Now the fact that the inclusion preserves the colimits $(\mathrm{colim}\,H, p_I)_{i\in\mathbf{I}}$ and $(\mathrm{colim}\,K, q_I)_{i\in\mathbf{I}}$ implies the existence of a unique map $\mathrm{colim}\,\beta = (n,g)\colon \mathrm{colim}\,K \to \mathrm{colim}\,H$ making the following diagram commute for each $I \in \mathbf{I}$:

$$
\begin{array}{ccc}
KI & \xrightarrow{(1,q_I)} & \mathrm{colim}\,K \\
{\scriptstyle(\alpha_I,1)}\big\downarrow & & \big\downarrow{\scriptstyle(n,g)} \\
HI & \xrightarrow{(1,p_I)} & \mathrm{colim}\,H \\
{\scriptstyle(1,\alpha_I)}\big\downarrow & & \big\downarrow{\scriptstyle(1,\mathrm{colim}\,\alpha)} \\
KI & \xrightarrow[(1,q_I)]{} & \mathrm{colim}\,K.
\end{array}
$$

Observe that the left composite $(1,\alpha_I)\circ(\alpha_I,1) = (\alpha_I,\alpha_I)$ is the component at I of a natural transformation $\varepsilon\colon \imath K \Rightarrow \imath K$ whose components are restriction idempotents. Therefore, by Lemma 3.13, the composite on the right $(1,\mathrm{colim}\,\alpha)\circ(n,g) = (n,(\mathrm{colim}\,\alpha)g)$ must be a restriction idempotent, and so $n = (\mathrm{colim}\,\alpha)g$.

On the other hand, the composite $(\alpha_I,1)\circ(1,\alpha_I) = (1,1)$ is the component of the identity natural transformation $\gamma\colon \imath H \Rightarrow \imath H$ at I, and so $\mathrm{colim}\,\gamma\colon \mathrm{colim}\,H \to \mathrm{colim}\,H$ must be $(1,1)$. However, as the following diagram also commutes, we must have $(n,g)\circ (1,\mathrm{colim}\,\alpha) = (1,1)$ by uniqueness:

$$
\begin{array}{ccc}
HI & \xrightarrow{(1,p_I)} & \mathrm{colim}\,H \\
{\scriptstyle(1,\alpha_I)}\big\downarrow & & \big\downarrow{\scriptstyle(1,\mathrm{colim}\,\alpha)} \\
KI & \xrightarrow{(1,q_I)} & \mathrm{colim}\,K \\
{\scriptstyle(\alpha_I,1)}\big\downarrow & & \big\downarrow{\scriptstyle(n,g)} \\
HI & \xrightarrow[(1,p_I)]{} & \mathrm{colim}\,H.
\end{array}
$$

So $(1,\mathrm{colim}\,\alpha)\circ(n,g) = (n,n)$ is a splitting of the restriction idempotent (n,n), which means that $(1,\mathrm{colim}\,\alpha)$ is a restriction monic. Therefore $\mathrm{colim}\,\alpha \in \mathbf{C}_{\mathcal{M}}$, proving the first part of the lemma.

Regarding the second part of the lemma, observe that $(n,g)\circ(1,\mathrm{colim}\,\alpha) = (1,1)$ implies g is an isomorphism (as $n = (\mathrm{colim}\,\alpha)$). Therefore, $(n,g) = (\mathrm{colim}\,\alpha, 1)$ and so the

following diagram commutes for all $I \in \mathbf{I}$:

$$
\begin{array}{ccc}
KI & \xrightarrow{(1,q_I)} & \mathrm{colim}\,K \\
{\scriptstyle(\alpha_I,1)}\Big\downarrow & & \Big\downarrow{\scriptstyle(\mathrm{colim}\,\alpha,1)} \\
HI & \xrightarrow[(1,p_I)]{} & \mathrm{colim}\,H.
\end{array}
$$

The result then follows by applying Lemma 3.12.   ■

3.15. LEMMA. *Let* $\mathbf{C}$ *be a cocomplete* $\mathcal{M}$-*category,* $H, K \colon \mathbf{I} \to \mathbf{C}$ *functors (with* $\mathbf{I}$ *small),* *and* $\alpha \colon H \Rightarrow K$ *a natural transformation such that each* $\alpha_I \in \mathbf{C}_{\mathcal{M}}$ *and all naturality* *squares are pullbacks. Let* $n \in \mathbf{C}_{\mathcal{M}}$, *and suppose* $x \colon \mathrm{colim}\,H \to X$ *and* $y \colon \mathrm{colim}\,K \to Y$ *make the right square commute, and make the outer square a pullback for all* $I \in \mathbf{I}$:

$$
\begin{array}{ccccc}
HI & \xrightarrow{p_I} & \mathrm{colim}\,H & \xrightarrow{x} & X \\
{\scriptstyle\alpha_I}\Big\downarrow & & {\scriptstyle\mathrm{colim}\,\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle n} \\
KI & \xrightarrow[q_I]{} & \mathrm{colim}\,K & \xrightarrow[y]{} & Y.
\end{array}
$$

Then the right square is also a pullback.

PROOF. By Lemma 3.12, showing that the right square is a pullback is the same as showing that the top-right square of the following diagram commutes:

$$
\begin{array}{ccccc}
KI & \xrightarrow{(1,q_I)} & \mathrm{colim}\,K & \xrightarrow{(1,y)} & Y \\
{\scriptstyle(\alpha_I,1)}\Big\downarrow & & {\scriptstyle(\mathrm{colim}\,\alpha,1)}\Big\downarrow & & \Big\downarrow{\scriptstyle(n,1)} \\
HI & \xrightarrow[(1,p_I)]{} & \mathrm{colim}\,H & \xrightarrow[(1,x)]{} & X \\
{\scriptstyle(1,\alpha_I)}\Big\downarrow & & {\scriptstyle(1,\mathrm{colim}\,\alpha)}\Big\downarrow & & \Big\downarrow{\scriptstyle(1,n)} \\
KI & \xrightarrow[(1,q_I)]{} & \mathrm{colim}\,K & \xrightarrow[(1,y)]{} & Y.
\end{array}
$$

Since $(\mathrm{colim}\,\alpha, x)$ and $(n,1)(1,y)$ are both maps out of $\mathrm{colim}\,K$, it is enough to show that

$$(\mathrm{colim}\,\alpha, x)(1, q_I) = (n, 1)(1, y)(1, q_I)$$

for all $I \in \mathbf{I}$. But the left-hand side is equal to (α_I, xp_I) by commutativity of the top-left square, and the right-hand side is also (α_I, xp_I) by assumption. Hence the result follows. ■

3.16. COROLLARY. *If* $(\mathbf{C}, \mathbf{C}_{\mathcal{M}})$ *is a cocomplete* $\mathcal{M}$-*category, then colimits in* $\mathbf{C}$ *are stable* *under pullback along* $\mathbf{C}_{\mathcal{M}}$-*maps.*

PROOF. Let $K \colon \mathbf{I} \to \mathbf{C}$ be a functor, P any object in $\mathbf{C}$, and suppose $\mu \colon P \to \operatorname{colim} K$ is a $\mathbf{C}_{\mathcal{M}}$-map. Since $\mu \in \mathbf{C}_{\mathcal{M}}$, for each $I \in \mathbf{I}$, we may take pullbacks of μ along the colimiting coprojections of $\operatorname{colim} K$, $(k_I \colon KI \to \operatorname{colim} K)_{I \in \mathbf{I}}$, and these we call $\alpha_I \colon HI \to KI$. This gives a functor $H \colon \mathbf{I} \to \mathbf{C}$, which on objects, takes I to HI, and on morphisms, takes $f \colon I \to J$ to the unique map making all squares in the following diagram pullbacks:

$$
\begin{array}{ccc}
HI \xrightarrow{Hf} HJ \xrightarrow{p_J} P \\
\alpha_I \downarrow \qquad \alpha_J \downarrow \qquad \mu \downarrow \\
KI \xrightarrow{Kf} KJ \xrightarrow{k_J} \operatorname{colim} K.
\end{array}
$$

By construction, $(P, p_I)_{I \in \mathbf{I}}$ is a cocone in $\mathbf{C}$ and $\alpha \colon H \to K$ is a natural transformation. Now form the colimit of H with colimiting coprojections $(h_I \colon HI \to \operatorname{colim} H)_{I \in \mathbf{I}}$. By the universal property of $\operatorname{colim} H$, there exists a unique $\gamma \colon \operatorname{colim} H \to P$ such that $p_I = \gamma h_I$ for all $I \in \mathbf{I}$, and by the functoriality of colimits, there is a map $\operatorname{colim} \alpha \colon \operatorname{colim} H \to \operatorname{colim} K$ making the left square of the following diagram commute for all $I \in \mathbf{I}$:

$$
\begin{array}{ccc}
HI \xrightarrow{h_I} \operatorname{colim} H \xrightarrow{\gamma} P \\
\alpha_I \downarrow \qquad \operatorname{colim} \alpha \downarrow \qquad \mu \downarrow \\
KI \longrightarrow \operatorname{colim} K = \operatorname{colim} K.
\end{array}
$$

It is easy to see that the right square commutes, and since the left square is a pullback for every $I \in \mathbf{I}$, the right square must be a pullback by Lemma 3.15. Therefore, because the pullback of the identity $1_{\operatorname{colim} K}$ is the identity, γ is invertible, so that colimits are preserved by pullbacks along $\mathbf{C}_{\mathcal{M}}$-maps. ∎

We now show that for any small $\mathcal{M}$-category $\mathbf{C}$, the Yoneda embedding $\mathbf{y} \colon \mathbf{C} \to \mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$ exhibits the $\mathcal{M}$-category of presheaves $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$ as the free cocompletion of $\mathbf{C}$.

3.17. THEOREM. *For any small $\mathcal{M}$-category $\mathbf{C}$ and cocomplete $\mathcal{M}$-category $\mathbf{D}$, the following is an equivalence of categories:*

$$(-) \circ \mathbf{y} \colon \mathcal{M}\mathbf{Cocomp}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}), \mathbf{D}) \to \mathcal{M}\mathbf{Cat}(\mathbf{C}, \mathbf{D}). \tag{3.1}$$

PROOF. Since $(-) \circ \mathbf{y} \colon \mathbf{Cocomp}(\mathsf{PSh}(\mathbf{C}), \mathbf{D}) \to \mathbf{Cat}(\mathbf{C}, \mathbf{D})$ is an equivalence of categories, we know that, given a functor $F \colon \mathbf{C} \to \mathbf{D}$, there is a cocontinuous $G \colon \mathsf{PSh}(\mathbf{C}) \to \mathbf{D}$ such that $G\mathbf{y} \cong F$. So the functor in (3.1) will be essentially surjective on objects if this G is an $\mathcal{M}$-functor whenever F is one.

