Theory and Applications of Categories, Vol. 35, No. 36, 2020, pp. 1379–1423.

LAX ORTHOGONAL FACTORISATIONS IN ORDERED STRUCTURES

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Abstract. We give an account of lax orthogonal factorisation systems on order-enriched categories. Among them, we define and characterise the KZ-reflective ones, in a way that mirrors the characterisation of reflective orthogonal factorisation systems. We use simple monads to construct lax orthogonal factorisation systems, such as one on the category of T_0 topological spaces closely related to continuous lattices.

1. Introduction

Weak factorisation systems (WFSs) have been a feature of Homotopy Theory even before Quillen's definition of model categories and the recognition of their importance. WFSs can be described as a pair of classes of morphisms $(\mathcal{L}, \mathcal{R})$ that satisfy three properties. First, each morphism of the category must be a composition of a morphism from $\mathcal{L}$ followed by one of $\mathcal{R}$, perhaps in a non-unique way. Secondly, each $r \in \mathcal{R}$ must have the *right lifting property* with respect to each $\ell \in \mathcal{L}$; in other words, each commutative square, as displayed, has a (not necessarily unique) diagonal filler.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\ell"'] & \cdot \arrow[d, "r"] \\ \cdot \arrow[r] \arrow[ur, dotted] & \cdot \end{tikzcd} \tag{1.1}$$

Thirdly, $(\mathcal{L}, \mathcal{R})$ is, in a precise way, maximal. Each one of Quillen's model categories comes equipped with two WFS: (cofibrations, trivial fibrations) and (trivial cofibrations, fibrations).

Orthogonal factorisations systems (OFS) can be described as WFSs in which the diagonal filler (1.1) not only exists but it is unique. This makes the factorisation of a morphism f as $f = r \cdot \ell$, with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$, unique up to unique isomorphism. Two typical

Research partially supported by the Centre for Mathematics of the University of Coimbra – UIDB/00324/2020, funded by the Portuguese Government through FCT/MCTES.
The second author was supported by a Research Fellowship of Gonville and Caius College, Cambridge, the Department of Mathematics and Mathematical Statistics of the University of Cambridge, PEDECIBA, ANII and Universidad de la República.
Received by the editors 2018-01-15 and, in final form, 2020-08-07.
Transmitted by Giuseppe Rosolini. Published on 2020-08-11.
2020 Mathematics Subject Classification: Primary 18D05, 18A32. Secondary 55U35.
Key words and phrases: lax orthogonal factorization system, lax idempotent monad, order-enriched category, weak factorization system, reflective factorization system, continuous lattice.

examples of OFSs are the factorisation of a function as a surjection followed by an injection, and of a continuous map between topological spaces as a surjection followed by an embedding (i.e., a homeomorphism onto its image). As far as we can discern, OFSs are coeval with WFSs but arose independently, inspired by different examples.

When the ambient category has a terminal object, denoted by 1, there is a case of (1.1) of special interest, namely:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\ell"'] & A \arrow[d] \\ \bullet \arrow[r] \arrow[ur, dotted] & 1 \end{tikzcd} \tag{1.2}$$

If the unique morphism $A \to 1$ has the right (unique) lifting property with respect to ℓ, one says that A is injective with respect (resp., orthogonal to) ℓ. Clearly each OFS $(\mathcal{L}, \mathcal{R})$ gives rise to a class of objects that are orthogonal to each member of $\mathcal{L}$: those objects A such that $A \to 1$ belongs to $\mathcal{R}$. The extent to which $(\mathcal{L}, \mathcal{R})$ is determined by this class of objects is the subject of study of [7]. The OFSs so determined are called *reflective*.

In addition to their widespread use in homological algebra, injective objects play a role in many other areas of mathematics. For example, *hyperconvex* spaces are the objects injective with respect to isometries in the category of metric spaces and non-expansive maps (see [2, 17]). Another example is provided by complete Boolean algebras, which are injective with respect to injections in the categories of Boolean algebras, Heyting algebras and distributive lattices [16, 4, 3].

There are examples, as those introduced by D. Scott [29], of squares (1.2) where the diagonal filler is not unique but there exists a smallest one, with respect to an ordering between morphisms. The main example from [29] consists of those topological spaces that arise from endowing *continuous lattices* with the Scott topology. These spaces are characterised by their injectivity with respect to topological embeddings. In fact, if ℓ is a topological embedding and A is a continuous lattice in (1.2), there is a diagonal filler that is the smallest with respect to the (opposite of) the pointwise specialisation order (see §18 for more details).

Another example comes from complete lattices, which can be characterised as those posets that are injective with respect to embeddings of posets. As in the previous example, in the situation (1.2) where A is a complete lattice and ℓ is a poset embedding, there exists a smallest diagonal filler.

Motivated by the above examples, one can generalise the existence of a smallest diagonal filler in the situation (1.2) to the situation of a commutative square (1.1). By doing so, one arrives to the notion of lax orthogonal factorisation system.

The present paper gives an account, in the context of order-enriched categories, of *lax orthogonal factorisation systems* (LOFS), a notion that sits between OFSs and WFSs.

orthogonal factorisation system	$\subset$	lax orthogonal factorisation system	$\subset$	weak factorisation system

LOFSs were introduced and studied in the context of 2-categories by the authors in [8]. We build on ibid. to prove new results on reflective LOFSs as well as to provide new examples.

In a LOFS, the existence of a diagonal filler (1.1) is replaced by the existence of a smallest diagonal filler. More precisely, there is a diagonal filler d with the property that $d \leqslant d'$ for any other diagonal filler d'.

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\ell"'] & \cdot \arrow[d, "r"] \\ \cdot \arrow[r] \arrow[ur, bend left, "d"] \arrow[ur, bend right, "d'"'] \arrow[ur, phantom, "\leqslant"] & \cdot \end{tikzcd}$$

Since morphisms between two objects in an order-enriched category form a poset, the above property uniquely defines the smallest diagonal filler. There are, however, advantages in providing these diagonals by means of an algebraic structure, instead of postulating the existence of a smallest diagonal filler. This algebraic structure is provided by the *algebraic weak factorisation systems* (AWFSs), introduced with a different name in [15] and slightly modified in [14]; we use the definition given in the latter.

An AWFS on an order-enriched category $\mathcal{C}$ consists of a locally monotone comonad L and a locally monotone monad R on $\mathcal{C}^{\mathbf{2}}$ interrelated by axioms, and that define a locally monotone functorial factorisation $f = Rf \cdot Lf$. Inspired by the observation of [15] that OFSs correspond to AWFSs whose monad and comonad are idempotent, we defined in [8] LOFSs as AWFSs whose monad and comonad are lax idempotent, or Kock-Zöberlein. We reprise this definition in the context of order-enriched categories, which enables some simplifications.

A fundamental example of LOFS on the order-enriched category of posets factors each morphism as a left adjoint right inverse (or LARI) followed by a split opfibration. This factorisation can be constructed on any order-enriched category with sufficient (finite) limits, and plays a similar role for LOFSs as the factorisation isomorphism–morphism (that factors f as $1_{\mathrm{dom}(f)}$ followed by f) plays for OFSs (§5).

WHAT IS NEW IN THIS PAPER?

1. We introduce KZ-*reflective* LOFSs as those LOFSs (L, R) that are determined by the restriction of the monad R on $\mathcal{C}^{\mathbf{2}}$ to $\mathcal{C}$ (here $\mathcal{C}$ is viewed as the full subcategory of $\mathcal{C}^{\mathbf{2}}$ with objects of the form $A \to 1$). We characterise KZ-reflective LOFSs as those satisfying the following properties: i) each LARI is an L-coalgebra and each morphism of the former is a morphism of the latter; ii) if $g \cdot f$ and g are L-coalgebras, then so is f; iii) a similar requirement at the level of morphisms of $\mathcal{C}^{\mathbf{2}}$ (see §12). For example, the LOFS of LARI–split opfibration mentioned above will be reflective with our definition.

2. After recalling the simple monads of [8] (§15), we obtain LOFSs on the category of T_0 topological spaces as a consequence of the simplicity of a certain monad: the *filter monad*, which associates to each topological space the space of filters of its open subsets endowed with a natural topology (§18). The algebras for the filter monad are precisely the continuous lattices (with the Scott topology). The induced LOFS on (T_0) topological spaces has an associated WFS that was considered in [6]. We also provide easy-to-verify conditions guaranteeing that a submonad of a simple lax idempotent monad enjoys these same properties (§16). When applied to the filter monad we obtain LOFSs closely related to continuous Scott domains, stably compact spaces and sober spaces.

3. Another example that we obtain from a simple monad is a LOFS on the order-enriched category of (skeletal) generalised metric spaces §19. The restriction of this LOFS to the category of metric spaces yields an OFS whose left class of morphisms are the dense inclusions. Further examples are explored in [9] in a very general framework that covers, for example, R. Lowen's *approach spaces* as well as the examples mentioned above.

4. A third example consists of a LOFS on the category of distributive lattices, induced by the monad that associates to each distributive lattice its frame of ideals. The left morphisms consist of inclusions of full sub-lattices $A \subseteq B$ with the property that, for all $a \in A$, if $a \leqslant b \vee b'$, then there are $x, x' \in A$ with $a \leqslant x \vee x'$ and $x \leqslant b$, $x' \leqslant b'$. Frames are characterised as the distributive lattices that are injective with respect to these inclusions.

5. We present some results on the structure of LOFSs. For example, we show that the category of LOFSs and morphisms of AWFSs is a preorder (§6) and a morphism $(\mathsf{L}, \mathsf{R}) \to (\mathsf{L}', \mathsf{R}')$ is the same as a morphism of comonads $\mathsf{L} \to \mathsf{L}'$ and the same as a morphism of monads $\mathsf{R}' \to \mathsf{R}$ (§7). We improve on results from [8] to obtain a more satisfactory characterisation of LOFSs in terms of KZ-lifting operations (Theorem 9.2). Furthermore, we show that, in our context of order-enriched categories, some axioms of LOFS are redundant. For example, the distributivity axiom, part of the definition of AWFS, can be omitted as long as the comonad and monad are lax idempotent (§10).

2. Lax idempotent monads

We shall denote the category of posets (sets with an order relation that is reflexive, transitive and anti-symmetric) and order-preserving morphisms by **Ord**. A category enriched in **Ord**, or **Ord**-category, amounts to a category whose homs are posets, and whose composition preserves the order. An **Ord**-functor between **Ord**-categories is the same as a functor that preserves the order of morphisms.

2.1. DEFINITION. A monad $\mathsf{T} = (T, \eta, \mu)$ on an **Ord**-category $\mathcal{C}$ is *lax idempotent*, or *Kock–Zöberlein*, if it satisfies any of the following equivalent conditions.

1. $T\eta \cdot \mu \leqslant 1$.
2. $1 \leqslant \eta T \cdot \mu$.
3. For any T-algebra $a\colon TA \to A$, the inequality $1_{TA} \leqslant \eta_A \cdot a$ holds.
4. A morphism $l : TA \to A$ defines a T-algebra structure (A, l) if and only if $l \dashv \eta_A$ with $l \cdot \eta_A = 1_A$.
5. $T\eta \leqslant \eta T$.

6. For any pair of T-algebras (A,a) and (B,b) and all morphisms $f\colon A \to B$ in $\mathcal{C}$, $b \cdot Tf \leqslant f \cdot a$ holds.
7. For any T-algebra (A,a) and any morphism $f\colon X \to A$ in $\mathcal{C}$, the equality $a \cdot Tf \cdot \eta_X = f$ exhibits $a \cdot Tf$ as a left extension of f along $\eta_X\colon X \to TX$.