To see that G takes monics in $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$ to monics in $\mathbf{D}_{\mathcal{M}}$, let $\mu\colon P \to Q$ be a map in $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$. Since every presheaf is a colimit of representables, we can write Q as a colimit $Q \cong \operatorname{colim} \mathbf{y}D$, where $D\colon \mathbf{I} \to \mathbf{C}$ is a functor (with $\mathbf{I}$ small). Since $\mu \in \mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$, we know that for every $I \in \mathbf{I}$, there is a map $m_I\colon C_I \to D_I$ in $\mathcal{M}$ making the following a pullback:

$$\begin{array}{ccc} \mathbf{y}C_I & \xrightarrow{p_I} & P \\ {\scriptstyle \mathbf{y}m_I}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathbf{y}D_I & \xrightarrow[q_I]{} & Q \end{array}$$

(where the maps q_I are the colimit coprojections). It follows there is a functor $C\colon \mathbf{I} \to \mathbf{C}$ which on objects takes I to C_I and on morphisms, takes $f\colon I \to J$ to the unique map Cf making the diagram below commute and the left square a pullback:

$$\begin{array}{ccccc} \mathbf{y}C_I & \dashrightarrow{\mathbf{y}Cf} & \mathbf{y}C_J & \xrightarrow{p_J} & P \\ {\scriptstyle \mathbf{y}m_I}\downarrow & & \downarrow{\scriptstyle \mathbf{y}m_J} & & \downarrow{\scriptstyle \mu} \\ \mathbf{y}D_I & \xrightarrow{\mathbf{y}Df} & \mathbf{y}D_J & \xrightarrow{q_J} & Q. \end{array} \qquad (3.2)$$

The fact colimits in $\mathsf{PSh}(\mathbf{C})$ are stable under pullback implies $(p_I\colon \mathbf{y}C_I \to P)_{I \in \mathbf{I}}$ is colimiting. Now applying G to the above diagram gives

$$\begin{array}{ccccc} G\mathbf{y}C_I & \dashrightarrow{G\mathbf{y}Cf} & G\mathbf{y}C_J & \xrightarrow{Gp_J} & GP \\ {\scriptstyle G\mathbf{y}m_I}\downarrow & & \downarrow{\scriptstyle G\mathbf{y}m_J} & & \downarrow{\scriptstyle G\mu} \\ G\mathbf{y}D_I & \xrightarrow{G\mathbf{y}Df} & G\mathbf{y}D_J & \xrightarrow{Gq_J} & GQ. \end{array} \qquad (3.3)$$

Since G is cocontinuous, both $(Gp_I)_{I \in \mathbf{I}}$ and $(Gq_I)_{I \in \mathbf{I}}$ are colimiting. Also, as $G\mathbf{y} \cong F$ and F is an $\mathcal{M}$-functor, the left square is a pullback for every pair $I, J \in \mathbf{I}$. Therefore, by Lemma 3.14, $G\mu$ must be in $\mathbf{D}_{\mathcal{M}}$.

Observe that the same lemma (Lemma 3.14) says that for every $I \in \mathbf{I}$, the outer square in (3.3) is a pullback for every $I \in \mathbf{I}$. In other words, G preserves pullbacks of the form

$$\begin{array}{ccc} \mathbf{y}C_I & \xrightarrow{p_I} & P \\ {\scriptstyle \mathbf{y}m_I}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathbf{y}D_I & \xrightarrow[q_I]{} & Q. \end{array} \qquad (3.4)$$

Now to see that G preserves $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$-pullbacks, consider the diagram below, where the right square is an $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$-pullback and the left square is a pullback for all $I \in \mathbf{I}$:

$$
\begin{array}{ccccc}
\mathbf{y}C_I & \xrightarrow{p_I} & P \cong \operatorname{colim} \mathbf{y}C & \longrightarrow & P' \\
{\scriptstyle \mathbf{y}m_I}\downarrow & & {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu'} \\
\mathbf{y}D_I & \xrightarrow[q_I]{} & Q \cong \operatorname{colim} \mathbf{y}D & \longrightarrow & Q'.
\end{array}
$$

The result then follows by applying G to the diagram and using Lemma 3.15. This proves that G is an $\mathcal{M}$-functor whenever F is, so that (3.1) is essentially surjective on objects.

Finally, to show that the functor in (3.1) is fully faithful, we need to show for any pair of cocontinuous $\mathcal{M}$-functors $F, F' \colon \mathsf{PSh}_{\mathcal{M}}(\mathbf{C}) \to \mathbf{D}$ and $\mathbf{C}_{\mathcal{M}}$-cartesian $\alpha \colon F\mathbf{y} \to F'\mathbf{y}$, there exists a unique $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$-cartesian $\tilde{\alpha} \colon F \to F'$ such that $\tilde{\alpha}\mathbf{y} = \alpha$. In other words, we must show that we have an isomorphism:

$$(-) \circ \mathbf{y} \colon \mathcal{M}\mathbf{Nat}(F, F') \to \mathcal{M}\mathbf{Nat}(F\mathbf{y}, F'\mathbf{y})$$

where $\mathcal{M}\mathbf{Nat}(F, F')$ is the set of $\mathcal{M}$-cartesian natural tranformations from F to F'. However, this condition may be reformulated as follows:

> For all natural transformations $\tilde{\alpha} \colon F \to F'$, $\tilde{\alpha}$ is $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$-cartesian if $\tilde{\alpha}\mathbf{y} \colon F\mathbf{y} \Rightarrow F'\mathbf{y}$ is $\mathbf{C}_{\mathcal{M}}$-cartesian. (3.5)

To see that these two statements are equivalent, observe that the second statement amounts to the following diagram being a pullback in **Set**:

$$
\begin{array}{ccc}
\mathcal{M}\mathbf{Nat}(F, F') & \longrightarrow & \mathcal{M}\mathbf{Nat}(F\mathbf{y}, F'\mathbf{y}) \\
\uparrow & & \uparrow \\
\mathbf{Nat}(F, F') & \xrightarrow[(-)\circ\mathbf{y}]{} & \mathbf{Nat}(F\mathbf{y}, F'\mathbf{y})
\end{array}
$$

where $\mathbf{Nat}(F, F')$ is the set of natural transformations between F and F'. However, as the bottom function is an isomorphism, by the universal property of ordinary free cocompletion, the top must also be an isomorphism and hence the two statements are equivalent. Therefore, we show the functor in (3.1) is fully faithful by proving (3.5).

So let $\mu \colon P \to Q$ be an $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$-map, and note that the left square below is a pullback for every $I \in \mathbf{I}$ as F preserves $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$-pullbacks:

$$
\begin{array}{ccccc}
F\mathbf{y}C_I & \xrightarrow{Fp_I} & FP & \xrightarrow{\tilde{\alpha}_P} & F'P \\
{\scriptstyle F\mathbf{y}m_I}\downarrow & & {\scriptstyle F\mu}\downarrow & & \downarrow{\scriptstyle F'\mu} \\
F\mathbf{y}D_I & \xrightarrow[Fq_I]{} & FQ & \xrightarrow[\tilde{\alpha}_Q]{} & F'Q.
\end{array}
\qquad (3.6)
$$

To show that the right square is a pullback, we will show that the outer square is a pullback for every $I \in \mathbf{I}$ and apply Lemma 3.15. Now by naturality of $\tilde{\alpha}$, this outer square is the outer square of the following diagram:

$$
\begin{array}{ccccc}
F\mathbf{y}C_I & \xrightarrow{\tilde{\alpha}_{\mathbf{y}C_I}} & F'\mathbf{y}C_I & \xrightarrow{F'p_I} & F'P \\
{\scriptstyle F\mathbf{y}m_I}\downarrow & & {\scriptstyle F'\mathbf{y}m_I}\downarrow & & \downarrow{\scriptstyle F'\mu} \\
F\mathbf{y}D_I & \xrightarrow[\tilde{\alpha}_{\mathbf{y}D_I}]{} & F'\mathbf{y}D_I & \xrightarrow[F'q_I]{} & F'Q.
\end{array}
$$

But $\tilde{\alpha} \circ \mathbf{y}$ being $\mathbf{C}_{\mathcal{M}}$-cartesian implies the left square is a pullback, and the right square is also a pullback by the fact F' preserves pullbacks of the form (3.4). Thus, by Lemma 3.15, each square on the right of (3.6) is a pullback, and so $\tilde{\alpha}$ is $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$-cartesian. ∎

3.18. COCOMPLETION OF RESTRICTION CATEGORIES. We have now explored in detail the notion of cocomplete $\mathcal{M}$-category, and wish to exploit this in order to investigate cocomplete restriction categories. Given that $\mathcal{M}\mathbf{Cat}$ and $\mathbf{rCat}_s$ are 2-equivalent, it makes sense to define a restriction category to be cocomplete in such a way that $\mathsf{Par}(\mathbf{C})$ will be cocomplete as a restriction category if and only if $\mathbf{C}$ is cocomplete as an $\mathcal{M}$-category.

3.19. DEFINITION. *A restriction category $\mathbb{X}$ is* cocomplete *if it is split, its subcategory* $\mathsf{Total}(\mathbb{X})$ *is cocomplete, and the inclusion* $\mathsf{Total}(\mathbb{X}) \hookrightarrow \mathbb{X}$ *preserves colimits. A restriction functor $F \colon \mathbb{X} \to \mathbb{Y}$ is* cocontinuous *if* $\mathsf{Total}(F) \colon \mathsf{Total}(\mathbb{X}) \to \mathsf{Total}(\mathbb{Y})$ *is cocontinuous. We denote by* $\mathbf{rCocomp}$ *the 2-category of cocomplete restriction categories, cocontinuous restriction functors and restriction transformations.*

As we said earlier, we would like $\mathsf{Par}(\mathbf{C})$ to be cocomplete as a restriction category if and only if $\mathbf{C}$ is cocomplete as an $\mathcal{M}$-category, and since $\mathsf{Par}(\mathbf{C})$ is always split, it makes sense to impose this as a condition of being cocomplete. Another reason why a cocomplete restriction category $\mathbb{X}$ ought to be split is because ordinary cocomplete categories have splittings of all idempotents, and so it makes sense for $\mathbb{X}$ to have splittings of all restriction idempotents. Observe that for any cocomplete restriction category $\mathbb{X}$, $\mathcal{M}\mathsf{Total}(\mathbb{X})$ is a cocomplete $\mathcal{M}$-category since $\mathsf{Total}(\mathbb{X})$ is cocomplete and $\mathsf{Total}(\mathbb{X}) \hookrightarrow \mathbb{X} \cong \mathsf{Par}(\mathcal{M}\mathsf{Total}(\mathbb{X}))$ preserves colimits.

3.20. EXAMPLE. For each class of examples from Example 3.7, $\mathsf{Par}(\mathcal{E}, \mathcal{E}_{\mathcal{M}})$ is a cocomplete restriction category. In particular, the restriction category of sets and partial functions $\mathbf{Set}_p$ is a cocomplete restriction category since $\mathbf{Set}_p = \mathsf{Par}(\mathbf{Set}, \mathbf{Inj})$.

On the other hand, since the $\mathcal{M}$-category $\mathbf{Ab}$ of abelian groups (with all monos) is not cocomplete as an $\mathcal{M}$-category, $\mathsf{Par}(\mathbf{Ab})$ is not cocomplete as a restriction category.

We know that for any small $\mathcal{M}$-category $\mathbf{C}$, $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$ is a cocomplete $\mathcal{M}$-category, and furthermore, $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}))$ is a cococomplete restriction category. In particular, the split restriction category $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))))$ is a cocomplete restriction category for any small restriction category $\mathbb{X}$. Moreover, we can, following [Cockett & Lack, 2002,

p. 252], embed $\mathbb{X}$ into this cocomplete restriction category via the composite:

$$\Lambda \colon \mathbb{X} \xrightarrow{J} \mathsf{K}_r(\mathbb{X}) \xrightarrow{\Phi_{\mathsf{K}_r(\mathbb{X})}} \mathsf{Par}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))) \xrightarrow{\mathsf{Par}(\mathbf{y})} \mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))) \ . \quad (3.7)$$

We will now show that this embedding exhibits $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))))$ as the *free restriction cocompletion* of $\mathbb{X}$.

3.21. THEOREM. *For any small restriction category $\mathbb{X}$ and cocomplete restriction category $\mathbb{E}$, the following is an equivalence of categories:*

$$(-) \circ \Lambda \colon \mathbf{rCocomp}(\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))), \mathbb{E}) \to \mathbf{rCat}(\mathbb{X}, \mathbb{E})$$

where Λ is the Cockett and Lack embedding introduced in (3.7).

PROOF. First note that $\mathbb{E} \cong \mathsf{Par}(\mathbf{D})$ for some cocomplete $\mathcal{M}$-category $\mathbf{D}$ (as $\mathbb{E}$ is split), and that

$$\mathbf{rCocomp}(\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})), \mathsf{Par}(\mathbf{D})) \simeq \mathcal{M}\mathbf{Cocomp}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}), \mathbf{D})$$

since Par and $\mathcal{M}\mathsf{Total}$ are 2-equivalences. Therefore,

$$(-) \circ \mathsf{Par}(\mathbf{y}) \colon \mathbf{rCocomp}(\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})), \mathbb{E}) \to \mathbf{rCat}(\mathsf{Par}(\mathbf{C}), \mathbb{E})$$

is an equivalence since

$$(-) \circ \mathbf{y} \colon \mathcal{M}\mathbf{Cocomp}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}), \mathbf{D}) \to \mathcal{M}\mathbf{Cat}(\mathbf{C}, \mathbf{D})$$

is an equivalence by Theorem 3.17. Therefore the following composite is an equivalence:

$$\mathbf{rCocomp}(\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))), \mathbb{E})$$
$$\downarrow {\scriptstyle (-) \circ \mathsf{Par}(\mathbf{y})}$$
$$\mathbf{rCocomp}(\mathsf{Par}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))), \mathbb{E})$$
$$\downarrow {\scriptstyle (-) \circ \Phi_{\mathsf{K}_r(\mathbb{X})} \circ J}$$
$$\mathbf{rCat}(\mathbb{X}, \mathbb{E})$$

as $\Phi_{\mathsf{K}_r(\mathbb{X})}$ is an isomorphism and J is the unit of the biadjunction $i \dashv \mathsf{K}_r$ at $\mathbb{X}$. ∎

4. Restriction presheaves

We have just seen that for any small restriction category $\mathbb{X}$, the Cockett–Lack embedding of (3.7) exhibits the restriction category $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))))$ as a free cocompletion of $\mathbb{X}$. However, this description of the free cocompletion seems rather unwieldy compared to the characterisation of $\mathsf{PSh}(\mathbf{C})$ and $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$ as the free cocompletions of ordinary categories and $\mathcal{M}$-categories respectively.

In this section, we give an alternate simpler definition in terms of a restriction category $\mathsf{PSh}_r(\mathbb{X})$ of *restriction presheaves*. The underlying category of $\mathsf{PSh}_r(\mathbb{X})$ will be a full subcategory of $\mathsf{PSh}(\mathbb{X})$ and the Yoneda embedding will factor through this subcategory to yield a restriction functor $\mathbf{y}_r \colon \mathbb{X} \to \mathsf{PSh}_r(\mathbb{X})$. We will show that the category $\mathsf{PSh}_r(\mathbb{X})$ is equivalent to $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))))$, so that it gives another, easier, way of describing free cocompletion in the restriction setting.

4.1. DEFINITION. *Let $\mathbb{X}$ be a restriction category. A* restriction presheaf *on $\mathbb{X}$ is an ordinary presheaf $P \colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ equipped with, for each $A \in \mathbb{X}$, a function $PA \to \mathbb{X}(A, A)$ sending each $x \in PA$ to a restriction idempotent $\bar{x} \colon A \to A$ in $\mathbb{X}$, all subject to the following three axioms:*

(A1) $x \cdot \bar{x} = x$;

(A2) $\overline{x \cdot \bar{f}} = \bar{x} \circ \bar{f}$, *where $\bar{f} \colon A \to A$ is a restriction idempotent in $\mathbb{X}$;*

(A3) $\bar{x} \circ g = g \circ \overline{x \cdot g}$, *where $g \colon B \to A$ in $\mathbb{X}$.*

We call the collection of functions $PA \to \mathbb{X}(A, A)$ above the restriction structure *of P.*

Unlike the restriction structure on a restriction category, the restriction structure on a restriction presheaf is uniquely determined:

4.2. LEMMA. *Let $\mathbb{X}$ be a restriction category and $P \colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ a presheaf. Suppose P has two restriction structures given by $x \mapsto \bar{x}$ and $x \mapsto \tilde{x}$. Then $\bar{x} = \tilde{x}$ for all $A \in \mathbb{X}$ and $x \in PA$.*

PROOF. We have
$$\bar{x} = \overline{x \cdot \tilde{x}} = \bar{x} \circ \tilde{x} = \tilde{x} \circ \bar{x} = \widetilde{x \cdot \bar{x}} = \tilde{x}$$
by the fact $\bar{x}$ and $\tilde{x}$ are restriction idempotents and using (A1),(A2). ∎

We also have the following analogues of basic results for restriction categories.

4.3. LEMMA. *Suppose P is a restriction presheaf on a restriction category $\mathbb{X}$, and let $A \in \mathbb{X}$, $x \in PA$ and $g \colon B \to A$. Then*

(1) $\bar{g} \circ \overline{x \cdot g} = \overline{x \cdot g}$;

(2) $\overline{\bar{x} \circ g} = \overline{x \cdot g}$.

PROOF. By (R2), (A2) and (R1),
$$\bar{g} \circ \overline{x \cdot g} = \overline{x \cdot g} \circ \bar{g} = \overline{(x \cdot g) \cdot \bar{g}} = \overline{x \cdot (g \circ \bar{g})} = \overline{x \cdot g}.$$

We also have
$$\overline{\bar{x} \circ g} = \overline{g \circ \overline{x \cdot g}} = \bar{g} \circ \overline{x \cdot g} = \overline{x \cdot g}$$
by (A3), (R3) and the previous result. ∎

The lemma above shows that (A2) and (A3) together imply $\overline{\overline{x} \circ g} = \overline{x \cdot g}$. However, what is perhaps surprising is that the converse is also true.

4.4. LEMMA. *Suppose* $P \colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ *is a presheaf, and let* $A \in \mathbb{X}$, $x \in PA$. *If* $\overline{\overline{x} \circ g} = \overline{x \cdot g}$ *is true for all maps* $g \colon B \to A$, *then* $\overline{x \cdot e} = \overline{x} \circ e$ *for all restriction idempotents* $e \colon A \to A$, *and also* $\overline{x} \circ g = g \circ \overline{x \cdot g}$.

PROOF. The fact $\overline{\overline{x} \circ g} = \overline{x \cdot g}$ implies $\overline{x \cdot e} = \overline{x} \circ e$ is straightforward, and by assumption, we have $g \circ \overline{x \cdot g} = g \circ \overline{\overline{x} \circ g} = \overline{\overline{x}} \circ g = \overline{x} \circ g$. ∎

So in fact, we may replace restriction presheaf axioms (A2) and (A3) by the condition that $\overline{\overline{x} \circ g} = \overline{x \cdot g}$ for all maps $g \colon B \to A$. Using this, we can give a further reformulation of the restriction presheaf notion. Let us introduce the presheaf $\mathcal{O} \colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$, sending each $A \in \mathbb{X}$ to the set $\mathcal{O}(A)$ of restriction idempotents on A, and whose action by a map $f \colon B \to A$ satisfies $x \cdot f = \overline{x \circ f}$ (cf. [Cockett & Lack, 2002, p. 253]).

Now for each presheaf P on $\mathbb{X}$, there is an action by $\mathcal{O}$ on P in the following sense: there is a natural transformation $\alpha \colon P \times \mathcal{O} \to P$ which on components, sends (x, e) to $x \cdot e$ (for each $A \in \mathbb{X}$, $x \in PA$ and each restriction idempotent $e \colon A \to A$). There is also another action on P given by $\pi \colon P \times \mathcal{O} \to P$, which sends (x, e) to x (the first projection). Since we now know that restriction structures are unique, we may characterise the restriction structure on any presheaf P in the following way.

4.5. PROPOSITION. *Let* $\mathbb{X}$ *be a restriction category. Then a presheaf* $P \colon \mathbb{X}^{\mathrm{op}} \to \mathbf{Set}$ *may be given a restriction structure if and only if there exists a (unique) section* $\sigma \colon P \to P \times \mathcal{O}$ *to both the actions* $\alpha, \pi \colon P \times \mathcal{O} \to P$ *described above.*

PROOF. The condition $x \cdot \overline{x} = x$ is given by the section σ, and the other necessary and sufficient property that $\overline{\overline{x} \circ g} = \overline{x \cdot g}$ is simply restating the fact that σ is natural. ∎

We now describe how to form restriction presheaves into a restriction category.

4.6. DEFINITION. *The category of restriction presheaves on* $\mathbb{X}$, $\mathsf{PSh}_r(\mathbb{X})$, *is the restriction category whose objects are restriction presheaves and whose maps are arbitrary natural transformations. The restriction of* $\alpha \colon P \to Q$ *is the natural transformation* $\bar{\alpha} \colon P \to P$ *given componentwise by* $\bar{\alpha}_A(x) = x \cdot \overline{\alpha_A(x)}$ *for every* $A \in \mathbb{X}$ *and* $x \in PA$.

We leave to the reader the straightforward calculations that the above does indeed define a restriction structure on $\mathsf{PSh}_r(\mathbb{X})$. We also emphasise that the underlying category of $\mathsf{PSh}_r(\mathbb{X})$ is a *full* subcategory of $\mathsf{PSh}(\mathbb{X})$; in particular, maps are not required to preserve restrictions. In fact, the restriction-preserving maps are precisely the total maps:

4.7. PROPOSITION. *A map* $\alpha \colon P \to Q$ *is total in* $\mathsf{PSh}_r(\mathbb{X})$ *if and only if* $\overline{\alpha_A(x)} = \bar{x}$ *for all* $A \in \mathbb{X}$ *and* $x \in PA$.

PROOF. Suppose $\alpha\colon P \to Q$ is total in $\mathsf{PSh}_r(\mathbb{X})$. Then $\bar{\alpha}_A(x) = 1_{PA}(x) = x$, or $x \cdot \overline{\alpha_A(x)} = x$. But this implies $\bar{x} \leq \overline{\alpha_A(x)}$ since

$$\bar{x} = \overline{x \cdot \overline{\alpha_A(x)}} = \bar{x} \circ \overline{\alpha_A(x)}.$$

On the other hand, $\overline{\alpha_A(x)} \leq \bar{x}$ as

$$\overline{\alpha_A(x)} = \overline{\alpha_A(x \cdot \bar{x})} = \overline{\alpha_A(x) \cdot \bar{x}} = \overline{\alpha_A(x)} \circ \bar{x}.$$

Therefore, α in $\mathsf{PSh}_r(\mathbb{X})$ is total if and only if α preserves restrictions. ∎

Now if $\mathbb{X}$ is a restriction category, then each representable $\mathbb{X}(-, A)$ has a restriction structure given by sending $f \in \mathbb{X}(B, A)$ to $\bar{f} \in \mathbb{X}$. In particular, this implies that the Yoneda embedding $\mathbf{y}\colon \mathbb{X} \to \mathsf{PSh}(\mathbb{X})$ factors (uniquely) as a functor $\mathbf{y}_r\colon \mathbb{X} \to \mathsf{PSh}_r(\mathbb{X})$:

$$\mathbb{X} \xrightarrow{\ \mathbf{y}_r\ } \mathsf{PSh}_r(\mathbb{X})$$

$$\mathbf{y} \searrow \ \ \ \ \downarrow$$

$$\mathsf{PSh}(\mathbb{X}).$$

4.8. LEMMA. *For any restriction category $\mathbb{X}$, the functor $\mathbf{y}_r\colon \mathbb{X} \to \mathsf{PSh}_r(\mathbb{X})$ is a restriction functor.*

PROOF. Let $f\colon A \to B$ be a map in $\mathbb{X}$. Then for all $X \in \mathbb{X}$ and $x \in \mathbb{X}(X, A)$, we have

$$\overline{\mathbf{y}_r f}_X(x) = x \cdot \overline{(\mathbf{y}_r f)_X(x)} = x \cdot \overline{f \circ x} = x \circ \overline{f \circ x} = \bar{f} \circ x = (\mathbf{y}_r \bar{f})_X(x)$$

and so $\mathbf{y}_r$ is a restriction functor. ∎

The restriction presheaf category has one more important property.