The equivalences of the above conditions can be found, in the more general case of 2-categories, in [22]. Morphisms f satisfying condition (6) are called *lax morphisms* of T-algebras, even for a monad T that is not lax idempotent; so condition (6) says that T is lax idempotent if any morphism in $\mathcal{C}$ between T-algebras is a lax morphism of T-algebras.

2.2. Definition. The notion of a lax idempotent comonad $\mathsf{G} = (G, \varepsilon, \delta)$ is a dual one: G is a lax idempotent comonad on $\mathcal{C}$ if $(G^{\mathrm{op}}, \varepsilon^{\mathrm{op}}, \delta^{\mathrm{op}})$, the corresponding monad on $\mathcal{C}^{\mathrm{op}}$, is lax idempotent.

2.3. Example. Given an ordered set X, denote by $P(X)$ the set of down-closed subsets of X, ordered by the inclusion. The assignment $X \mapsto P(X)$ can be extended to a functor whose value on a monotone function $f\colon X \to Y$ is

$$P(X) \xrightarrow{f_*} P(Y) \qquad f_*(Z) = \{y \in Y : (\exists x \in Z)(y \leqslant f(x))\} = \bigcup_{x \in Z} {\downarrow} f(x).$$

Clearly, $f_* \leqslant g_*$ if $f \leqslant g$, so P is an **Ord**-functor. It is well-known that $X \mapsto P(X)$ defines a monad on **Ord**, with unit

$$\eta_X\colon X \longrightarrow P(X) \qquad x \mapsto {\downarrow} x = \{y \in X : y \leqslant x\}$$

and multiplication $\mu\colon P^2(X) \to P(X)$ given by $\big(\mathcal{U} \subseteq P(X)\big) \mapsto \bigcup \mathcal{U}$. This **Ord**-monad on the **Ord**-category $\underline{\mathbf{Ord}}$ is lax idempotent, since

$$P\eta_X(Z) = \bigcup_{x \in Z} {\downarrow}({\downarrow} x) \subseteq {\downarrow} Z = \eta_{P(X)}(Z).$$

The **Ord**-category P-Alg is the category of complete lattices with morphisms those monotone maps that preserve arbitrary suprema.

2.4. Example. If X is a topological space, there is a preorder on the set X given by $x \leqslant y$ when $y \in \overline{\{x\}}$. This is the opposite of the so-called *specialisation order*, and makes X into a poset precisely when X is a T_0 topological space.

For $\mathbf{Top}_0$ the category of T_0 topological spaces, $\underline{\mathbf{Top}}_0$ is the associated **Ord**-category, with ordering $f \leqslant g\colon X \to Y$ if $f(x) \leqslant g(x)$ for all $x \in X$. There is an endo-**Ord**-functor $F\colon \underline{\mathbf{Top}}_0 \to \underline{\mathbf{Top}}_0$ that sends X to the set $F(X)$ of filters of open sets of X, with topology generated by the subsets $U^\# = \{\varphi \in F(X) : U \in \varphi\}$, for $U \in \mathcal{O}X$. This is in fact the functor part of the lax idempotent *filter monad* on $\underline{\mathbf{Top}}_0$ that will be considered in Section 18.

There is a well-known result about algebras for lax idempotent monads on **Ord**-categories (see [24] and [13]) that can be summarised by saying that algebras are closed under retracts. More precisely:

2.5. Lemma. *If $\mathsf{T} = (T, \eta, \mu)$ is a lax idempotent monad on an* **Ord***-category, the following conditions on an object A are equivalent.*

1. *A admits a (unique) T-algebra structure (we simply say that A is a T-algebra).*
2. *There is an adjunction $a \dashv \eta_A \colon A \to TA$ with $a \cdot \eta_A = 1$.*
3. *$\eta_A \colon A \to TA$ has a retract.*
4. *A is a retract of TA.*
5. *A is a retract of a T-algebra.*

In this case, $a \colon TA \to A$ is a T-algebra structure on A.

2.6. Remark. The lemma means that, if T is a lax idempotent monad on $\mathcal{A}$, then the **Ord**-category T-Alg may be regarded as a locally full sub-**Ord**-category of $\mathcal{A}$. Part of the proof consists in showing that if $b \colon TB \to B$ is an algebra structure and $r \colon B \to A$ is a retract with section s, then $r \cdot b \cdot Ts$ is an algebra structure for A, and s becomes a morphism of algebras $A \to B$. One could say that $\mathsf{T}\text{-Alg} \subset \mathcal{A}$ is a *discrete $\mathcal{S}$-fibration* where $\mathcal{S}$ is the class of sections in $\mathcal{A}$.

Note that η_A may have at most one retract, which is *a fortiori* a T-algebra structure for A (just set $B = TA$ and $b = \mu_A$ in the previous paragraph). Furthermore, the inclusion $\mathsf{T}\text{-Alg} \subset (T, \eta)\text{-Alg}$ into the **Ord**-category of algebras for the pointed endo-**Ord**-functor (T, η) is an equality.

In the following lemma, the term monad morphism is used in strict sense of a natural transformation that is compatible with the multiplications and units, i.e., a monoid morphism.

2.7. Lemma. *Let T and S be monads on an* **Ord***-category. Then there is at most one monad morphism $\mathsf{T} \to \mathsf{S}$ if T is lax idempotent.*

Proof. Suppose that $\varphi_X \colon TX \to SX$ are the components of a monad morphism. The morphism

$$\psi_X \colon TSX \xrightarrow{\varphi_{SX}} S^2X \xrightarrow{\mu^S_X} SX$$

is a T-algebra structure on SX, and therefore it is uniquely defined as the left adjoint to the unit $SX \to TSX$. Therefore, $\varphi_X = \psi_X \cdot T(\eta^S_X)$ is uniquely determined. ∎

3. Orthogonal factorisations and simple reflections, revisited

In this section we revisit some of the material of Cassidy–Hébert–Kelly work on simple reflections [7] from a slightly different perspective, more amenable to generalisation.

Suppose that $T \colon \mathcal{A} \to \mathcal{A}$ is a reflection, with unit $\eta_A \colon A \to TA$, on the category $\mathcal{A}$, which we assume to admit pullbacks. The corresponding reflective subcategory will be denoted by T-Alg, as it consists of the algebras for the idempotent monad T associated to T, whose invertible multiplication we denote by $\mu\colon T^2 \Rightarrow T$.

We say that a morphism f in $\mathcal{A}$ is a *T-isomorphism*, or is *T-invertible*, if Tf is an isomorphism.

Each morphism $f\colon A \to B$ can be factorised through a pullback square, as displayed.

$$f = Rf \cdot Lf \qquad \begin{tikzcd} A \arrow[dr, "Lf"] \arrow[drr, bend left, "\eta_A"] \arrow[ddr, bend right, "f"'] & & \\ & Kf \arrow[r, "q_f"] \arrow[d, "Rf"] \arrow[dr, phantom, "\text{pb}"] & TA \arrow[d, "Tf"] \\ & B \arrow[r, "\eta_B"] & TB \end{tikzcd}$$

3.1. Remark. The factorisation $f = Rf \cdot Lf$ is *functorial*, in the sense that, if $(h,k)\colon f \to g$ is a morphism in the arrow category $\mathcal{A}^{\mathbf{2}}$, then there is a morphism $K(h,k)\colon Kf \to Kg$

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "k"] & \cdot \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "Lf"'] & \cdot \arrow[d, "Lg"] \\ Kf \arrow[r, "K(h,k)"] \arrow[d, "Rf"'] & Kg \arrow[d, "Rg"] \\ \cdot \arrow[r, "k"] & \cdot \end{tikzcd}$$

yielding a functor $K\colon \mathcal{A}^{\mathbf{2}} \to \mathcal{A}$.

3.2. Remark. The assignment that sends a morphism $f \mapsto Lf$ is part of an endofunctor on $\mathcal{A}^{\mathbf{2}}$, given on morphisms by

$$f \xrightarrow{(h,k)} g \quad \longmapsto \quad Lf \xrightarrow{(h,K(h,k))} Lg.$$

Furthermore, there is a natural transformation $\Phi\colon L \Rightarrow 1$ with components

$$\Phi_f \quad = \quad \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "Rf"] & \cdot \end{tikzcd} .$$

3.3. Remark. The assignment $f \mapsto Rf$ underlies a monad on the arrow category $\mathcal{A}^{\mathbf{2}}$. Its unit and multiplication are given by

$$\Lambda_f \quad = \quad \begin{tikzcd} \cdot \arrow[r, "Lf"] \arrow[d, "f"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd} \qquad \Pi_f \quad = \quad \begin{tikzcd} \cdot \arrow[r, "\pi_f"] \arrow[d, "R^2 f"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

where the morphism $\pi_f\colon KRf \to Kf$ is the unique morphism into the pullback Kf such that $q_f \cdot \pi_f = \mu_{\mathrm{dom}(f)} \cdot Tq_f \cdot q_{Rf}$ and $Rf \cdot \pi_f = RRf$.

3.4. Definition. [7] The reflection $\mathsf{T} = (T, \eta)$ is *simple* if Lf is a T-isomorphism.

As pointed out in [7], if T is simple then the factorisation $f = Rf \cdot Lf$ defines an orthogonal factorisation system, with left class of morphisms that of T-isomorphisms. To say only a few words about this fact, any morphism of the form Tf is orthogonal to T-isomorphisms, and so Rf, as a pullback of Tf, is also orthogonal to T-isomorphisms; together with the simplicity hypothesis that Lf be a T-isomorphism, we obtain an orthogonal factorisation.

We recall that a copointed endofunctor $\Gamma\colon G \Rightarrow 1$ is *well-copointed* if $G\Gamma = \Gamma G$ (see [19, p. 4] for the definition in the pointed context). For example, the copointed endofunctor of an idempotent comonad is well-copointed. The category of coalgebras for a well-copointed endofunctor (G, Γ) on a category $\mathcal{C}$ is the full subcategory defined by those objects $C \in \mathcal{C}$ for which Γ_C is invertible [19, Prop. 5.2].

A well-copointed endofunctor (G, Γ) underlies a comonad precisely when each $G(C)$ is a (G, Γ)-coalgebra, i.e., when $\Gamma G = G\Gamma$ is invertible. In this case, the comultiplication has components $\Gamma^{-1}_{G(C)}\colon G(C) \to G^2(C)$. The category of coalgebras for the comonad coincides with that of (G, Γ).

An example we shall use is given by the full subcategory $\mathcal{C}^{\mathbf{2}}$ consisting of isomorphisms. It is isomorphic to (I, Υ)-Coalg, where $\Upsilon\colon I \Rightarrow 1_{\mathcal{C}^{\mathbf{2}}}$ is the well-copointed endofunctor given by $I(f) = 1_{\mathrm{dom}(f)}$ and $\Upsilon_f = (1, f)\colon 1_{\mathrm{dom}(f)} \to f$. Furthermore, (I, Υ) underlies an idempotent comonad I on $\mathcal{C}^{\mathbf{2}}$.