4.9. PROPOSITION. *Let $\mathbb{X}$ be a restriction category. Then $\mathsf{PSh}_r(\mathbb{X})$ is a split restriction category.*

PROOF. Let $\bar{\alpha}\colon P \to P$ be a restriction idempotent in $\mathsf{PSh}_r(\mathbb{X})$. Since all idempotents in $\mathsf{PSh}(\mathbb{X})$ split, we may write $\bar{\alpha} = \mu\rho$ for some maps $\mu\colon Q \to P$ and $\rho\colon P \to Q$ such that $\rho\mu = 1$. Componentwise, we may take μ_A to be the inclusion $QA \hookrightarrow PA$ with $QA = \{x \in PA \mid \bar{\alpha}_A(x) = x\}$. Therefore, to show $\mathsf{PSh}_r(\mathbb{X})$ is split, it is enough to show that Q is a restriction presheaf. However, P is a restriction presheaf and Q is a subfunctor of P. Therefore, imposing the restriction structure of P onto Q will make Q a restriction presheaf. Hence $\mathsf{PSh}_r(\mathbb{X})$ is a split restriction category. ∎

Before moving on to the main theorems in this section, let us recall the split restriction category $\mathsf{K}_r(\mathbb{X})$, whose objects are pairs (A, e) (with e a restriction idempotent on $A \in \mathbb{X}$). Also recall the unit of the biadjunction $i \dashv \mathsf{K}_r$ at $\mathbb{X}$, $J: \mathbb{X} \to \mathsf{K}_r(\mathbb{X})$, which sends objects A to $(A, 1_A)$ and morphisms $f: A \to B$ to $f: (A, 1_A) \to (B, 1_B)$.

4.10. PROPOSITION. $\mathsf{PSh}_r(\mathbb{X})$ and $\mathsf{PSh}_r(\mathsf{K}_r(\mathbb{X}))$ are equivalent as restriction categories.

PROOF. Since $\mathsf{K}_r(\mathbb{X})$ is a full subcategory of $\mathsf{Split}(\mathbb{X})$, the idempotent completion of $\mathbb{X}$, and idempotent completion does not affect categories of presheaves, the functor $(-) \circ J^{\mathrm{op}}: \mathsf{PSh}(\mathsf{K}_r(\mathbb{X})) \to \mathsf{PSh}(\mathbb{X})$ is an equivalence. Therefore, the result will follow if we can show this functor restricts back to an equivalence between $\mathsf{PSh}_r(\mathsf{K}_r(\mathbb{X}))$ and $\mathsf{PSh}_r(\mathbb{X})$. In other words, we must show that $(-) \circ J^{\mathrm{op}}$ sends restriction presheaves on $\mathsf{K}_r(\mathbb{X})$ to restriction presheaves on $\mathbb{X}$; that it is essentially surjective on objects; and that it is a restriction functor.

So let P be a restriction presheaf on $\mathsf{K}_r(\mathbb{X})$. Then PJ^{op} is a restriction presheaf on $\mathbb{X}$ if we define the restriction of $x \in (PJ^{\mathrm{op}})(A) = P(A, 1_A)$ to be the same as in $P(A, 1_A)$ for all $A \in \mathbb{X}$. Moreover, for any $\alpha: P \Rightarrow Q$ in $\mathsf{PSh}_r(\mathsf{K}_r(X))$, we have that

$$(\overline{\alpha} \circ J^{\mathrm{op}})_A(x) = \overline{\alpha}_{(A,1_A)}(x) = x \cdot \overline{\alpha_{(A,1_A)}(x)} = x \cdot \overline{(\alpha \circ J^{\mathrm{op}})_A(x)} = \left(\overline{\alpha \circ J^{\mathrm{op}}} \right)_A(x)$$

so that $(-) \circ J^{\mathrm{op}}$ preserves restrictions. All that remains is to show essential surjectivity.

Let Q be a restriction presheaf on $\mathbb{X}$, and define the restriction presheaf Q' on $\mathsf{K}_r(\mathbb{X})$ by taking $Q'(A, e) = \{x \in QA \mid x \cdot e = x\}$ and $Q'(f)(y) = Q(f)(y)$ for all $f: (A', e') \to (A, e)$ in $\mathsf{K}_r(X)$. Note that the action on maps is well-defined since

$$Q(f)(y) = Q(f)(y \cdot e') = Q(f)Q(e')(y) = Q(e'f)(y) = Q(fe)(y) = Q(f)(y) \cdot e \ .$$

Moreover Q' is a restriction presheaf under the same restriction structure as Q. Obviously $Q'(A, 1_A) = Q(A)$, and so $Q' \circ J^{\mathrm{op}} = Q$. Hence, $(-) \circ J^{\mathrm{op}}: \mathsf{PSh}(\mathsf{K}_r(\mathbb{X})) \to \mathsf{PSh}(\mathbb{X})$ is essentially surjective on objects, and therefore $\mathsf{PSh}_r(\mathbb{X})$ and $\mathsf{PSh}_r(\mathsf{K}_r(\mathbb{X}))$ are equivalent. ∎

4.11. THEOREM. Let $\mathbf{C}$ be an $\mathcal{M}$-category. Then $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$ and $\mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C})))$ are equivalent.

PROOF. Our approach will be to find a pair of functors $F: \mathsf{PSh}(\mathbf{C}) \to \mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C})))$ and $G: \mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}))) \to \mathsf{PSh}(\mathbf{C})$, and natural isomorphisms $\eta: 1 \Rightarrow GF$ and $\varepsilon: FG \Rightarrow 1$, and then show that F and G are in fact $\mathcal{M}$-functors. (Note that η and ε must necessarily be $\mathcal{M}$-cartesian).
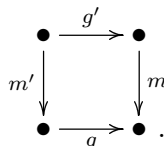
We define F on objects as follows. Let P be a presheaf on $\mathbf{C}$. If $X \in \mathsf{Par}(\mathbf{C})$, then $(FP)(X)$ is the set of equivalence classes

$$(FP)(X) = \{(m, f) \mid m: Y \to X \in \mathbf{C}_{\mathcal{M}}, f \in PY\}$$

where $(m, f) \sim (n, g)$ if and only if there exists an isomorphism φ such that $n = m\varphi$ and $g = f \cdot \varphi$. To define FP on morphisms, given $(n, g): Z \to X$ in $\mathsf{Par}(\mathbf{C})$ and an element $(m, f) \in (FP)(X)$, define

$$\left((FP)(n, g) \right)(m, f) = (nm', f \cdot g')$$

where (m', g') is the pullback of (m, g), as in:

Defining the restriction of $(m, f) \in (FP)(X)$ to be (m, m) makes $FP \colon \mathsf{Par}(\mathbf{C})^{\mathrm{op}} \to \mathbf{Set}$ a restriction presheaf. This defines F on objects. Now suppose $\alpha \colon P \to Q$ is a map in $\mathsf{PSh}(\mathbf{C})$. Define $F\alpha \colon FP \to FQ$ componentwise as follows:

$$(F\alpha)_X(m, f) = (m, \alpha_{\mathrm{dom}\ m}(f)).$$

Then $F\alpha$ is natural (by naturality of α) and also total, making F a functor from $\mathsf{PSh}(\mathbf{C})$ to $\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C})))$.

We now give the data for the functor G from $\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C})))$ to $\mathsf{PSh}(\mathbf{C})$. Let P be a restriction presheaf on $\mathsf{Par}(\mathbf{C})$, and define $GP \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ as follows. If $X \in \mathbf{C}$, then

$$(GP)(X) = \{x \mid x \in PX, \bar{x} = (1, 1)\}.$$

And if $f \colon Z \to X$ is a map in $\mathbf{C}$, define

$$(GP)(f) = P(1, f).$$

Note that $(GP)(f)$ is well-defined since for every $x \in (GP)(X)$,

$$\overline{P(1, f)(x)} = \overline{x \cdot (1, f)} = \bar{x} \circ (1, f) = (1, 1),$$

and so $(GP)(f)$ is a function from $(GP)(X)$ to $(GP)(Z)$.

Finally, if $\alpha \colon P \to Q$ is a total map in $\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}))$, define $G\alpha \colon GP \to GQ$ componentwise by $(G\alpha)_X(x) = \alpha_X(x)$. for every $X \in \mathbf{C}$ and $x \in (GP)(X)$. Again, to see that $G\alpha$ is well-defined, note that α total implies $\overline{\alpha_X(x)} = \bar{x} = 1$ (Proposition 4.7) and so $\alpha_X(x) \in (GQ)(X)$. This makes G a functor from $\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C})))$ to $\mathsf{PSh}(\mathbf{C})$.

The next step is defining isomorphisms $\eta \colon 1 \Rightarrow GF$ and $\varepsilon \colon FG \Rightarrow 1$. To define η, we need to give components for every presheaf P on $\mathbf{C}$, and this involves giving isomorphisms $(\eta_P)_X \colon PX \to (GFP)(X)$. But $(GFP)(X) = \{(1, f) \mid f \in PX\}$. Therefore, defining $(\eta_P)_X(f) = (1, f)$ makes η an isomorphism, and naturality is easy to check.

Similarly, to define ε, we need to define isomorphisms $(\varepsilon_P)_X \colon (FGP)(X) \to PX$ for every restriction presheaf P on $\mathsf{Par}(\mathbf{C})$ and object $X \in \mathsf{Par}(\mathbf{C})$. Since

$$(FGP)(X) = \{(m, f) \mid m \colon Y \to X \in \mathbf{C}_\mathcal{M}, f \in PY, \bar{f} = (1, 1)\},$$

we may define $(\varepsilon_P)_X(m, f) = f \cdot (m, 1)$. Its inverse $(\varepsilon_P)_X^{-1} \colon PX \to (FGP)(X)$ is then given by

$$(\varepsilon_P)_X^{-1}(x) = (n, x \cdot (1, n))$$

where $\bar{x} = (n, n)$ (as P is a restriction presheaf on $\mathsf{Par}(\mathbf{C})$). Checking the naturality of ε is again straightforward. All that remains is to show that both $F \colon \mathsf{PSh}_{\mathcal{M}}(\mathbf{C}) \to \mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C})))$ and $G \colon \mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}))) \to \mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$ are $\mathcal{M}$-functors. However, as F and G are equivalences in $\mathbf{Cat}$, they necessarily preserve limits, and so it suffices to show that they preserve $\mathcal{M}$-maps.

So let $\mu \colon P \to Q$ be in $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$. To show $F\mu$ is a restriction monic, we need to show $F\mu$ is the equaliser of 1 and some restriction idempotent $\alpha \colon FQ \to FQ$. To define this α, let $X \in \mathsf{Par}(\mathbf{C})$ and $(n, g) \in (FQ)(X)$ where $n \colon Z \to X$, say. Now as $g \in QZ$, there exists a corresponding natural transformation $\hat{g} \colon \mathbf{y}Z \to Q$ by Yoneda. However, as μ is in $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$, there exists an $m_g \colon B \to Z$ in $\mathbf{C}_{\mathcal{M}}$ making the following a pullback:

$$
\begin{array}{ccc}
\mathbf{y}B & \longrightarrow & P \\
{\scriptstyle \mathbf{y}m_g} \downarrow & & \downarrow {\scriptstyle \mu} \\
\mathbf{y}Z & \xrightarrow[\hat{g}]{} & Q.
\end{array}
$$

So define α by its components as follows,

$$
\alpha_X(n, g) = (nm_g, g \cdot m_g).
$$

It is then not difficult to show this α is well-defined, is a natural transformation and is a restriction idempotent.

Now to show that $F\mu$ equalises 1 and α, we need to show $(F\mu)_X \colon (FP)(X) \to (FQ)(X)$ is an equaliser of 1 and $\alpha_{(FQ)(X)}$ in $\mathbf{Set}$ for all $X \in \mathsf{Par}(\mathbf{C})$. In other words, that $(F\mu)_X$ is injective, and that:

> $(n, g) \in (FQ)(X)$ satisfies $(n, g) = (F\mu)_X(m, f) = (m, \mu_{\mathrm{dom}\, m}(f))$ for some $(m, f) \in (FP)(X)$ if and only if $\alpha_X(n, g) = (n, g)$. $\qquad (4.1)$

To show $(F\mu)_X$ is injective, suppose $(F\mu)_X(m, f) = (F\mu)_X(m', f')$, or equivalently, $(m, \mu_{\mathrm{dom}\, m}(f)) = (m', \mu_{\mathrm{dom}\, m'}(f'))$. That is, there exists an isomomorphism φ such that $m' = m\varphi$ and $\mu_{\mathrm{dom}\, m'}(f') = \mu_{\mathrm{dom}\, m}(f) \cdot \varphi$. But the naturality of μ implies $\mu_{\mathrm{dom}\, m'}(f \cdot \varphi) = \mu_{\mathrm{dom}\, m}(f) \cdot \varphi = \mu_{\mathrm{dom}\, m'}(f')$. Therefore, as μ is monic, we must have $f \cdot \varphi = f'$. Hence $(m, f) = (m', f')$, and so $(F\mu)_X$ is injective.

To prove (4.1), let $(n, g) \in (FQ)(X)$ and suppose $\alpha_X(n, g) = (n, g)$. This says that $(nm_g, g \cdot m_g) = (n, g)$, or equivalently that m_g is an isomorphism. This happens if and only if $\mathbf{y}m_g$ is an isomorphism; but since $\mathbf{y}m_g$ is a pullback of μ along $\hat{g}$, this happens in turn if and only if $\hat{g} = \mu\hat{h}$ for some $\hat{h} \colon \mathbf{y}Z \to P$:

$$
\begin{array}{ccc}
\mathbf{y}B & \longrightarrow & P \\
{\scriptstyle \mathbf{y}m_g} \downarrow & {\scriptstyle \hat{h}} \nearrow & \downarrow {\scriptstyle \mu} \\
\mathbf{y}Z & \xrightarrow[\hat{g}]{} & Q.
\end{array}
$$

But by Yoneda, the statement $\hat{g} = \mu\hat{h}$ is equivalent to the statement that $g = \mu_Z(h)$ for some $h \in PZ$, which is the same as saying $(n, g) = (n, \mu_Z(h)) = (F\mu)_X(n, h)$, with $(n, h) \in (FP)(X)$. Therefore, $(F\mu)_X$ is an equaliser of 1 and $\alpha_{(FQ)(X)}$ in **Set** for all $X \in \mathsf{Par}(\mathbf{C})$, and hence, $F\mu$ equalises 1 and α.

Now to see that G is also an $\mathcal{M}$-functor, let $\mu\colon P \to Q$ be a restriction monic in $\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}))$. To show $G\mu$ is in $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$, we need to show that for any given $\hat{\theta}\colon \mathbf{y}C \to Q$, there exists a monic $m\colon D \to C$ in $\mathbf{C}_{\mathcal{M}}$ and a map $\hat{\delta}\colon \mathbf{y}D \to P$ making the following a pullback:

$$\begin{array}{ccc} \mathbf{y}D & \xrightarrow{\hat{\delta}} & GP \\ {\scriptstyle \mathbf{y}m}\downarrow & & \downarrow{\scriptstyle G\mu} \\ \mathbf{y}C & \xrightarrow{\hat{\theta}} & GQ. \end{array}$$

Here we make two observations. First, commutativity says m and δ must satisfy $G\mu \circ \hat{\delta} = \hat{\theta} \circ \mathbf{y}m$. On the other hand, Yoneda tells us that $\hat{\theta} \circ \mathbf{y}m = \widehat{\theta \cdot m}$ and $G\mu \circ \hat{\delta} = \widehat{(G\mu)_D}(\delta)$, where $\theta \in QC$ and $\delta \in PD$ are the unique transposes of $\hat{\theta}$ and $\hat{\delta}$ respectively. Therefore, m and δ must satisfy the following condition:

$$(G\mu)_D(\delta) = \theta \cdot_{GQ} m. \tag{4.2}$$

That is, $\mu_D(\delta) = \theta \cdot_Q (1, m)$. Secondly, m and δ must make the following a pullback in **Set** (for all objects $X \in \mathbf{C}$):

$$\begin{array}{ccc} \mathbf{C}(X, D) & \xrightarrow{\hat{\delta}_X = \delta \cdot_{GP}(-)} & (GP)(X) \\ {\scriptstyle m\circ(-)}\downarrow & & \downarrow{\scriptstyle (G\mu)_X} \\ \mathbf{C}(X, C) & \xrightarrow[\hat{\theta}_X = \theta \cdot_{GQ}(-)]{} & (GQ)(X). \end{array}$$

In other words, for any $f \in \mathbf{C}(X, C)$ and $x \in (GP)(X)$ such that $\theta \cdot_{GQ} f = (G\mu)_X(x)$ (i.e., such that $\theta \cdot_Q (1, f) = \mu_X(x)$), there exists a unique $g \in \mathbf{C}(X, D)$ such that

$$\delta \cdot_{GP} g = x, \text{ and } mg = f. \tag{4.3}$$

Alternatively, $\delta \cdot_P (1, g) = x$ and $mg = f$.

We now find m and δ satisfying (4.2) and (4.3). To find m that because μ is a restriction monic, there exists a ρ such that $\underline{\mu\rho = \bar{\rho}}$ and $\rho\mu = 1$. Since $\theta \in QC$, applying ρ_C to θ and then taking its restriction gives $\overline{\rho_C(\theta)} = (m, m)$ for some $m \in \mathbf{C}_{\mathcal{M}}$.

This gives us m. To get δ, note that $P(1, m)$ is a function from PC to PD. So define

$$\delta = \rho_C(\theta) \cdot_P (1, m).$$

Then $\delta \in (GP)(D)$ since

$$\bar{\delta} = \overline{\overline{\rho_C(\theta)} \circ (1, m)} = \overline{(m, m) \circ (1, m)} = \overline{(1, m)} = (1, 1).$$

So all that remains is to show m and δ satisfy (4.2) and (4.3). To show m and δ satisfy (4.2), one simply substitutes the given values into the equation, using the fact $\mu\rho = \bar{\rho}$. To see that (4.3) is also satisfied, suppose there exist $f \in \mathbf{C}(X, C)$ and $x \in (GP)(X)$ such that $\theta \cdot_P (1, f) = \mu_X(x)$. Then applying ρ_X to both sides gives

$$\rho_C(\theta) \cdot_P (1, f) = x$$

since $\rho\mu = 1$. We need to show there exists a g such that $mg = f$ and $\delta \cdot_P (1, g) = x$. But $mg = f$ implies

$$x = \rho_C(\theta) \cdot_P (1, f) = \rho_C(\theta) \cdot_P (1, mg) = \rho_C(\theta) \cdot_P (1, m) \cdot_P (1, g) = \delta \cdot_P (1, g).$$

Therefore, we just need to find g satisfying $mg = f$.

Consider the composite $(m, m) \circ (1, f) = (m', mf')$, where (m', f') is the pullback of (m, f):

$$
\begin{array}{ccc}
X \times_C D & \xrightarrow{f'} & D \\
{\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\
X & \xrightarrow{f} & C.
\end{array}
$$

Note that if m' is an isomorphism, then $g = f'(m')^{-1}$ will satisfy the condition $mg = f$. Now by restriction presheaf axioms and naturality of $\bar{\rho}$, we have $\theta \cdot_Q (m', mf') = \theta \cdot_Q (1, f)$. But $\theta \in (GQ)(C)$ implies

$$\overline{\theta \cdot_Q (m', mf')} = \overline{\bar{\theta} \circ (m', mf')} = \overline{(m', mf')} = (m', m')$$

and

$$\overline{\theta \cdot_Q (1, f)} = \overline{\bar{\theta} \circ (1, f)} = \overline{(1, f)} = (1, 1).$$

Therefore, m' must be an isomorphism, which means m and δ satisfy (4.3). Hence, G is also an $\mathcal{M}$-functor and $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$ and $\mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C})))$ are equivalent. ∎

We now use the above theorem to prove the following result.

4.12. PROPOSITION. *Let $\mathbf{C}$ be an $\mathcal{M}$-category. There is an equivalence of restriction categories $L\colon \mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})) \to \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}))$ satisfying the relation $\mathbf{y}_r = L \circ \mathsf{Par}(\mathbf{y})$.*

PROOF. Since Par and $\mathcal{M}\mathsf{Total}$ are 2-equivalences, the following is an isomorphism of categories:

$$\mathcal{M}\mathbf{Cat}\Big(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C}), \mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C})))\Big) \cong \mathbf{rCat}\Big(\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})), \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}))\Big).$$

We know from Theorem 4.11 that $F\colon \mathsf{PSh}_{\mathcal{M}}(\mathbf{C}) \to \mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C})))$ is an equivalence. So define $L = \tilde{F}$, the transpose of F. Explicitly, $\tilde{F} = \Phi^{-1}_{\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}))} \circ \mathsf{Par}(F)$, where $\Phi_{\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}))}$ is the unit of the 2-equivalence between Par and $\mathcal{M}\mathsf{Total}$.