If we denote by $F^T\colon \mathcal{A} \to \mathsf{T}\text{-Alg}$ the left adjoint of the inclusion $\mathsf{T}\text{-Alg} \subset \mathcal{A}$, then we can consider the full subcategory $\mathsf{T}\text{-Iso} \subset \mathcal{A}^{\mathbf{2}}$ whose objects are those morphisms of $\mathcal{A}$ that are T-isomorphisms (equivalently, those morphisms f such that $F^T(f)$ is an isomorphism) as a pullback.

$$\begin{tikzcd}
\mathsf{T}\text{-Iso} \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & \mathrm{Iso} \arrow[d] \\
\mathcal{A}^{\mathbf{2}} \arrow[r, "(F^T)^{\mathbf{2}}"] & \mathsf{T}\text{-Alg}^{\mathbf{2}}
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\mathrm{pb}"] & (U^T)^{\mathbf{2}} I (F^T)^{\mathbf{2}} \arrow[d, "(U^T)^{\mathbf{2}}\Upsilon(F^T)^{\mathbf{2}}"] \\
1_{\mathcal{A}^{\mathbf{2}}} \arrow[r, "(F^T)^{\mathbf{2}}"] & T^{\mathbf{2}}
\end{tikzcd}$$

The pullback diagram on the right above defines the well-copointed endofunctor (L, Φ) on $\mathcal{A}^{\mathbf{2}}$, and (L, Φ)-Coalg is isomorphic to T-Iso over $\mathcal{A}^{\mathbf{2}}$ (see [19, Prop. 9.2]).

3.5. Lemma. *The following assertions are equivalent.*

1. *The monad* T *is simple.*
2. *The copointed endofunctor* (L, Φ) *underlies a comonad whose category of coalgebras is* $\mathsf{T}\text{-Iso} \hookrightarrow \mathcal{A}^{\mathbf{2}}$.
3. *The copointed endofunctor* (L, Φ) *can be extended to a comonad.*

PROOF. (1)⇒(2) The simplicity of T means that $L(f)$ is a T-isomorphism, that is, it is an (L, Φ)-coalgebra. By the well-copointeness of (L, Φ) this means that each $\Phi_{L(f)}$ is invertible. (L, Φ) underlies an idempotent comonad whose coalgebras are the T-isomorphisms. See the comments on well-copointed endofunctors before this lemma.

(2)⇒(3) is obvious. Finally, if (L, Φ) can be extended to a comonad, as we have already mentioned, this comonad has the same coalgebras as (L, Φ). Then, $L(f) \in \mathsf{T}$-Iso, and we have (3)⇒(1). ∎

3.6. REMARK. The full subcategory $\mathsf{T}\text{-Iso} \subset \mathcal{A}^{\mathbf{2}}$ may be coreflective when T is not simple. See [7, Thm. 3.3].

4. Algebraic weak factorisation systems

Algebraic weak factorisation systems (AWFSs) where first introduced by M. Grandis and W. Tholen in [15], with an extra distributivity condition later added by R. Garner in [14]. In this section we shall give the definition of AWFSs on order-enriched categories, which is the case we will need, even though the definitions remain virtually unchanged.

4.1. DEFINITION. An **Ord**-functorial factorisation on an **Ord**-category $\mathcal{C}$ consists of a factorisation

$$\mathrm{dom} \overset{\lambda}{\Rightarrow} E \overset{\rho}{\Rightarrow} \mathrm{cod}$$

in the category of locally monotone functors $\mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ of the natural transformation dom ⇒ cod with component at $f \in \mathcal{C}^{\mathbf{2}}$ equal to $f\colon \mathrm{dom}(f) \to \mathrm{cod}(f)$. It is important that in this factorisation E should be a locally monotone functor.

As in the case of functorial factorisations on ordinary categories, an **Ord**-functorial factorisation as the one described in the previous paragraph can be equivalently described as:

- A copointed endo-**Ord**-functor $\Phi\colon L \Rightarrow 1_{\mathcal{C}^{\mathbf{2}}}$ on $\mathcal{C}^{\mathbf{2}}$ with $\mathrm{dom}(\Phi) = 1$.
- A pointed endo-**Ord**-functor $\Lambda\colon 1_{\mathcal{C}^{\mathbf{2}}} \Rightarrow R$ on $\mathcal{C}^{\mathbf{2}}$ with $\mathrm{cod}(\Lambda) = 1$.

The three descriptions of an **Ord**-functorial factorisation are related by:

$$\mathrm{dom}(\Lambda_f) = Lf = \lambda_f \qquad \mathrm{cod}(\Phi_f) = Rf = \rho_f. \tag{4.1}$$

4.2. DEFINITION. An *algebraic weak factorisation system*, abbreviated AWFS, on an **Ord**-category $\mathcal{C}$ consists of a pair (L, R), where $\mathsf{L} = (L, \Phi, \Sigma)$ is an **Ord**-comonad and $\mathsf{R} = (R, \Lambda, \Pi)$ is an **Ord**-monad on $\mathcal{C}^{\mathbf{2}}$, such that (L, Φ) and (R, Λ) represent the same **Ord**-functorial factorisation on $\mathcal{C}$ (i.e., the equalities (4.1) hold), plus a distributivity condition that we proceed to explain.

The unit axiom $\Pi\cdot(\Lambda R) = 1$ of the monad R implies, since $\mathrm{cod}(\Lambda) = 1$, that $\mathrm{cod}(\Pi) = 1$; dually $\mathrm{dom}(\Sigma) = 1$, so these transformations have components that look like:

$$\Sigma_f = \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "L^2 f"] \\ \cdot \arrow[r, "\sigma_f"] & \cdot \end{tikzcd} \quad \text{and} \quad \Pi_f = \begin{tikzcd} \cdot \arrow[r, "\pi_f"] \arrow[d, "R^2 f"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd} \tag{4.2}$$

One can form a transformation

$$\Delta\colon LR \Longrightarrow RL \qquad \Delta_f = \begin{tikzcd} Kf \arrow[r, "\sigma_f"] \arrow[d, "LRf"'] \arrow[dr, dotted, "1"] & KLf \arrow[d, "RLf"] \\ KRf \arrow[r, "\pi_f"] & Kf \end{tikzcd}$$

The distributivity axiom requires Δ to be a mixed distributive law between the comonad L and the monad R; this amounts to the commutativity of the following diagrams.

$$\begin{tikzcd} LR^2 \arrow[r, "\Delta R"] \arrow[d, "L\Pi"'] & RLR \arrow[r, "R\Delta"] & R^2L \arrow[d, "\Pi L"] \\ LR \arrow[rr, "\Delta"] & & RL \end{tikzcd} \qquad \begin{tikzcd} LR \arrow[rr, "\Delta"] \arrow[d, "\Sigma R"'] & & RL \arrow[d, "R\Sigma"] \\ L^2R \arrow[r, "L\Delta"] & LRL \arrow[r, "\Delta L"] & RL^2 \end{tikzcd} \tag{4.3}$$

(The two axioms of a mixed distributive law that involve the unit of the monad and the counit of the comonad automatically hold.)

4.3. Example. Each OFS $(\mathscr{E},\mathscr{M})$ on $\mathcal{C}$ gives rise (upon choosing an $(\mathscr{E},\mathscr{M})$-factorisation for each morphism) to an AWFS (L,R), where L is the idempotent comonad associated to the coreflective subcategory $\mathscr{E}\subset\mathcal{C}^{\mathbf{2}}$ and R is the idempotent monad associated to the reflective inclusion $\mathscr{M}\subset\mathcal{C}^{\mathbf{2}}$. Conversely, an AWFS (L,R) with both L and R idempotent induces an OFS. This was first shown in [15, Thm. 3.2], and [5, Prop. 3] further shows that it suffices that either L or R be idempotent.

If (L,R) is an AWFS on $\mathcal{C}$, an L-coalgebra structure on f and an R-algebra structure on g can be depicted by commutative squares

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "f"'] & \cdot \arrow[d, "Lf"] \\ \cdot \arrow[r, "s"] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "Rg"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

and the (co)algebra axioms can be written in the following way (where the morphisms σ_f and π_g are those described in Definition 4.2).

$$\begin{aligned} Rf\cdot s &= 1 \qquad & K(1,s)\cdot s &= \sigma_f\cdot s \\ p\cdot Lg &= 1 \qquad & p\cdot K(p,1) &= p\cdot\pi_g \end{aligned}$$

A morphism of L-coalgebras $(f,s) \to (f',s')$ is a morphism $(h,k)\colon f \to f'$ in $\mathcal{C}^{\mathbf{2}}$ that is compatible with the coalgebra structures in the usual way:

$$K(h,k) \cdot s = s' \cdot k.$$

Similarly, a morphism of R-algebras $(g,p) \to (g',p')$ is a morphism $(u,v)\colon g \to g'$ such that

$$p' \cdot K(u,v) = u \cdot p.$$

With the obvious composition and identities we obtain categories L-Coalg and R-Alg, equipped with forgetful functors into $\mathcal{C}^{\mathbf{2}}$. These are **Ord**-categories by stipulating that the ordering of morphisms of (co)algebras is inherited from the ordering of morphisms in $\mathcal{C}^{\mathbf{2}}$; as a consequence, the forgetful functors from L-Coalg and R-Alg to $\mathcal{C}^{\mathbf{2}}$ become **Ord**-enriched.

4.4. Remark. Each AWFS (L,R) (enriched or not) has an *underlying* WFS $(\mathcal{L},\mathcal{R})$. The class $\mathcal{L}$ consists of all those morphisms that admit a structure of coalgebra over the copointed endofunctor (L,Φ) that underlies L; similarly, $\mathcal{R}$ consists of all those morphisms that admit a structure of an algebra over the pointed endofunctor (R,Λ) that underlies R.

5. LARIs and AWFSs

One of the most important examples of AWFSs for us will be provided by the so-called LARIs.

5.1. Definition. A *left adjoint right inverse*, or LARI, in an **Ord**-category is a morphism f that is part of an adjunction $f \dashv g$ with $1 = g \cdot f$. In the same situation, we say that g is a *right adjoint left inverse*, or RALI.

Suppose given another adjunction $f' \dashv g'$ with $1 = g' \cdot f'$, and morphisms h and k as in the displayed diagram.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, shift right, "f"'] \arrow[d, phantom, "\dashv"] & X' \arrow[d, shift right, "f'"'] \arrow[d, phantom, "\dashv"] \\ Y \arrow[r, "k"'] \arrow[u, shift right, "g"'] & Y' \arrow[u, shift right, "g'"'] \end{tikzcd}$$

We say that (h,k) is a *morphism of* LARIs $f \to f'$, and that (h,k) is a *morphism of* RALIs $g \to g'$, if $f' \cdot h = k \cdot f$ and $g' \cdot k = h \cdot g$. With the obvious notion of composition, LARIs and RALIs form categories that come equipped with forgetful functors into $\mathcal{C}^{\mathbf{2}}$. Furthermore, if $\mathcal{C}$ is an **Ord**-category, there are **Ord**-categories $\mathrm{Lari}(\mathcal{C})$ and $\mathrm{Rali}(\mathcal{C})$ with objects and morphisms described above, and ordering between morphisms those of $\mathcal{C}^{\mathbf{2}}$.

Since LARIs are dual to the LALIs described in some detail in [5, §4.3], we feel free to give only a summary description of the associated AWFS, which is in fact a LOFS.

The inclusion **Ord**-functor $\mathrm{Lari}(\mathcal{C}) \subset \mathcal{C}^{\mathbf{2}}$ is comonadic if $\mathcal{C}$ admits certain **Ord**-enriched limits: the so-called *limits of morphisms*, or in other words, comma-objects of the form $f \downarrow 1_B$, where $f\colon A \to B$. This allows the construction of the free (split) opfibration monad M on $\mathcal{C}$ given by $M(f) = f \downarrow 1_B$. The unit of the monad M is given by the morphism $E(f)\colon A \to Kf$ such that $r_f \cdot E(f) = 1_A$ and $M(f) \cdot E(f) = f$.

$$
\begin{tikzcd}
Kf \arrow[r, "r_f"] \arrow[d, "Mf"'] \arrow[dr, phantom, "\geqslant"] & A \arrow[d, "f"] \\
B \arrow[r, equal] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "E(f)"] \arrow[d, "f"'] & Kf \arrow[d, "M(f)"] \\
B \arrow[r, equal] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal, "1"] \arrow[d, "E(f)"'] \arrow[dr, phantom, "\Sigma_f"] & A \arrow[d, "E^2(f)"] \\
Kf \arrow[r, "\sigma_f"] & KE(f)
\end{tikzcd}
$$

The endofunctor E has a copoint with components $\Phi^E_f\colon (1, M(f))\colon E(f) \to f$, and a comultiplication Σ_f as depicted above, where $\sigma_f\colon K(f) \to KE(f)$ is the unique morphism defined by the conditions $M(E(f)) \cdot \sigma_f = 1_{Kf}$ and $r_{E(f)} \cdot \sigma_f = r_f$.