Now define $\widetilde{\mathbf{y}}_r \colon \mathbf{C} \to \mathcal{M}\mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C})))$ as the transpose of $\mathbf{y}_r \colon \mathsf{Par}(\mathbf{C}) \to \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}))$. Explicitly, $\widetilde{\mathbf{y}}_r$ is the unique map whose underlying functor (also called $\widetilde{\mathbf{y}}_r$ by an abuse of notation) makes the following diagram commute:

$$
\begin{array}{ccc}
\mathbf{C} & \xrightarrow{\;\widetilde{\mathbf{y}}_r\;} & \mathsf{Total}(\mathsf{PSh}_r(\mathsf{Par}(\mathbf{C}))) \\
\downarrow & & \uparrow \\
\mathsf{Par}(\mathbf{C}) & \xrightarrow[\mathbf{y}_r]{} & \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C})).
\end{array}
$$

Since $\widetilde{\mathbf{y}}_r = F\mathbf{y}$ will imply $\mathbf{y}_r = L \circ \mathsf{Par}(\mathbf{y})$, we prove the former. So let $A \in \mathsf{Par}(\mathbf{C})$. Then $\widetilde{\mathbf{y}}_r(A) = \mathsf{Par}(\mathbf{C})(-, A)$ by definition. On the other hand, $(F\mathbf{y})(A)$ defined on objects $B \in \mathsf{Par}(\mathbf{C})$ is the following set:

$$(F\mathbf{y}A)(B) = \{(m, f) \mid m \colon Y \to B \in \mathbf{C}_{\mathcal{M}}, f \in \mathbf{C}(Y, A)\}.$$

In other words, elements of $(F\mathbf{y}A)(B)$ are spans $B \xleftarrow{m} Y \xrightarrow{f} A$ with $m \in \mathbf{C}_{\mathcal{M}}$.

Clearly $(F\mathbf{y}A)(B) = \mathsf{Par}(\mathbf{C})(B, A) = (\widetilde{\mathbf{y}}_r A)(B)$. Likewise, if $(n, g) \colon C \to B$ is a map in $\mathsf{Par}(\mathbf{C})$, then $(F\mathbf{y}A)(n, g) = (-) \circ (n, g) = (\widetilde{\mathbf{y}}_r A)(n, g)$, and so $\widetilde{\mathbf{y}}_r(A) = (F\mathbf{y})(A)$.

Now let $h \colon B \to C$ be a map in $\mathbf{C}$. Then $(F\mathbf{y})(h) \colon \mathsf{Par}(\mathbf{C})(-, B) \Rightarrow \mathsf{Par}(\mathbf{C})(-, C)$ has components given by

$$(F\mathbf{y}h)_D(n, g) = (n, (\mathbf{y}h)_{\mathrm{dom}\, n}(g)) = (n, hg) = (1, h) \circ (n, g)$$

for all $D \in \mathsf{Par}(\mathbf{C})$ and $(n, g) \in \mathsf{Par}(\mathbf{C})(D, C)$. But $\widetilde{\mathbf{y}}_r(h) = \mathbf{y}_r(1, h)$ also has components given by $\big(\mathbf{y}_r(1, h)\big)_D = (1, h) \circ (-)$ at $D \in \mathsf{Par}(\mathbf{C})$. Therefore, $(F\mathbf{y})(h) = \widetilde{\mathbf{y}}_r(h)$ and so $F\mathbf{y} = \widetilde{\mathbf{y}}_r$. Hence, $\mathbf{y}_r = L \circ \mathsf{Par}(\mathbf{y})$.   ∎

We now prove the main result of this section.

4.13. THEOREM. *Let* $\mathbb{X}$ *be a restriction category. Then*

$$\mathsf{PSh}_r(\mathbb{X}) \simeq \mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))))$$

and the following diagram commutes up to isomorphism:

$$
\begin{array}{ccc}
 & \mathbb{X} & \\
{\scriptstyle \mathbf{y}_r}\swarrow & {\scriptstyle \cong} & \searrow{\scriptstyle \Lambda} \\
\mathsf{PSh}_r(\mathbb{X}) & \xleftarrow[\;\simeq\;]{} & \mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))))
\end{array}
$$

where Λ *is the Cockett–Lack embedding of* (3.7).

PROOF. Consider the following diagram, where $\mathbf{C} = \mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))$ and the top composite is the Cockett–Lack embedding Λ from (3.7):

$$
\begin{array}{ccccc}
\mathbb{X} & \xrightarrow{\;\;\Phi_{\mathsf{K}_r(\mathbb{X})} \circ J\;\;} & \mathsf{Par}(\mathbf{C}) & \xrightarrow{\;\;\mathsf{Par}(\mathbf{y})\;\;} & \mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})) \\
{\scriptstyle \mathbf{y}_r}\downarrow & & {\scriptstyle \mathbf{y}_r}\downarrow & & \downarrow{\scriptstyle L} \\
\mathsf{PSh}_r(\mathbb{X}) & \xleftarrow[(-)\circ(\Phi_{\mathsf{K}_r(\mathbb{X})}\circ J)^{\mathrm{op}}]{} & \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C})) & = = = = & \mathsf{PSh}_r(\mathsf{Par}(\mathbf{C})).
\end{array}
$$

By Proposition 4.12, the right square commutes up to isomorphism. However, the left square also commutes up to isomorphism as $\Phi_{\mathsf{K}_r(\mathbb{X})} \circ J$ is fully faithful. Hence the result follows. ∎

4.14. COROLLARY. *For any small restriction category* $\mathbb{X}$, *the embedding* $\mathbf{y}_r \colon \mathbb{X} \to \mathsf{PSh}_r(\mathbb{X})$ *exhibits* $\mathsf{PSh}_r(\mathbb{X})$ *as the* free restriction cocompletion *of* $\mathbb{X}$.

5. Free cocompletion of locally small restriction categories

So far in our discussions, we have considered the free cocompletion of a small $\mathcal{M}$-category $\mathbf{C}$ and of a small restriction category $\mathbb{X}$, given by $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$ and $\mathsf{PSh}_r(\mathbb{X})$ respectively. We now turn our attention to the cases where our categories may not necessarily be small, but only locally small. When $\mathbf{C}$ is an ordinary locally small category, we can construct its free cocompletion as the full subcategory $\mathcal{P}(\mathbf{C})$ of $\mathsf{PSh}(\mathbf{C})$ on the small presheaves. (Recall that a presheaf on $\mathbf{C}$ is called *small* if it can be written as a small colimit of representables [Day & Lack, 2007].)

In an entirely analogous way, we would like to define, for each locally small $\mathcal{M}$-category, an $\mathcal{M}$-category of small presheaves which will be its free cocompletion, and then transfer this result across to locally small restriction categories. To begin, we define what we mean by a locally small $\mathcal{M}$-category.

5.1. DEFINITION. *An $\mathcal{M}$-category* $(\mathbf{C}, \mathbf{C}_{\mathcal{M}})$ *is called* locally small *if $\mathbf{C}$ is locally small and $\mathcal{M}$-well-powered. That is, for any object $C \in \mathbf{C}$, the $\mathcal{M}$-subobjects of C form a small partially ordered set.*

5.2. REMARK. Note that this definition is exactly what is required for $\mathsf{Par}(\mathbf{C})$ to be a locally small category, as noted by [Robinson & Rosolini, 1988, p. 99].

By analogy with the case of locally small categories, we define for any locally small $\mathcal{M}$-category $(\mathbf{C}, \mathbf{C}_{\mathcal{M}})$, the $\mathcal{M}$-category of small presheaves $\mathcal{P}_{\mathcal{M}}(\mathbf{C}) = (\mathcal{P}(\mathbf{C}), \mathcal{P}(\mathbf{C})_{\mathcal{M}})$, where $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$ is defined in exactly the same way as for $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$. We begin by showing that $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$ is a stable system of monics.

5.3. LEMMA. *Let $\mathbf{C}$ be a locally small $\mathcal{M}$-category, and let $\mu \colon P \to Q$ be a map in $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$. If $\gamma \colon Q' \to Q$ is a map in $\mathcal{P}(\mathbf{C})$, then the pullback of μ along γ calculated in $\mathsf{PSh}(\mathbf{C})$ is in*

$\mathcal{P}(\mathbf{C})_{\mathcal{M}}$:

$$
\begin{array}{ccc}
P' & \longrightarrow & P \\
\mu' \downarrow & & \downarrow \mu \\
Q' & \xrightarrow{\gamma} & Q.
\end{array}
$$

PROOF. Certainly μ' exists and is in $\mathsf{PSh}(\mathbf{C})_{\mathcal{M}}$ by the fact that $\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})$ is an $\mathcal{M}$-category. So all we need to show is that P' is a small presheaf. Since Q' is small, we may express Q' as $\operatorname{colim} \mathbf{y} D$ for some functor $D \colon \mathbf{I} \to \mathbf{C}$ with $\mathbf{I}$ small. Let us write the colimiting coprojections as $q_I \colon \mathbf{y} D_I \to Q'$. Now μ is a map in $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$, which means that for each $I \in \mathbf{I}$ and composite $\gamma \circ q_I$, there exists an $m_I \colon C_I \to D_I$ in $\mathbf{C}_{\mathcal{M}}$ making the outer square a pullback in:

$$
\begin{array}{ccccc}
\mathbf{y} C_I & \xrightarrow{p_I} & P' & \longrightarrow & P \\
\mathbf{y} m_I \downarrow & & \downarrow \mu' & & \downarrow \mu \\
\mathbf{y} D_I & \xrightarrow{q_I} & Q' & \xrightarrow{\gamma} & Q.
\end{array}
$$

By the same argument as in the proof of Theorem 3.17, it follows that there is a functor $C \colon \mathbf{I} \to \mathbf{C}$ which on objects, takes I to C_I, and that there is a unique map $p_I \colon \mathbf{y} C_I \to P'$ making the left square a pullback for every $I \in \mathbf{I}$. However, because colimits are stable under pullback in $\mathsf{PSh}(\mathbf{C})$, this means $(p_I \colon \mathbf{y} C_I \to P')_{I \in \mathbf{I}}$ is colimiting, which ensures that P' is a small presheaf. ∎

5.4. REMARK. Note that the previous result implies that $\mathcal{P}(\mathbf{C})$ admits pullbacks along $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$-maps, and that these are computed pointwise.

Having now shown that $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$ is a stable system of monics, so that $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is an $\mathcal{M}$-category, we claim that $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is indeed the free cocompletion of $\mathbf{C}$. To do so, however, will first require showing that $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is both locally small and cocomplete.

5.5. LEMMA. *If* $\mathbf{C}$ *is a locally small* $\mathcal{M}$-*category, then* $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ *is locally small.*

PROOF. Since $\mathcal{P}(\mathbf{C})$ is a locally small category [Day & Lack, 2007], all we need to do is show that $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is $\mathcal{M}$-well-powered. So let Q be a small presheaf, and rewrite $Q \cong \operatorname{colim} \mathbf{y} D$, where $D \colon \mathbf{I} \to \mathbf{C}$ is a functor with $\mathbf{I}$ small. Again denote the colimiting coprojections by $(q_I \colon \mathbf{y} D_I \to Q)_{I \in \mathbf{I}}$.

As before, if $\mu \colon P \to Q$ is an $\mathcal{M}$-subobject of Q, then μ induces a functor $C \colon \mathbf{I} \to \mathbf{C}$, which on objects, takes $I \to C_I$, and takes maps $f \colon I \to J$ to the unique map Cf making the diagram in (3.2) commute and the left square of that diagram a pullback. Note that $P \cong \operatorname{colim} \mathbf{y} C$ as colimits are stable under pullback in $\mathcal{P}(\mathbf{C})$. There is also a natural transformation $\alpha \colon C \Rightarrow D$, given componentwise on I by $m_I \in \mathbf{C}_{\mathcal{M}}$ and whose naturality squares are pullbacks for every $I \in \mathbf{I}$.

So given a small presheaf Q, the functors $C \colon \mathbf{I} \to \mathbf{C}$ induced by the $\mathcal{M}$-subobjects of Q (together with $D \colon \mathbf{I} \to \mathbf{C}$ from Q) form an $\mathcal{M}$-category $([\mathbf{I}, \mathbf{C}], [\mathbf{I}, \mathbf{C}]_{\mathcal{M}})$, with the maps in $[\mathbf{I}, \mathbf{C}]_{\mathcal{M}}$ being just the natural transformations whose components are maps in $\mathbf{C}_{\mathcal{M}}$. It is easy to see that $([\mathbf{I}, \mathbf{C}], [\mathbf{I}, \mathbf{C}]_{\mathcal{M}})$ is locally small.

Let $\mathbf{Sub}_{\mathcal{M}}\colon \mathbf{Sub}_{\mathcal{P}(\mathbf{C})_{\mathcal{M}}}(Q) \to \mathbf{Sub}_{[\mathbf{I},\mathbf{C}]_{\mathcal{M}}}(D)$ be the function taking the $\mathcal{M}$-subobjects of Q to the $\mathcal{M}$-subobjects of D. So to show that $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is $\mathcal{M}$-well-powered, it is enough to show that $\mathbf{Sub}_{\mathcal{M}}$ is injective. Let $\mu\colon P \to Q$ and $\mu\colon P' \to Q$ be two $\mathcal{M}$-subobjects of Q which are mapped to the same $\mathcal{M}$-subobject of D. That is, there is an isomorphism from C to C' making the following diagram commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\ \cong\ } & C' \\
& \searrow{\scriptstyle \alpha} \quad \swarrow{\scriptstyle \alpha'} & \\
& D. &
\end{array}
$$

But because $P \cong \operatorname{colim} \mathbf{y}C \cong \operatorname{colim} \mathbf{y}C' \cong P'$, this induces an isomorphism between P and P' making the following diagram commute:

$$
\begin{array}{ccccc}
\mathbf{y}C_I & \xrightarrow{\ p_I\ } & & P & \\
{\scriptstyle \mathbf{y}\alpha_I}\Big\downarrow & \cong\searrow & & \Big\downarrow{\scriptstyle \mu} & \searrow{\scriptstyle \cong} \\
& \mathbf{y}C'_I & \xrightarrow{\ p'_I\ } & & P' \\
& \mathbf{y}\alpha'_I\searrow & & & \swarrow{\scriptstyle \mu'} \\
\mathbf{y}D_I & \xrightarrow{\ q_I\ } & & Q. &
\end{array}
$$

In other words, μ and μ' are the same $\mathcal{M}$-subobject of Q, and so the function $\mathbf{Sub}_{\mathcal{M}}$ is injective. Hence, if $\mathbf{C}$ is a locally small $\mathcal{M}$-category, then so is $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$. ∎

Next, to show that $\mathcal{P}_{\mathcal{M}}(\mathbf{C})$ is cocomplete, we exploit Proposition 3.8 and the following two lemmas.

5.6. LEMMA. *Let $\mathbf{C}$ be a locally small $\mathcal{M}$-category and $\mathcal{I}$ a small set. If $\{\mu_i\colon P_i \to Q_i\}_{i\in\mathcal{I}}$ is a family of maps in $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$, then their coproduct $\sum_{i\in\mathcal{I}} \mu_i$ is also in $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$.*

PROOF. To show that $\sum_{i\in\mathcal{I}} \mu_i$ is in $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$, we need to show that for any map $h\colon \mathbf{y}D \to \sum_{i\in\mathcal{I}} Q_i$ in $\mathcal{P}(\mathbf{C})$ there is a map $m\colon C \to D$ in $\mathbf{C}_{\mathcal{M}}$ making the following diagram a pullback:

$$
\begin{array}{ccc}
\mathbf{y}C & \longrightarrow & \sum_{i\in\mathcal{I}} P_i \\
{\scriptstyle \mathbf{y}m}\Big\downarrow & & \Big\downarrow{\scriptstyle \sum_{i\in\mathcal{I}} \mu_i} \\
\mathbf{y}D & \xrightarrow[\ h\ ]{} & \sum_{i\in\mathcal{I}} Q_i.
\end{array}
$$

Since $\mathcal{P}(\mathbf{C})\big(\mathbf{y}D, \sum_{i\in\mathcal{I}} Q_i\big) \cong \big(\sum_{i\in\mathcal{I}} Q_i\big)(D)$ by the Yoneda lemma, and $\big(\sum_{i\in\mathcal{I}} Q_i\big)(D) \cong \sum_{i\in\mathcal{I}} Q_i D$ as coproducts in $\mathcal{P}(\mathbf{C})$ are taken pointwise, this means h corresponds uniquely with some element in $\sum_{i\in\mathcal{I}} Q_i D$. This, together with the naturality of the bijection $\mathcal{P}(\mathbf{C})(\mathbf{y}D, Q_i) \cong Q_i D$ for each $i \in \mathcal{I}$, imply that $h\colon \mathbf{y}D \to \sum_{i\in\mathcal{I}} Q_i$ factors through exactly one of the coproduct injections $\imath_{Q_j}\colon Q_j \to \sum_{i\in\mathcal{I}} Q_i$. By extensivity of the presheaf category $\mathsf{PSh}(\mathbf{C})$, the pullback of $\sum_{i\in\mathcal{I}} \mu_i$ along $\imath_{Q_i}$ must be μ_i. However, as μ_j is an

$\mathcal{P}(\mathbf{C})_{\mathcal{M}}$-map, there exists an $m\colon C \to D$ in $\mathbf{C}_{\mathcal{M}}$ making the left square of the following diagram commute:

$$
\begin{array}{ccccc}
\mathbf{y}C & \longrightarrow & P_j & \xrightarrow{\iota_{P_j}} & \sum_{i\in\mathcal{I}} P_i \\
{\scriptstyle \mathbf{y}m}\downarrow & & {\scriptstyle \mu_j}\downarrow & & \downarrow{\scriptstyle \sum_{i\in\mathcal{I}}\mu_i} \\
\mathbf{y}D & \xrightarrow{h'} & Q_j & \xrightarrow{\iota_{Q_j}} & \sum_{i\in\mathcal{I}} Q_i.
\end{array}
$$
$$\underbrace{\phantom{\mathbf{y}D \xrightarrow{h'} Q_j}}_{h}$$

Therefore, as both squares are pullbacks, $\mathbf{y}m$ is a pullback of $\sum_{i\in\mathcal{I}}\mu_i$ along h, which means $\sum_{i\in\mathcal{I}}\mu_i \in \mathcal{P}(\mathbf{C})_{\mathcal{M}}$. ∎

5.7. **LEMMA.** *Let $\mathbf{C}$ be a locally small $\mathcal{M}$-category, and suppose m is a map in $\mathcal{P}(\mathbf{C})$. If the pullback of m along some epimorphism is an $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$-map, then m must also be in $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$.*

PROOF. Let $m\colon P \to Q$ be a map in $\mathcal{P}(\mathbf{C})$, and suppose $m'\colon P' \to Q'$ is a pullback of m along some epimorphism $f\colon Q' \to Q$. To show that m is an $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$, let $g\colon \mathbf{y}D \to Q$ be any map in $\mathcal{P}(\mathbf{C})$. Again by Yoneda, there is a bijection $\mathcal{P}(\mathbf{C})(\mathbf{y}D, Q) \cong QD$, giving a corresponding element $\tilde{g} \in QD$. Since f is an epimorphism in $\mathcal{P}(\mathbf{C})$, its component at D, $f_D\colon Q'D \to QD$, must also be an epimorphism, which means there exists some element $\tilde{f}' \in Q'D$ such that $f_D(\tilde{f}') = \tilde{g}$. The naturality of the bijection $\mathcal{P}(\mathbf{C})(\mathbf{y}D, Q) \cong QD$ then implies there is a map $f'\colon \mathbf{y}D \to Q'$ such that $g = ff'$. Now using the fact m' is an $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$-map, there exists a map $n \in \mathbf{C}_{\mathcal{M}}$ such that $\mathbf{y}n$ is the pullback of m' along f':

$$
\begin{array}{ccccc}
\mathbf{y}C & \longrightarrow & P' & \longrightarrow & P \\
{\scriptstyle \mathbf{y}n}\downarrow & & {\scriptstyle m'}\downarrow & & \downarrow{\scriptstyle m} \\
\mathbf{y}D & \xrightarrow{f'} & Q' & \xrightarrow{f} & Q.
\end{array}
$$
$$\underbrace{\phantom{\mathbf{y}D \xrightarrow{f'} Q'}}_{g}$$

Then as both squares are pullbacks, $\mathbf{y}n$ must be the pullback of m along $g = ff'$, making m an $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$-map. ∎

5.8. **LEMMA.** *Let $(\mathbf{C}, \mathbf{C}_{\mathcal{M}})$ be a locally small $\mathcal{M}$-category. Then $(\mathcal{P}(\mathbf{C}), \mathcal{P}(\mathbf{C})_{\mathcal{M}})$ is a cocomplete $\mathcal{M}$-category.*

PROOF. We begin by noting that the category of small presheaves on $\mathbf{C}$, $\mathcal{P}(\mathbf{C})$, is cocomplete. Therefore, it remains to show that the inclusion $\mathcal{P}(\mathbf{C}) \hookrightarrow \mathsf{Par}(\mathcal{P}(\mathbf{C}), \mathcal{P}(\mathbf{C})_{\mathcal{M}})$ is cocontinuous. However, by Proposition 3.8, it is enough to show that the following conditions hold:

(a) If $\{m_i\colon P_i \to Q_i\}_{i\in\mathcal{I}}$ is a family of maps in $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$ indexed by a small set $\mathcal{I}$, then

$\sum_{i \in \mathcal{I}} m_i$ is also in $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$ and the following squares are pullbacks for each $i \in \mathcal{I}$:

$$
\begin{array}{ccc}
P_i & \xrightarrow{\iota_{P_i}} & \sum_{i \in \mathcal{I}} P_i \\
\downarrow m_i & & \downarrow \sum_{i \in \mathcal{I}} m_i \\
Q_i & \xrightarrow{\iota_{Q_i}} & \sum_{i \in \mathcal{I}} Q_i.
\end{array}
$$

(b) Given the following diagram,

$$
\begin{array}{ccccc}
P' & \overset{f'}{\underset{g'}{\rightrightarrows}} & P & \xrightarrow{c'} & G \\
\downarrow m' & & \downarrow m & & \downarrow n \\
Q' & \overset{f}{\underset{g}{\rightrightarrows}} & Q & \xrightarrow{c} & H
\end{array}
$$

if $m \in \mathcal{P}(\mathbf{C})_{\mathcal{M}}$ and the left two squares are pullbacks, and c, c' are the coequalisers of f, g and f', g' respectively, then the unique map n making the right square commute is in $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$ and the right square is also a pullback.

(c) Colimits in $\mathcal{P}(\mathbf{C})$ are stable under pullback along $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$-maps.

To see that (c) holds, recall that $\mathcal{P}(\mathbf{C})$ admits pullbacks along $\mathcal{P}(\mathbf{C})_{\mathcal{M}}$-maps, and that these are calculated pointwise as in **Set** (Remark 5.4). The result then follows from the fact that colimits in $\mathcal{P}(\mathbf{C})$ are also calculated pointwise together with the fact colimits are stable under pullback in **Set**.