It is not hard to show that $E(f) \dashv r_f$ and this will be the LARI structure that makes $E(f)$ the cofree LARI on f, so $\mathrm{Lari}(\mathcal{C})$ is isomorphic to E-Coalg. See [5, §4.2] for the dual construction for LALIs. If $(1, s)\colon f \to E(f)$ is an E-coalgebra structure, the right adjoint of f is obtained as $r_f \cdot s\colon \mathrm{cod}(f) \to K(f) \to \mathrm{dom}(f)$.

5.2. LEMMA. *The comonad* E *is lax idempotent.*

PROOF. Suppose that $f\colon A \to B$ and $g\colon C \to D$ are E-coalgebras with structures $(1, s)$ and $(1, t)$. A morphism $(h, k)\colon f \to g$ in $\mathcal{C}^{\mathbf{2}}$ is a lax morphism of coalgebras if $E(h, k) \cdot (1, s) \leqslant (1, t) \cdot (h, k)$, or, equivalently, if

$$K(h, k) \cdot s \leqslant t \cdot k\colon D \to K(g) \tag{5.1}$$

since the domain component of the inequality is trivially an equality $h = h$. The required inequality will hold precisely when it does after composing with the projections $M(g)\colon K(g) \to D$ and $r_g\colon K(g) \to C$. Composition with $M(g)$ yields in fact an equality, as $M(g) \cdot K(h, k) \cdot s = k \cdot M(f) \cdot s = k = M(g) \cdot t \cdot k$. Composition with r_g of the left hand side of (5.1) equals $r_g \cdot K(h, k) \cdot s = h \cdot r_f \cdot s$, while the right hand side equals $r_g \cdot t \cdot k$. From the comments above this lemma, $f \dashv (r_f \cdot s)$ and $g \dashv (r_g \cdot t)$. Taking mates in $g \cdot h = k \cdot f$ we obtain $h \cdot r_f \cdot s \leqslant r_g \cdot t \cdot k$, and we deduce that (5.1) holds. We have shown that any morphism $f \to g$ in $\mathcal{C}^{\mathbf{2}}$ is a lax morphism of coalgebras, which is to say that E is lax idempotent. ■

Remark 2.6 immediately yields the following corollary.

5.3. COROLLARY. *The inclusion* E-Coalg $\subset (E, \Phi^E)$-Coalg *is an equality.*

6. Lax orthogonal factorisation systems

6.1. DEFINITION. An AWFS (L, R) on an **Ord**-category $\mathcal{C}$ is a *lax orthogonal factorisation system* (abbreviated LOFS) if either of the following equivalent conditions holds:

- The comonad L is lax idempotent.
- The monad R is lax idempotent.

The equivalence of the two conditions was established in [8].

According to our notation, the unit and multiplication of R and the counit and comultiplication of L are depicted as morphisms in $\mathcal{C}^{\mathbf{2}}$ as follows.

$$
\begin{tikzcd}
\cdot \arrow[r, "Lf"] \arrow[d, "f"'] \arrow[dr, phantom, "\Lambda_f"] & \cdot \arrow[d, "Rf"] \\
\cdot \arrow[r, equal] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "\pi_f"] \arrow[d, "R^2f"'] \arrow[dr, phantom, "\Pi_f"] & \cdot \arrow[d, "Rf"] \\
\cdot \arrow[r, equal] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "Lf"'] \arrow[dr, phantom, "\Phi_f"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "Rf"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "Lf"'] \arrow[dr, phantom, "\Sigma_f"] & \cdot \arrow[d, "L^2f"] \\
\cdot \arrow[r, "\sigma_f"'] & \cdot
\end{tikzcd}
$$

Then, (L, R) is lax orthogonal if and only if any of the following conditions hold:

$$K(Lf, 1) \cdot \pi_f \leqslant 1 \qquad 1 \leqslant LRf \cdot \pi_f \qquad 1 \leqslant \sigma_f \cdot RLf \qquad \sigma_f \cdot K(1, Rf) \leqslant 1. \tag{6.1}$$

In terms of R-algebras and L-coalgebras, the lax idempotency of (L, R) is described as follows. If (f, s) is an L-coalgebra and (g, p) is an R-algebra, as displayed below,

$$
\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "(f{,}s)"] & \cdot \arrow[d, "Lf"] \\
\cdot \arrow[r, "s"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "p"] \arrow[d, "Rg"'] \arrow[dr, phantom, "(g{,}p)"] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, equal] & \cdot
\end{tikzcd}
$$

then the AWFS is lax orthogonal if and only if any of the following two equivalent conditions hold, for all (f, s) and (g, p):

$$1 \leqslant s \cdot Rf \quad \text{and} \quad 1 \leqslant Lg \cdot p.$$

6.2. EXAMPLE. The AWFS (E, M) of §5, for which M-algebras are opfibrations and E-coalgebras are LARIs, is lax orthogonal. Indeed, the monad M is well-known to be lax idempotent.

There is a category $\mathbf{AWFS}(\mathcal{C})$ whose objects are AWFSs on the $\mathbf{Ord}$-category $\mathcal{C}$. A morphism $(\mathsf{L}, \mathsf{R}) \longrightarrow (\mathsf{L}', \mathsf{R}')$ is a natural family of morphisms φ_f that make the following diagrams commute.

$$
\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "L'f"] \\
Kf \arrow[r, "\varphi_f"] \arrow[d, "Rf"'] & K'f \arrow[d, "R'f"] \\
\cdot \arrow[r, equal] & \cdot
\end{tikzcd}
\tag{6.2}
$$

Furthermore, the morphisms $(1, \varphi_f) \colon Lf \to L'f$ must form a comonad morphism $\mathsf{L} \to \mathsf{L}'$, and the morphisms $(\varphi_f, 1) \colon Rf \to R'f$ must form a monad morphism $\mathsf{R} \to \mathsf{R}'$.

There is a full subcategory $\mathbf{LOFS}(\mathcal{C})$ of $\mathbf{AWFS}(\mathcal{C})$ consisting of the LOFSs.

6.3. LEMMA. $\mathbf{LOFS}(\mathcal{C})$ *is a preorder.*

PROOF. If the morphisms φ_f as in (6.2) form a morphism from (L,R) to $(\mathsf{L}',\mathsf{R}')$, then the morphisms $(\varphi_f, 1)\colon Rf \to R'f$ define a morphism of monads. By Lemma 2.7, there can be only one morphism of monads from a lax idempotent monad. ∎

7. Lifting operations

In this section we introduce KZ-lifting operations and explain the motivation behind the definition of lax orthogonal factorisation systems. Before all that, we must say something about how lifting operations work in relation to AWFSs on **Ord**-categories.

Suppose that $U\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}} \leftarrow \mathcal{B}\colon V$ are locally monotone functors between **Ord**-categories. A *lifting operation* from U to V can be described as a choice of a diagonal filler $\phi_{a,b}(h,k)$ for each morphism $(h,k)\colon Ua \to Vb$ in $\mathcal{C}^{\mathbf{2}}$.

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "Ua"'] & \cdot \arrow[d, "Vb"] \\ \cdot \arrow[r, "k"'] \arrow[ur, dotted, "\phi_{a,b}(h,k)"] & \cdot \end{tikzcd}$$

These diagonal fillers must satisfy a naturality condition with respect to morphisms in $\mathcal{A}$ and $\mathcal{B}$. If $\alpha\colon a' \to a$ and $\beta\colon b \to b'$ are morphisms in $\mathcal{A}$ and $\mathcal{B}$ respectively, then

$$\phi_{a',b'}\big(\operatorname{dom} V\beta \cdot h \cdot \operatorname{dom} U\alpha, \operatorname{cod} V\beta \cdot k \cdot \operatorname{cod} U\alpha\big) = (\operatorname{dom} V\beta) \cdot \phi_{a,b}(h,k) \cdot (\operatorname{cod} U\alpha)$$

as depicted in the following diagram.

$$\begin{tikzcd} \cdot \arrow[r, "\operatorname{dom} U\alpha"] \arrow[d, "Ua'"'] & \cdot \arrow[r, "h"] \arrow[d, "Ua"'] & \cdot \arrow[r, "\operatorname{dom} V\beta"] \arrow[d, "Vb"] & \cdot \arrow[d, "Vb'"] \\ \cdot \arrow[r, "\operatorname{cod} U\alpha"'] \arrow[rrru, dotted] & \cdot \arrow[r, "k"'] \arrow[ur, dotted] & \cdot \arrow[r, "\operatorname{cod} V\beta"'] & \cdot \end{tikzcd}$$

So far, the definition of lifting operation is the one given in [14], but our categories are enriched in **Ord** and the functors U and V are locally monotone, so we require that the diagonal filler satisfies: if (h,k) and $(h',k')\colon Ua \to Vb$ are commutative squares in $\mathcal{C}$ with $(h,k) \leqslant (h',k')$ (i.e., $h \leqslant h'$ and $k \leqslant k'$) then

$$\phi_{a,b}(h,k) \leqslant \phi_{a,b}(h',k').$$

The idea of a functorial factorisation $\operatorname{dom} \Rightarrow E \Rightarrow \operatorname{cod}$, as defined in Definition 4.1, is that it induces a canonical lifting operation between the forgetful **Ord**-functors U and V

$$U\colon (L,\Phi)\text{-Coalg} \longrightarrow \mathcal{C}^{\mathbf{2}} \longleftarrow (R,\Lambda)\text{-Alg}\colon V.$$

Here $\Phi\colon L \Rightarrow 1_{\mathcal{C}^{\mathbf{2}}}$ and $\Lambda\colon 1_{\mathcal{C}^{\mathbf{2}}} \Rightarrow R$ are, respectively, the copointed endo-**Ord**-functor and the pointed endo-**Ord**-functor on $\mathcal{C}^{\mathbf{2}}$ associated to the given **Ord**-functorial factorisation.

A coalgebra for (L, Φ) can be depicted as the commutative square on the left below, while an algebra for (R, Λ) is a commutative square on the right

$$(f, s) \quad = \quad \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "f"'] & \cdot \arrow[d, "Lf"] \\ \cdot \arrow[r, "s"] & \cdot \end{tikzcd} \qquad (g, p) \quad = \quad \begin{tikzcd} \cdot \arrow[r, "p"] \arrow[d, "Rg"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$$

satisfying $Rf \cdot s = 1$ and $p \cdot Lg = 1$. Given a commutative square $(h, k)\colon f \to g$, there is a canonical diagonal filler

$$\phi_{(f,s),(g,p)}(h, k) = p \cdot K(h, k) \cdot s. \tag{7.1}$$

It is immediate to see that these diagonal fillers form a lifting operation from U to V.

This is a good point to include the following result, which will be useful in the proof of Theorem 10.1.

7.1. Lemma. *For any* AWFS (L, R), *the diagonals* $\phi_{Lf,Rf}(Lf, Rf)$ *are identity morphisms.*

$$\begin{tikzcd} \cdot \arrow[r, "Lf"] \arrow[d, "Lf"'] & \cdot \arrow[d, "Rf"] \\ \cdot \arrow[r, "Rf"'] \arrow[ur, "1"] & \cdot \end{tikzcd}$$

Proof. For $f\colon A \to B$, the L-coalgebra structure of Lf is given by the comultiplication $(1_A, \sigma_f)\colon Lf \to L^2 f$, while the R-algebra structure of Rf is given by the multiplication $(\pi_f, 1_B)\colon R^2 f \to Rf$. See (4.2) for a depiction of the comultiplication Σ and the multiplication Π. According to (7.1), the diagonal filler of the square of the statement is $\pi_f \cdot K(Lf, Rf) \cdot \sigma_f$. Factorise the morphism $(Lf, Rf)\colon Lf \to Rf$ in $\mathcal{C}^{\mathbf{2}}$ as $(1_A, Rf)\colon Lf \to f$ followed by $(Lf, 1_B)\colon f \to Rf$. Thus, the diagonal filler is $\pi_f \cdot K(Lf, 1_B) \cdot K(1_A, Rf) \cdot \sigma_f$. Applying the domain functor to the monad axiom $\Pi_f \cdot R\Lambda_f = 1_f$ we obtain $\pi_f \cdot K(Lf, 1_B) = 1$. Similarly, applying the codomain functor to the comonad axiom $L\Phi_f \cdot \Sigma_f = 1_f$ we obtain $K(1_A, Rf) \cdot \sigma_f = 1$. ∎

7.2. Remark. As pointed out in [5, §2.5], the commutativity of the two diagrams (4.3) that express the fact that $\Delta\colon LR \Rightarrow RL$ is a mixed distributive law is equivalent to the requirement that the diagonal filler of the displayed square should be $\sigma_f \cdot \pi_f$.

$$\begin{tikzcd} Kf \arrow[r, "\sigma_f"] \arrow[d, "LRf"'] & KLf \arrow[d, "RLf"] \\ KRf \arrow[r, "\pi_f"'] \arrow[ur, dotted, "\sigma_f \cdot \pi_f"] & Kf \end{tikzcd}$$

8. KZ-lifting operations

In the previous section we saw that each functorial factorisation canonically induces a lifting operation. It is logical to expect that lifting operations that arise from LOFSs should have an extra property. In this section we identify this property.

8.1. DEFINITION. Suppose given a lifting operation ϕ from $U\colon \mathcal{A} \to \mathcal{C}^{\mathbf{2}}$ to $V\colon \mathcal{B} \to \mathcal{C}^{\mathbf{2}}$ on an **Ord**-category $\mathcal{C}$ as defined in §7. We say that ϕ is a *KZ-lifting operation* if, for all $a \in \mathcal{A}$, $b \in \mathcal{B}$ and each commutative diagram as on the left, the inequality on the right holds.

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "Ua"'] & \cdot \arrow[d, "Vb"] \\ \cdot \arrow[r, "k"'] \arrow[ur, "d"] & \cdot \end{tikzcd} \quad \Longrightarrow \quad \phi_{a,b}(h,k) \leqslant d$$

In other words, the diagonal filler given by the lifting operation ϕ is a lower bound of all possible diagonal fillers.

8.2. EXAMPLE. Consider the monotone map $0\colon \mathbf{1} \to \mathbf{2}$ that includes the terminal ordered set as the initial element of the ordered set $\mathbf{2} = (0 \leqslant 1)$. There is a bijection between opfibration structures on a morphism $g\colon X \to Y$ in **Ord** and KZ-lifting operations on g against the morphism 0. To see this, first notice that a commutative square

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d, "0"'] & X \arrow[d, "g"] \\ \mathbf{2} \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

is equally well given by an element $x \in X$ and an element $y \in Y$ such that $g(x) \leqslant y$. The existence of a diagonal filler is the existence of an element $x_y \in X$ with $x \leqslant x_y$ and $g(x_y) = y$. This diagonal filler is a lower bound if for any other $x \leqslant \bar{x}$ with $g(\bar{x}) = y$ there is an inequality $x_y \leqslant \bar{x}$. The element x_y is unique and the assignment $(x, y) \mapsto x_y$ defines a split opfibration structure on g.

8.3. THEOREM. [8, Thm. 9.10] *The following conditions are equivalent for an AWFS* (L, R) *on an* **Ord**-*category* $\mathcal{C}$.

1. *The AWFS is a LOFS.*
2. *The lifting operation from the forgetful functor* $U\colon \mathsf{L}\text{-Coalg} \to \mathcal{C}^{\mathbf{2}}$ *to the forgetful functor* $V\colon \mathsf{R}\text{-Alg} \to \mathcal{C}^{\mathbf{2}}$ *is a KZ-lifting operation.*

8.4. THEOREM. *Let* (L, R) *be a LOFS on an* **Ord**-*category* $\mathcal{C}$. *Then, the following statements about a morphism* f *of* $\mathcal{C}$ *are equivalent:*

1. f *is an* R-*algebra.*

2. *f is injective with respect to* L*-coalgebras, in the sense that any commutative square*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "\ell"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

with $\ell \in \mathsf{L}$-Coalg *has a diagonal filler.*

3. *f admits a (necessarily unique) (R, Λ)-algebra structure.*

4. *f is a retract in $\mathcal{C}^{\mathbf{2}}$ of an* R*-algebra.*

The WFS *that underlies* (L, R) *has as left part the* L*-coalgebras and as right part the* R*-algebras.*

PROOF. We have seen in §7 that (1) implies (2). To prove that (2) implies (3), consider the diagonal filler below, which shows that $(p, 1)\colon Rf \to f$ is an (R, Λ)-algebra structure.

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "Rf"'] \arrow[ur, dotted, "p"] & \cdot \end{tikzcd}$$

The implications (3)⇒(4)⇒(1) are particular instances of part of Lemma 2.5 and Remark 2.6, since R is lax idempotent.

As mentioned in Remark 4.4, the underlying WFS $(\mathcal{L}, \mathcal{R})$ of (L, R) has as right class the algebras for the pointed endofunctor (R, Λ). Then, $f \in \mathcal{R}$ (or, by duality, $f \in \mathcal{L}$) precisely when f is an R-algebra (an L-coalgebra). ∎

9. Categories of lifting operations

If $U\colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ is an **Ord**-functor, there is a sub-**Ord**-category $\mathcal{J}^{\pitchfork_{\text{KZ}}}$ of $\mathcal{C}^{\mathbf{2}}$ whose objects are those morphisms f of $\mathcal{C}$ that admit a KZ-lifting operation against U, i.e., those for which the following **Ord**-natural transformation between **Ord**-functors $\mathcal{J}^{\text{op}} \to \mathbf{Ord}$ is a RALI.

$$\phi_{-,f}\colon \mathcal{C}(\operatorname{cod} U(-), \operatorname{dom} f) \longrightarrow \mathcal{C}^{\mathbf{2}}(U(-), f).$$

A morphism is a morphism in $\mathcal{C}^{\mathbf{2}}$ that is compatible with the left adjoints of the $\phi_{j,f}$ and the counits of these adjunctions in the obvious way. The subcategory $\mathcal{J}^{\pitchfork_{\text{KZ}}} \subset \mathcal{C}^{\mathbf{2}}$ is replete and the inclusion is full on isomorphisms.

The construction $(\mathcal{J}, U) \mapsto \mathcal{J}^{\pitchfork_{\text{KZ}}}$ is part of a functor

$$(-)^{\pitchfork_{\text{KZ}}}\colon (\mathbf{Ord\text{-}Cat}/\mathcal{C}^{\mathbf{2}})^{\text{op}} \longrightarrow \operatorname{Sub}(\mathcal{C}^{\mathbf{2}})$$

whose codomain is the poset of replete sub-**Ord**-categories of $\mathcal{C}^{\mathbf{2}}$ that are full on isomorphisms, denoted here simply by $\operatorname{Sub}(\mathcal{C}^{\mathbf{2}})$. Explicitly, if $S\colon \mathcal{J} \to \mathcal{I}$ is an **Ord**-functor over $\mathcal{C}^{\mathbf{2}}$, then $\mathcal{I}^{\pitchfork_{\text{KZ}}} \subset \mathcal{J}^{\pitchfork_{\text{KZ}}}$.

Given an **Ord**-functor $U\colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$, there is another subcategory ${}^{\pitchfork_{\mathrm{KZ}}}\mathcal{J} \subset \mathcal{C}^{\mathbf{2}}$ that is constructed dually to $\mathcal{J}^{\pitchfork_{\mathrm{KZ}}}$. More explicitly, ${}^{\pitchfork_{\mathrm{KZ}}}\mathcal{J}$ has objects $(f, \phi_{f,-})$ where $f \in \mathcal{C}^{\mathbf{2}}$ and ϕ is a KZ-lifting operation from f to U.

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "Uj"] \\ \cdot \arrow[r, "k"'] \arrow[ur, dotted, "\phi_{f,j}(h,k)"] & \cdot \end{tikzcd}$$

The KZ-lifting operation $\phi_{f,-}$ is a RALI structure on the **Ord**-natural transformation morphisms $\mathcal{C}(\mathrm{cod}(f), \mathrm{dom}\, U(-)) \to \mathcal{C}^{\mathbf{2}}(f, U(-))$.

9.1. THEOREM. *Suppose given* **Ord**-*functors*

$$\mathcal{J} \xrightarrow{U} \mathcal{C}^{\mathbf{2}} \xleftarrow{V} \mathcal{I}.$$

There is a bijection between:

- KZ-*lifting operations from U to V;*
- *Factorisations of V through $\mathcal{J}^{\pitchfork_{\mathrm{KZ}}}$;*
- *Factorisations of U through ${}^{\pitchfork_{\mathrm{KZ}}}\mathcal{I}$.*

These correspondences yield a Galois connection on the poset $\mathrm{Sub}(\mathcal{C}^{\mathbf{2}})$.

Suppose that (L, R) is a LOFS on the **Ord**-category $\mathcal{C}$. There is an **Ord**-enriched inclusion of subcategories of $\mathcal{C}^{\mathbf{2}}$

$$\mathsf{R}\text{-Alg} \subset \mathsf{L}\text{-Coalg}^{\pitchfork_{\mathrm{KZ}}} \tag{9.1}$$

introduced in [8], that equips each R-algebra with its canonical KZ-lifting operation against L-coalgebras (see Theorem 8.3). Using [5, §6.3] one could deduce that (9.1) is an isomorphism. Since we can treat L-Coalg and R-Alg are subcategories of $\mathcal{C}^{\mathbf{2}}$, one has the following simpler proof.

9.2. THEOREM. *The inclusion* (9.1) *induced by a* LOFS (L, R) *is an equality.*

PROOF. Supposing that $(g, \phi_{-,g})$ is a KZ-lifting operation against the forgetful **Ord**-functor $U\colon \mathsf{L}\text{-Coalg} \to \mathcal{C}^{\mathbf{2}}$, we want to construct an R-algebra structure on $g\colon A \to B$. It suffices to exhibit g as a retract in $\mathcal{C}^{\mathbf{2}}$ of Rg, by Theorem 8.4, and this can be done by means of the KZ-diagonal filler $p = \phi_{Lg,g}(1, Rg)$ as depicted below.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "Lg"'] & A \arrow[d, "g"] \\ Kg \arrow[r, "Rg"'] \arrow[ur, dotted, "p"] & B \end{tikzcd}$$

Now that we know that (9.1) is an equality on objects, it remains to prove that it is fully faithful, in the **Ord**-enriched sense. Suppose that $(h,k)\colon (f,\phi_{-,f}) \to (g,\phi_{-,g})$ is a morphism in $\mathsf{L}\text{-Coalg}^{\pitchfork_{\mathrm{KZ}}}$, and let $p_f\colon Rf \to f$ and $p_g\colon Rg \to g$ be the associated algebra structures. We have the following string of equalities

$$h\cdot p_f = h\cdot \phi_{Lf,f}(1,Rf) = \phi_{Lf,g}(h,k\cdot Rf) = \phi_{Lg,g}(1,Rg)\cdot K(h,k) = p_g\cdot K(h,k),$$

which are a result of the definition of lifting operations.