For (b), it will be enough to show that the square on the right in (b) is a pullback (by Lemma 5.7). Now the right square is a pullback in $\mathcal{P}(\mathbf{C})$ if and only if componentwise for every $A \in \mathbf{C}$, it is a pullback in **Set**. So consider the diagram in (b) componentwise at $A \in \mathbf{C}$:

$$
\begin{array}{ccccc}
P'A & \overset{f'_A}{\underset{g'_A}{\rightrightarrows}} & PA & \xrightarrow{c'_A} & GA \\
\downarrow m'_A & & \downarrow m_A & & \downarrow n_A \\
Q'A & \overset{f_A}{\underset{g_A}{\rightrightarrows}} & QA & \xrightarrow{c_A} & HA.
\end{array}
$$

The two left squares remain pullbacks in **Set**, and c_A, c'_A remain coequalisers of f_A, g_A and f'_A, g'_A respectively since colimits in $\mathcal{P}(\mathbf{C})$ are calculated pointwise. Observe also that m_A is a monomorphism as maps between small presheaves in $\mathcal{P}(\mathbf{C})$ are monic if and only if they are componentwise monic for every $A \in \mathbf{C}$ (by a Yoneda argument). Now we know that the $\mathcal{M}$-category $(\mathbf{Set}, \mathbf{Inj})$ (where **Inj** are all the injective functions) is a cocomplete $\mathcal{M}$-category (Example 3.4), and since m_A is monic, the square on the right must be a pullback in **Set**. Therefore, as pullbacks in $\mathcal{P}(\mathbf{C})$ are calculated pointwise, the square on the right of (b) must also be a pullback.

For (a), we know that $\sum_{i \in I} m_I \in \mathcal{P}(\mathbf{C})_{\mathcal{M}}$ from Lemma 5.6. Then, as $(\mathbf{Set}, \mathbf{Inj})$ is cocomplete and both pullbacks and colimits in $\mathcal{P}(\mathbf{C})$ are computed pointwise as in $\mathbf{Set}$, the result follows by an analogous argument to (b).

Therefore, $(\mathcal{P}(\mathbf{C}), \mathcal{P}(\mathbf{C})_{\mathcal{M}})$ is a cocomplete $\mathcal{M}$-category. ∎

5.9. THEOREM. *Let $\mathbf{C}$ be a locally small $\mathcal{M}$-category, and let $\mathbf{D}$ be a locally small, cocomplete $\mathcal{M}$-category. Then the following is an equivalence of categories:*

$$(-) \circ \mathbf{y} \colon \mathcal{M}\mathbf{Cocomp}(\mathcal{P}_{\mathcal{M}}(\mathbf{C}), \mathbf{D}) \to \mathcal{M}\mathbf{CAT}(\mathbf{C}, \mathbf{D})$$

where $\mathcal{M}\mathbf{CAT}$ is the 2-category of locally small $\mathcal{M}$-categories.

PROOF. The proof follows exactly the same arguments presented in the proof of Theorem 3.17. ∎

5.10. COROLLARY. *For any locally small restriction category $\mathbb{X}$ and locally small, cocomplete restriction category $\mathbb{E}$, the following is an equivalence of categories:*

$$(-) \circ \Lambda \colon \mathbf{rCocomp}(\mathsf{Par}(\mathcal{P}_{\mathcal{M}}(\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))), \mathbb{E}) \to \mathbf{rCAT}(\mathbb{X}, \mathbb{E})$$

where Λ is the Cockett and Lack embedding introduced in (3.7) and $\mathbf{rCAT}$ is the 2-category of locally small restriction categories.

Given that a small presheaf on an ordinary category is one that can be written as a colimit of small representables, it is natural to ask whether there is a similar notion of *small restriction presheaf.* So let $\mathbb{X}$ be a locally small restriction category, and denoting the $\mathcal{M}$-category $\mathcal{M}\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))$ by $\mathbf{C}$, the previous corollary says that $\mathsf{Par}(\mathcal{P}_{\mathcal{M}}(\mathbf{C}))$ is the free cocompletion of $\mathbb{X}$. Since $\mathcal{P}(\mathbf{C})$ is a full replete subcategory of $\mathsf{PSh}(\mathbf{C})$ and $\mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})) \simeq \mathsf{PSh}_r(\mathbb{X})$, there exists a full subcategory $\mathcal{P}_r(\mathbb{X}) \subset \mathsf{PSh}_r(\mathbb{X})$ which is equivalent to $\mathsf{Par}(\mathcal{P}_{\mathcal{M}}(\mathbf{C}))$:

$$\begin{array}{ccc} \mathcal{P}_r(\mathbb{X}) & \xrightarrow{\simeq} & \mathsf{Par}(\mathcal{P}_{\mathcal{M}}(\mathbf{C})) \\ \downarrow & & \downarrow \\ \mathsf{PSh}_r(\mathbb{X}) & \xrightarrow[\simeq]{} & \mathsf{Par}(\mathsf{PSh}_{\mathcal{M}}(\mathbf{C})) \end{array}$$

where the above square is a pullback and the bottom map is the equivalence from Theorem 4.13.

To see what objects should be in $\mathcal{P}_r(\mathbb{X})$, it is enough to apply Total to the above diagram, giving the following pullback:

$$\begin{array}{ccc} \mathsf{Total}(\mathcal{P}_r(\mathbb{X})) & \longrightarrow & \mathcal{P}(\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))) \\ \downarrow & & \downarrow \\ \mathsf{Total}(\mathsf{PSh}_r(\mathbb{X})) & \xrightarrow[G]{} & \mathsf{PSh}(\mathsf{Total}(\mathsf{K}_r(\mathbb{X}))) \end{array}$$

where G is an equivalence. Since the above diagram is a pullback, an object P will be in $\mathsf{Total}(\mathcal{P}_r(\mathbb{X}))$ (and hence in $\mathcal{P}_r(\mathbb{X})$) if GP is an object in $\mathcal{P}(\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))$; that is, $GP \cong \mathrm{colim}\,\mathbf{y}C_I$, where $C\colon \mathbf{I} \to \mathsf{Total}(\mathsf{K}_r(\mathbb{X}))$ is a functor with $\mathbf{I}$ small. If we define H to be a pseudo-inverse for G, then an object will be in $\mathcal{P}_r(\mathbb{X})$ if it is of the form $P \cong \mathrm{colim}\,H\mathbf{y}C_I$, for some small $\mathbf{I}$ and functor $C\colon \mathbf{I} \to \mathsf{Total}(\mathsf{K}_r(\mathbb{X}))$. We call such a P a *small restriction presheaf*.

We also give an explicit description of a small restriction presheaf as follows. Since GP is an object in $\mathcal{P}(\mathsf{Total}(\mathsf{K}_r(\mathbb{X})))$, it will be the colimit of a small diagram whose vertices are of the form $\mathbf{y}(A, e)$, where (A, e) is an object in $\mathsf{K}_r(\mathbb{X})$. Now given $(A, e) \in \mathsf{K}_r(\mathbb{X})$, note the following splitting in $\mathsf{PSh}_r(\mathbb{X})$:

$$
\begin{array}{ccc}
 & Q(A,e) & \\
 \nearrow & & \searrow \\
\mathbf{y}_r A & \xrightarrow{\quad \mathbf{y}_r e \quad} & \mathbf{y}_r A.
\end{array}
$$

This gives a functor $Q\colon \mathsf{K}_r(\mathbb{X}) \to \mathsf{PSh}_r(\mathbb{X})$. Then a restriction presheaf is called *small* if it is the colimit of some functor $D\colon \mathbf{I} \to \mathsf{PSh}_r(\mathbb{X})$ ($\mathbf{I}$ small), where each DI is of the form $Q(A, e)$ for some $(A, e) \in \mathsf{K}_r(\mathbb{X})$, and each $D(f\colon I \to J)$ is total. We denote by $\mathcal{P}_r(\mathbb{X})$ the restriction category whose objects are small restriction presheaves on $\mathbb{X}$. By construction, it is also the free cocompletion of $\mathbb{X}$. It is not difficult to check that when $\mathbb{X}$ is a small restriction category, restriction presheaves on $\mathbb{X}$ are small, and so $\mathcal{P}_r(\mathbb{X}) = \mathsf{PSh}_r(\mathbb{X})$.

References

[Booth & Brown, 1978] P. Booth & R. Brown (1978). Spaces of partial maps, fibred mapping spaces and the compact-open topology. Topology Appl. *8*, 181-195.

[Carboni, 1987] A. Carboni (1987). Bicategories of partial maps, Cahiers Topologie Géom. Différentielle Catég. *28*, 111-126.

[Cockett & Cruttwell, 2014] J. Cockett & G. Cruttwell (2014). Differential structure, tangent structure, and SDG. Appl. Categ. Structures *22*, 331-417.

[Cockett & Lack, 2002] J. Cockett & S. Lack (2002). Restriction categories I: categories of partial maps. Theoret. Comput. Sci. *270*, 223-259.

[Cockett & Lack, 2003] J. Cockett & S. Lack (2003). Restriction categories II: partial map classification. Theoret. Comput. Sci. *294*, 61-102.

[Cockett & Lack, 2007] J. Cockett & S. Lack (2007). Restriction categories III: colimits, partial limits and extensivity. Math. Structures Comput. Sci. *17*, 775-817.

[Day & Lack, 2007] B. Day & S. Lack (2007). Limits of small functors. J. Pure Appl. Algebra *210*, 651-663.

[Di Paola & Heller, 1987] R. Di Paola & A. Heller (1987). Dominical categories: recursion theory without elements. J. Symbolic Logic *52*, 594-635.

[Ehresmann, 1957] C. Ehresmann (1957). Gattungen von lokalen Strukturen. Jahresbericht der Deutschen Mathematiker-Vereinigung *60*, 49–77.

[Freyd, 1964] P. Freyd (1964). Abelian categories: An Introduction to the Theory of Functors. Harper & Row. Republished in Reprints in Theory and Applications of Categories, No. 3 (2003), 23-164.

[Grandis, 1990] M. Grandis (1990). Cohesive categories and manifolds. Ann. Mat. Pura Appl. (4) *157*, 199-244.

[Heller, 1983] A. Heller (1983). Dominical categories and recursion theory. In C. Bernadi & P. Pagli (eds), Atti degli incontri di Logica Matematica Volume *2*, 339-344. Università di Siena.

[Johnstone, 2002] P. Johnstone (2002). Sketches of an elephant: a topos theory compendium, Volume 1. Oxford University Press.

[Lawson, 1998] M. Lawson (1998). Inverse semigroups: the theory of partial symmetries. World Scientific Publishing.

[Lin, 2019] D. Lin (2019). Presheaves over a join restriction category. Appl. Categ. Structures *27*, 289-310.

[Longo & Moggi, 1984] G. Longo & E. Moggi (1984). Cartesian closed categories of enumerations and effective type structures. In G. Kahn, D.B. MacQueen & G. Plotkin (eds), Semantics of Data Types SDT 1984. Lecture Notes in Computer Science *173*, 235-255. Springer Berlin Heidelberg.

[Mulry, 1994] P. Mulry (1994). Partial map classifiers and partial cartesian closed categories. Theoret. Comput. Sci. *136*, 109-123.

[Robinson & Rosolini, 1988] E. Robinson & G. Rosolini (1988). Categories of partial maps. Inform. and Comput. *79*, 95-130.

[Rosolini, 1986] G. Rosolini (1986). Continuity and effectiveness in topoi. PhD thesis, University of Oxford.

[Rosolini, 1987] G. Rosolini (1987). Categories and effective computations. In D.H. Pitt, A. Poigné & D.E. Rydeheard (eds), Category Theory and Computer Science. Lecture Notes in Computer Science *283*, 1-11. Springer Berlin Heidelberg.

[Rosolini, 1988] G. Rosolini (1988). Representation theorems for p-categories. In F. Borceux (ed), Categorical Algebra and its Applications. Lecture Notes in Mathematics *1348*, 307-315. Springer Berlin Heidelberg.

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