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[r, "h"] \arrow[d, "f"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "Rf"'] \arrow[ur, dotted, "p_f"] & \cdot \arrow[r, "k"'] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "Lf"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "k\cdot Rf"'] \arrow[ur, dotted, "\phi_{Lf,g}(h,k\cdot Rf)"] & \cdot \end{tikzcd} = \begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "Lf"'] & \cdot \arrow[r, equal] \arrow[d, "Lg"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "K(h,k)"'] & \cdot \arrow[r, "Rg"'] \arrow[ur, dotted, "p_g"] & \cdot \end{tikzcd}$$

This shows that (9.1) is full on morphisms. ∎

We conclude the section with a result on morphisms of LOFSs.

9.3. PROPOSITION. *Suppose that* (L,R) *and* $(\mathsf{L}',\mathsf{R}')$ *are* LOFS*s on the* **Ord**-*category* $\mathcal{C}$. *There is a bijection between the following sets, which, moreover, can have at most one element.*

(a) Morphisms of LOFSs $(\mathsf{L},\mathsf{R}) \longrightarrow (\mathsf{L}',\mathsf{R}')$.

(b) Comonad morphisms $\mathsf{L}\to\mathsf{L}'$.

(c) Monad morphisms $\mathsf{R}\to\mathsf{R}'$.

PROOF. First, there is at most one morphism of the kind in (a), (b) and (c) by Lemma 6.3, Lemma 2.7 and its dual form (i.e., the version for comonads). Clearly, if there is a morphism as in (a), then there are morphisms as in (b) and (c), just by definition of morphism of AWFSs (§6).

Suppose there is a morphism of comonads Q from L to L'. It induces an inclusion $\mathsf{L}\text{-Coalg} \subset \mathsf{L}'\text{-Coalg}$ of subcategories of $\mathcal{C}^{\mathbf{2}}$. Applying the functor $(-)^{\pitchfork_{\mathrm{KZ}}}$ and employing Theorem 9.2 we obtain an inclusion

$$\mathsf{R}'\text{-Alg} = \mathsf{L}'\text{-Coalg}^{\pitchfork_{\mathrm{KZ}}} \subset \mathsf{L}\text{-Coalg}^{\pitchfork_{\mathrm{KZ}}} = \mathsf{R}\text{-Alg},$$

which is necessarily induced by an **Ord**-monad morphism $\mathsf{R}\to\mathsf{R}'$.

We have seen that (c) has a member if (b) has a member. By a duality argument, i.e., by taking the opposite **Ord**-category of $\mathcal{C}$, we deduce the converse: (b) has a member if (c) does.

To complete the proof, it suffices to produce a morphism of LOFSs from a comonad morphism Q as above. Even tough this can be achieved by means of double categories and [5, Prop. 2], we prefer to avoid introducing double categories at this point, giving instead an elementary proof.

Due to the counit axiom, $(1, R'f) \cdot Q_f = (1, Rf)$, we have that Q_f is of the form $(1, \varphi_f)$ for a morphism $\varphi_f \colon Kf \to K'f$. Using the construction in the proof of Theorem 9.2, one can describe the R-algebra structure on an R'-algebra $(p, 1) \colon R'f \to f$. Its associated KZ-lifting operation $\phi_{-,f}$ defines a diagonal filler for each commutative square

$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "\ell"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "k"'] \arrow[ur, dotted, "\phi_{\ell,f}(h,k)"] & \cdot \end{tikzcd} \qquad \phi_{\ell,f}(h,k) = p \cdot K(h,k) \cdot s$$

for any L'-coalgebra $(1, s) \colon \ell \to L\ell$. Upon applying $\mathsf{L}'\text{-Coalg}^{\pitchfork_{\mathrm{KZ}}} \subset \mathsf{L}\text{-Coalg}^{\pitchfork_{\mathrm{KZ}}}$ we obtain a KZ-lifting operation $\psi_{-,f}$ of f against all L-coalgebras. If $(1, t) \colon g \to Lg$ is an L-coalgebra, its L'-coalgebra structure is

$$g \xrightarrow{(1,t)} Lg \xrightarrow{(1,\varphi_g)} L'g$$

and therefore $\psi_{g,f}(h, k)$ is the form

$$\psi_{g,f}(h, k) = p \cdot K'(h, k) \cdot \varphi_g \cdot t = p \cdot \varphi_f \cdot K(h, k) \cdot t.$$

We now obtain the R-algebra structure on f by $\psi_{Lf,f}(1, Rf)$,

$$\psi_{Lf,f}(1, Rf) = p \cdot \varphi_f \cdot K(1, Rf) \cdot \sigma_f = p \cdot \varphi_f.$$

In conclusion, (9.1) sends an R'-algebra $(p, 1) \colon R'f \to f$ to the R-algebra $(p \cdot \varphi_f, 1) \colon Rf \to f$. This implies that the monad morphism $\mathsf{R} \to \mathsf{R}'$ (whose existence we showed above) has components of the form $(\varphi_f, 1) \colon Rf \to R'f$. Therefore, the morphisms $\varphi_f \colon Kf \to K'f$ form a morphism of LOFSs as in (a), completing the proof. ∎

The above proposition is a reminder of the differences that exist between general AWFSs and LOFSs. In the general case, the proposition does not hold; see [28, Lemma 6.9] or [5, Prop. 2].

10. The definition of LOFS revisited

Lax orthogonal factorisation systems on **Ord**-categories were defined in §6 as **Ord**-enriched AWFSs (L, R) whose comonad L is lax idempotent, or equivalently, whose monad R is lax idempotent. The definition of AWFS includes a mixed distributive law $\Delta \colon LR \Rightarrow RL$, with components $(\sigma_f, \pi_f) \colon LRf \to RLf$. The axioms of a mixed distributive law in this case amount to the commutativity of the diagrams in (4.3), and they are equivalent, as mentioned in Remark 7.2, to the requirement that the diagonal filler of the square below should be $\sigma_f \cdot \pi_f$.

$$\begin{tikzcd} Kf \arrow[r, "\sigma_f"] \arrow[d, "LRf"'] & KLf \arrow[d, "RLf"] \\ KRf \arrow[r, "\pi_f"'] \arrow[ur, dotted, "\sigma_f \cdot \pi_f"] & Kf \end{tikzcd} \tag{10.1}$$

The main result of the section is the following.

10.1. Theorem. *The distributive law axiom in the definition of* LOFS *is redundant. More precisely, given a domain-preserving* **Ord**-*comonad* L *and a codomain-preserving* **Ord**-*monad* R *on* $\mathcal{C}^{\mathbf{2}}$ *that induce the same* **Ord**-*functorial factorisation* $f = Rf \cdot Lf$, *the following two statements are equivalent, and when they hold we are in the presence of a* LOFS.

- *One of* L, R *is lax idempotent and the distributive law axiom holds.*
- *Both* L *and* R *are lax idempotent.*

Proof. All we need to show is that $\sigma_f \cdot \pi_f$ is the diagonal filler of the square (10.1). The existence of a KZ-lifting operation for R-algebras against L-coalgebras does not depend on the distributivity axiom but it suffices that both L and R be lax idempotent. Then, we only need to show that

$$\sigma_f \cdot \pi_f \leqslant d \tag{10.2}$$

for the KZ-diagonal filler d of the square (10.1), for, in this case, the inequality is necessarily an equality. There are adjunctions $\sigma_f \dashv K(1, Rf)$ and $K(Lf, 1) \dashv \pi_f$ since L and R are lax idempotent. Thus, the inequality (10.2) is equivalent to $1 \leqslant K(1, Rf) \cdot d \cdot K(Lf, 1)$, due to the inequalities (6.1) of §6. Consider the following diagram, where $(Lf, K(Lf, 1)) = L(Lf, 1)$ is a morphism of L-coalgebras and $(K(1, Rf), Rf) = R(1, Rf)$ is a morphism of R-algebras.

$$\begin{tikzcd}
\cdot \arrow[r, "Lf"] \arrow[d, "Lf"'] & \cdot \arrow[r, "\sigma_f"] \arrow[d, "LRf"'] & \cdot \arrow[r, "K(1,Rf)"] \arrow[d, "RLf"] & \cdot \arrow[d, "Rf"] \\
\cdot \arrow[r, "K(Lf,1)"'] & \cdot \arrow[r, "\pi_f"'] \arrow[ur, "d"] & \cdot \arrow[r, "Rf"'] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "Lf"] \arrow[d, "Lf"'] & \cdot \arrow[d, "Rf"] \\
\cdot \arrow[r, "Rf"'] \arrow[ur, dotted, "1"] & \cdot
\end{tikzcd}$$

By the naturality of the diagonal fillers with respect to morphisms of L-coalgebras and morphism of R-algebras, we deduce that $K(1, Rf) \cdot d \cdot K(Lf, 1)$ is the diagonal filler of the square on the right hand side, and hence equal to the identity morphism (see Lemma 7.1). Therefore the inequality (10.2) holds, completing the proof. ■

11. Embeddings with respect to a monad

Embeddings with respect to a lax idempotent monad were extensively exploited in [12, 13] and in [11], where topological embeddings were exhibited as an example (more on this in §18). In this section we begin our analysis of the interplay between these embeddings and LOFSs.

11.1. Definition. If $S\colon \mathcal{C} \to \mathcal{B}$ is an **Ord**-functor between **Ord**-categories, a morphism f in $\mathcal{C}$ is an S-*embedding* if Sf is a LARI in $\mathcal{B}$. A morphism of S-embeddings $f \to g$ is a morphism $(h, k)\colon f \to g$ in $\mathcal{C}^{\mathbf{2}}$ that is compatible with the right adjoints of Sf and Sg: if $Sf \dashv r$ and $Sg \dashv t$, then $Sh \cdot r = t \cdot Sk$ must hold. This defines a sub-**Ord**-category

S-Emb of $\mathcal{C}^{\mathbf{2}}$, which is locally full and fits in a pullback square of **Ord**-functors.

$$\begin{tikzcd} S\text{-Emb} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\mathrm{pb}"] & \mathrm{Lari}(\mathcal{B}) \arrow[d, hook] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "S^{\mathbf{2}}"'] & \mathcal{B}^{\mathbf{2}} \end{tikzcd} \tag{11.1}$$

11.2. Lemma. *The forgetful* **Ord**-*functor* S-Emb $\to \mathcal{C}^{\mathbf{2}}$ *creates colimits, provided that* $\mathcal{C}$ *has and* S *preserves colimits.*

Proof. In the pullback diagram (11.1), the leftmost vertical **Ord**-functor creates any colimit that is preserved by S (and thus by $S^{\mathbf{2}}$), since the rightmost vertical **Ord**-functor creates colimits. ∎

11.3. Definition. If T is an **Ord**-monad on $\mathcal{C}$, we shall call F^{T}-embeddings T-*embeddings*, and denote the **Ord**-category F^{T}-Emb by T-Emb. Here $F^{\mathsf{T}}\colon \mathcal{C} \to \mathsf{T}$-Alg denotes the free T-algebra functor.

If $\mathcal{A}$ is an **Ord**-category with a terminal object, we write $\mathcal{A}/1$ for the full subcategory of $\mathcal{A}^{\mathbf{2}}$ consisting of morphisms $A \to 1$.

11.4. Proposition. *Let* T *be a lax idempotent monad on an* **Ord**-*category with a terminal object. There is an equality of sub-***Ord**-*categories of* $\mathcal{C}^{\mathbf{2}}$

$$\mathsf{T}\text{-Emb} = {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{T}\text{-Alg}/1\big). \tag{11.2}$$

Proof. An object of ${}^{\pitchfork_{\mathrm{KZ}}}(\mathsf{T}\text{-Alg}/1)$ is a morphism $f\colon X \to Y$ of $\mathcal{C}$ for which the following is a rali

$$\mathcal{C}(f, A)\colon \mathcal{C}(Y, A) \longrightarrow \mathcal{C}^{\mathbf{2}}(f, (A \to 1)) \cong \mathcal{C}(X, A) \tag{11.3}$$

naturally in the T-algebra A, in the sense that, each T-algebra morphism $A \to A'$ must induce a morphism of ralis. In other words, each morphism $X \to A$ has a left Kan extension along f, and these are preserved by T-algebra morphisms.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & A \\ Y \arrow[ur, dotted] & \end{tikzcd}$$

The morphism (11.3) can be written as

$$\mathcal{C}(Y, A) \cong \mathsf{T}\text{-Alg}(F^{\mathsf{T}}Y, A) \xrightarrow{\mathsf{T}\text{-Alg}(F^{\mathsf{T}}f, 1)} \mathsf{T}\text{-Alg}(F^{\mathsf{T}}X, A) \cong \mathcal{C}(X, V(A)) \tag{11.4}$$

which is a rali, naturally in the T-algebra A, if and only if $F^{\mathsf{T}}f$ is a lari. This gives an equality on objects in (11.2).

Both sides of (11.2) are locally full subcategories of $\mathcal{C}^{\mathbf{2}}$, so it only remains to verify the equality at the level of morphisms. Suppose that f and g are T-embeddings. A morphism

$(h,k)\colon f \to g$ is a morphism on the right hand side of (11.2) if it is compatible with the RALI structures on the morphisms (11.3) corresponding to f and g; in other words, if (h,k) induces a morphism of RALIs. This is equivalent to requiring that (h,k) should induce a morphism of RALIs between the RALIs (11.4) that correspond to f and g. By Yoneda lemma, this means that (h,k) is a morphism of T-embeddings, and concludes the proof. ∎

11.5. COROLLARY. *Assume that the* **Ord**-*category* $\mathcal{C}$ *has a terminal object. The assignment* $\mathsf{T} \mapsto \mathsf{T}\text{-Emb}$ *is a functor from the category of lax idempotent monads on* $\mathcal{C}$ *to the poset of sub-***Ord**-*categories of* $\mathcal{C}^{\mathbf{2}}$.

PROOF. If $\varphi\colon \mathsf{S} \to \mathsf{T}$ is a monad morphism between lax idempotent monads, then $T\text{-Alg} \subset \mathsf{S}\text{-Alg}$. Applying ${}^{\pitchfork_{\mathrm{KZ}}}(-)$ and Proposition 11.4, we obtain $\mathsf{S}\text{-Emb} \subset \mathsf{T}\text{-Emb}$. ∎

11.6. PROPOSITION. *Let* T *be a lax idempotent monad on an* **Ord**-*category with a terminal object. The obvious inclusion of* **Ord**-*categories*

$$\mathsf{T}\text{-Alg}/1 \subset (\mathsf{T}\text{-Emb})^{\pitchfork_{\mathrm{KZ}}}$$

identifies $\mathsf{T}\text{-Alg}$ *with the fibre of* $\mathrm{cod}\colon (\mathsf{T}\text{-Emb})^{\pitchfork_{\mathrm{KZ}}} \longrightarrow \mathcal{C}$ *over* 1.

PROOF. We will show that a morphism $A \to 1$ is in $(\mathsf{T}\text{-Emb})^{\pitchfork_{\mathrm{KZ}}}$ if and only if A is a T-algebra. The components $\eta_X\colon X \to TX$ of the unit of the monad T are T-embeddings due to the adjunction $T\eta_X \dashv \mu_X$. Furthermore, for any morphism $u\colon X \to Y$, there is a morphism $(u, Tu)\colon \eta_X \to \eta_Y$ in $\mathsf{T}\text{-Emb}$ because $Tu \cdot \mu_X = \mu_Y \cdot T^2 u$.

Suppose that $A \to 1$ has a KZ-lifting operation against T-embeddings, which provides a diagonal filler to the square displayed below.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "\eta_A"'] & A \arrow[d] \\ TA \arrow[r] \arrow[ur, dotted, "a"] & 1 \end{tikzcd}$$

Being a retract for η_A, the map a is a T-algebra structure for A; see Remark 2.6. We leave to the reader the verification that the **Ord**-functor of the statement is full and faithful. ∎

11.7. COROLLARY. *In the conditions of Proposition 11.6,*

$$\mathsf{T}\text{-Emb} = {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{T}\text{-Emb}^{\pitchfork_{\mathrm{KZ}}}\big)$$

PROOF. There always is an inclusion of the left into the right hand side of the equality above, by the Galois connection of Theorem 9.1. Let us denote by $\mathsf{T}\text{-Emb}_1^{\pitchfork_{\mathrm{KZ}}}$ the fibre of $\mathrm{cod}\colon \mathsf{T}\text{-Emb}^{\pitchfork_{\mathrm{KZ}}} \subset \mathcal{C}$ over 1. We have

$$\mathsf{T}\text{-Emb} = {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{T}\text{-Alg}/1\big) = {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{T}\text{-Emb}_1^{\pitchfork_{\mathrm{KZ}}}\big) \supset {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{T}\text{-Emb}^{\pitchfork_{\mathrm{KZ}}}\big).$$

∎

11.8. Corollary. *If* (L,R) *is a* LOFS *on an* **Ord***-category* $\mathcal{C}$ *with a terminal object, then*

$$\mathsf{L}\text{-Coalg} \subset \mathsf{R}_1\text{-Emb}$$

where R_1 *is the* **Ord***-monad on* $\mathcal{C} \cong \mathcal{C}/1$ *that is the restriction of* R.

Proof. The inclusion of R_1-Alg $\hookrightarrow$ R-Alg, given by $A \mapsto (A \to 1)$, induces the inclusion arrow in the following,

$$\mathsf{L}\text{-Coalg} = {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{R}\text{-Alg}\big) \subset {}^{\pitchfork_{\mathrm{KZ}}}\big(\mathsf{R}_1\text{-Alg}/1\big) = \mathsf{R}_1\text{-Emb}$$

where the last isomorphism is provided by Proposition 11.4. ∎

The **Ord**-functor of Corollary 11.8 may be described more explicitly. If $f\colon X \to Y$ is an L-coalgebra, then the corresponding R_1-embedding structure is given by the adjunction $R_1 f \dashv r\colon R_1 Y \to R_1 X$ where r is the unique morphism of R_1-algebras that composed with the unit $\eta_Y\colon Y \to R_1 Y$ equals the KZ-lifting corresponding to the square displayed below.

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & R_1 X \arrow[d, "!=R(!)"] \\ Y \arrow[r, "!"'] \arrow[ur, dotted, "r\cdot\eta_Y"] & 1 \end{tikzcd}$$

12. KZ-reflective LOFSs

We begin by summarising the most basic definitions of [7] around reflective factorisation systems.

An OFS $(\mathscr{E},\mathscr{M})$ on a category with a terminal object $\mathcal{C}$ induces a reflective subcategory of $\mathcal{C}$ formed by those objects X for which $X \to 1$ belongs to $\mathscr{M}$. In the other direction, each reflective subcategory $\mathcal{B} \subseteq \mathcal{C}$ induces a pre-factorisation system $(\mathscr{E},\mathscr{M})$ whose $\mathscr{E}$ is formed by all the morphisms that are orthogonal to each object of $\mathcal{B}$. With an obvious ordering on reflective subcategories and pre-factorisation systems, these two constructions form an adjunction (a Galois correspondence). Those pre-factorisation systems obtained from reflective subcategories are called *reflective*, and are characterised as those for which $g \cdot f \in \mathscr{E}$ and $g \in \mathscr{E}$ implies $f \in \mathscr{E}$.

In this section we consider the analogous notion of KZ-reflective LOFS and find a characterisation that mirrors the case of OFSs.

12.1. Definition. We say that the **Ord**-monad T on $\mathcal{C}$ is *fibrantly* KZ*-generating* if the locally full inclusion of **Ord**-categories T-Emb $\subset \mathcal{C}^{\mathbf{2}}$ has a right adjoint (in the **Ord**-enriched sense).

12.2. Proposition. *Let* T *be an* **Ord***-monad on* $\mathcal{C}$*. Then* T *is fibrantly* KZ*-generating if and only if there exists an* **Ord***-enriched* AWFS (L,R) *for which* L-Coalg $=$ T-Emb*. Furthermore, if* $\mathcal{C}$ *is cocomplete and has limits of morphisms, this* AWFS *is lax orthogonal.*

Proof. The implication in one direction is clear; indeed, if T-Emb equals L-Coalg, then the condition of Definition 12.1 holds.

Assume that T is fibrantly KZ-generating. The inclusion Lari(T-Alg) $\subset$ T-Alg$^{\mathbf{2}}$ is comonadic by §5. The inclusion T-Emb $\subset \mathcal{C}^{\mathbf{2}}$ is a pullback of the comonadic inclusion mentioned, therefore, it satisfies all the hypotheses of (the **Ord**-enriched version) of Beck's comonadicity theorem, except perhaps for the hypothesis of being a left adjoint. Together with Definition 12.1, we deduce that T-Emb is comonadic over $\mathcal{C}^{\mathbf{2}}$.

The **Ord**-category of T-embeddings forms part of a double category. In other words, T-embeddings are closed under composition, just as LARIS are. We will be able to apply the dual of [5, Thm. 6] if we show the following: if f is a T-embedding, then the square on the left is a morphism of T-embeddings $1 \to f$. This is equivalent to saying that the square on the right is a morphism of LARIS $1 \to F^T f$, which is easily seen to hold.

$$\begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "1"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "f"] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "1"'] & \cdot \arrow[d, "F^T f"] \\ \cdot \arrow[r, "F^T f"] & \cdot \end{tikzcd}$$

We deduce, by a dual form of [5, Thm. 6], that the underlying category of T-Emb is L-Coalg for an AWFS (L, R). We leave to the reader the verification that this isomorphism is not only one of categories but one of **Ord**-categories.

It remains to show that, in the presence of limits of morphisms and colimits, this AWFS is a LOFS. For this we appeal to the dual version of [27, Cor. A.3], which we explain here. The setup consists of a pullback square as displayed on the left hand side below, for **Ord**-monads P and S, the respective forgetful **Ord**-functors as vertical arrows, and a **Ord**-functor $W\colon \mathcal{A} \to \mathcal{B}$ with the property that right Kan extensions along W exist (e.g., when W is a right adjoint). The completeness conditions are that $\mathcal{A}$ should be complete and W continuous ([27, Cor. A.3] says *cocomplete* where it should say of *complete*; the proof is unaltered). The thesis states that P is lax idempotent if S is so.

$$\begin{tikzcd} \mathsf{P}\text{-Alg} \arrow[r] \arrow[d] & \mathsf{S}\text{-Alg} \arrow[d] \\ \mathcal{A} \arrow[r, "W"] & \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{T}\text{-Emb} \arrow[r] \arrow[d, "U"'] & \mathsf{E}\text{-Coalg} \arrow[d] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "(F^T)^{\mathbf{2}}"] & \mathsf{T}\text{-Alg}^{\mathbf{2}} \end{tikzcd}$$

Our situation is the pullback square displayed on the right hand side above, where $\mathcal{C}$ is cocomplete and the free algebra **Ord**-functor F^T is a left adjoint. The lax idempotent comonad E on T-Alg$^{\mathbf{2}}$ is the one of §5 and exists since $\mathcal{C}$, and thus T-Alg, has limits of morphisms. Therefore, the comonad corresponding to the comonadic U is lax idempotent by the dual of [27, Cor. A.3] recalled above. ∎

12.3. Definition. The **Ord**-category of lax idempotent monads on the **Ord**-category $\mathcal{C}$, denoted by $\mathbf{LIMnd}(\mathcal{C})$, has morphisms $\mathsf{T} \to \mathsf{S}$ natural transformations that are compatible with the multiplication and unit of the monads, in the usual manner. This **Ord**-category is a (possibly large) *preordered set,* i.e., it has at most one morphism between any pair of objects, by Lemma 2.7.

We will denote by $\mathbf{LIMnd}_{\mathrm{fib}}(\mathcal{C})$ the full subcategory of $\mathbf{LIMnd}(\mathcal{C})$ consisting of those monads that are fibrantly KZ-generating, in the sense of Definition 12.1.

When $\mathcal{C}$ is cocomplete, has limits of morphisms and terminal object, we have a situation that can be summarised by the following diagram of preordered sets and order-preserving maps.

$$\begin{tikzcd} \mathbf{LOFS}(\mathcal{C}) \arrow[d, hook, "(-)\text{-Coalg}"'] \arrow[dr, "\tilde{\Phi}"] & \mathbf{LIMnd}_{\mathrm{fib}}(\mathcal{C}) \arrow[l, dotted, "\Psi"'] \arrow[d, hook, "I"] \\ \mathbf{Sub}(\mathcal{C}^{\mathbf{2}}) & \mathbf{LIMnd}(\mathcal{C}) \arrow[l, "\tilde{\Psi}"'] \end{tikzcd} \tag{12.1}$$

The vertical maps are full, the one on the right being an inclusion. The one on the left sends each LOFS on $\mathcal{C}$ to the sub-**Ord**-category L-Coalg of $\mathcal{C}^{\mathbf{2}}$. The map $\tilde{\Psi}$ sends a lax idempotent monad T on $\mathcal{C}$ to ${}^{\pitchfork_{\mathrm{KZ}}}(\mathsf{T}\text{-Alg}/1)$, and has a lifting to a map Ψ that sends a fibrantly KZ-generating T to the LOFS (L,R) on $\mathcal{C}$ that satisfies L-Coalg $=$ T-Emb – see Proposition 12.2. Finally, $\tilde{\Phi}$ sends (L,R) to R_1, the restriction of R to the slice $\mathcal{C}/1 \cong \mathcal{C}$.

It will be convenient to use the following relaxed notion of adjunction. Suppose given a diagram of functors and a natural transformation, that may be enriched as needed, as displayed.

$$\begin{tikzcd} \mathcal{A} \arrow[dr, bend right, "F"'] & \mathcal{B} \arrow[l, "G"'] \arrow[d, "I"] \arrow[dl, phantom, "\overset{\theta}{\Rightarrow}"] \\ & \mathcal{D} \end{tikzcd}$$

12.4. Definition. Following [31, §2], we say that θ *exhibits G as a I-right adjoint of F*, and F *as a I-left adjoint of G* denoted by $F \dashv_I G$, if

$$\mathcal{A}(A, G(B)) \xrightarrow{F} \mathcal{D}(F(A), FG(B)) \xrightarrow{\mathcal{D}(1,\theta_B)} \mathcal{D}(F(A), I(B))$$

is invertible, for all objects $A \in \mathcal{A}$ and $B \in \mathcal{B}$

It is easy to prove that if $I \colon \mathcal{B} \to \mathcal{D}$ is fully faithful and θ is an isomorphism, then G is fully faithful.

12.5. Theorem. *When $\mathcal{C}$ is cocomplete, has limits of morphisms and terminal object, the functor $\tilde{\Phi}$ in (12.1) is a I-left adjoint of Ψ. Moreover, Ψ is fully faithful.*

Proof. Let us denote the monad part of the LOFS $\Psi(\mathsf{T})$ by S. The counit of the relative adjunction will have components $\mathsf{S}_1 \to \mathsf{T}$, where S_1 is the restriction of S to $\mathcal{C} \cong \mathcal{C}/1$. We want to define this counit to be an isomorphism, for which we need $\mathsf{S}_1 \cong \mathsf{T}$, or, equivalently, S_1-Alg $=$ T-Alg. The left hand side of the equality is the fibre over 1 of the codomain **Ord**-functor S-Alg $\to \mathcal{C}$, while the right hand side is the fibre over 1 for the codomain **Ord**-functor $\mathsf{T}\text{-Emb}^{\pitchfork_{\mathrm{KZ}}} \to \mathcal{C}$, by Proposition 11.6. It suffices, thus, to show S-Alg $= \mathsf{T}\text{-Emb}^{\pitchfork_{\mathrm{KZ}}}$. By hypothesis, T-Emb $=$ L-Coalg, and the required equality follows from Theorem 9.2.

We next prove that the function

$$\mathbf{LOFS}(\mathcal{C})((\mathsf{L},\mathsf{R}),\Psi(\mathsf{T})) \xrightarrow{\tilde{\Phi}} \mathbf{LIMnd}(\mathcal{C})(\mathsf{R}_1,\mathsf{T}),$$

induced by the counit described in the previous paragraph, is bijective. Domain and codomain are either empty or singletons, so it suffices to prove that the existence of a morphism $\mathsf{R}_1 \to \mathsf{T}$ implies that of another $(\mathsf{L},\mathsf{R}) \to \Psi(\mathsf{T})$. Equivalently, it suffices to prove that T-Alg $\subset \mathsf{R}_1$-Alg implies L-Coalg $\subset \mathsf{T}$-Emb. Furthermore, it suffices to prove that R_1-Emb $\subset \mathsf{T}$-Emb, by Corollary 11.8. The latter inclusion follows from applying ${}^{\pitchfork_{\mathrm{KZ}}}(-)$ to T-Alg $\subset \mathsf{R}_1$-Alg and applying Proposition 11.4. ∎

12.6. Definition. We call a LOFS (L,R) on $\mathcal{C}$ *KZ-reflective* if L-Coalg $= \mathsf{T}$-Emb for a lax idempotent monad T on $\mathcal{C}$. In other words, the KZ-reflective LOFSs are those LOFSs that arise from lax idempotent monads that are fibrantly KZ-generating; see Proposition 12.2.

When $\mathcal{C}$ is cocomplete, has limits of morphisms and terminal object, so we have the situation (12.1), KZ-reflective LOFSs are those isomorphic to one of the form $\Psi(\mathsf{T})$, for a lax idempotent monad T.

12.7. Proposition. *Let* (L,R) *be a KZ-reflective LOFS on a cocomplete* **Ord**-*category with lax limits of morphisms and terminal object. Then,* L-Coalg $= \mathsf{R}_1$-Emb *and* $(\mathsf{L},\mathsf{R}) \cong \Psi(\mathsf{R}_1)$.

Proof. Suppose that $(\mathsf{L},\mathsf{R}) \cong \Psi(\mathsf{T})$ for a lax idempotent monad T. This means that L-Coalg $= \mathsf{T}$-Emb for an **Ord**-monad T on $\mathcal{C}$. On the other hand, R-Alg $= \mathsf{L}\text{-Coalg}^{\pitchfork_{\mathrm{KZ}}}$ for any LOFS, as we saw in Theorem 9.2. Therefore,

$$\mathsf{R}_1\text{-Alg} = \mathsf{R}\text{-Alg}_1 = \mathsf{T}\text{-Emb}_1^{\pitchfork_{\mathrm{KZ}}} = \mathsf{T}\text{-Alg}$$

where the subscript 1 denotes the fibre of the various categories fibred over $\mathcal{C}$ via the codomain functor. The last equality of the sequence is provided by Proposition 11.6. We obtain an isomorphism of **Ord**-monads between R_1 and T. ∎

12.8. Definition. An object A in the **Ord**-category $\mathcal{C}$, which we assume has a terminal object, is *KZ-injective* with respect to an **Ord**-functor $U\colon \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ if $(A \to 1) \in \mathcal{J}^{\pitchfork_{\mathrm{KZ}}}$.

KZ-injectivity can be expressed in the absence of a terminal object. The universal property of the diagonal filler is that of a left Kan extension, plus there should be compatibility with the morphisms of $\mathcal{J}$. We will always have terminal objects available, so we can safely leave this formulation to the reader. The notion was extensively used in [1]; see [11] for an early appearance (where the term *right injective* was used for a dual notion).

12.9. Corollary. *If* (L,R) *is KZ-reflective on* $\mathcal{C}$*, then*

1. *An object of* $\mathcal{C}$ *is an* R_1*-algebra if and only if it is KZ-injective, equivalently, injective, with respect to* R_1*-embeddings*
2. *A morphism* f *of* $\mathcal{C}$ *is an* L*-coalgebra if and only if each* R_1*-algebra is KZ-injective with respect to* f.

PROOF. Let $(\mathcal{L}, \mathcal{R})$ be the underlying WFS of (L, R) (see Remark 4.4). We already know that any R_1-algebra is KZ-injective with respect to L-coalgebras, by Theorem 9.2, and, by hypothesis, with respect to R_1-embeddings too.

It remains to prove that each object C that is injective with respect to $\mathcal{L}$ is an R_1-algebra. The unit $\eta_C \colon C \to \mathsf{R}_1(C)$ is an R_1-embedding, since R_1 is lax idempotent, and therefore it is an L-coalgebra, and *a fortiori* it is in $\mathcal{L}$. Then, η_C has a section and C is an R_1-algebra; see Lemma 2.5.

To say that each R_1-algebra A is KZ-injective with respect to f is equivalent to saying that f is an R_1-embedding, i.e., f is an L-coalgebra. ∎

12.10. REMARK. In part (2) of the above corollary, KZ-injectivity of R_1-algebras cannot be replaced by plain injectivity. Indeed, any split monic is injective with respect to all objects, but there are many OFSs (in particular, LOFSs) whose left morphisms do not include the split monics. To wit, the OFS on the category of sets whose left morphisms are the surjections.

13. A characterisation of KZ-reflective lax factorisations

In this section we will denote by (E, M) the LOFS on $\mathcal{C}$ whose E-coalgebras are LARIs in $\mathcal{C}$ and whose M-algebras are split opfibrations in $\mathcal{C}$.

13.1. DEFINITION. We will refer to those LOFSs (L, R) that admit a morphism $(\mathsf{E}, \mathsf{M}) \to (\mathsf{L}, \mathsf{R})$ as *sub-LARI LOFSs*. If such morphism exists, it is unique. Another way of putting the definition is to require $\mathrm{Lari}(\mathcal{C}) \subset \mathsf{L}\text{-Coalg}$.

Not all LOFSs are sub-LARI. For example, the initial AWFS (the one that factors a morphism f as $f = Rf \cdot Lf$ with $Lf = 1_{\mathrm{dom}(f)}$ and $Rf = f$) is orthogonal and, thus, lax orthogonal. Coalgebras for the associated comonad are the invertible morphisms in $\mathcal{C}$. It is clear that not every LARI is an isomorphism, so this LOFS is not sub-LARI.

13.2. PROPOSITION. *KZ-reflective LOFSs are sub-LARI.*

PROOF. By definition, L-Coalg equals T-Emb, for a certain T. We have to show that $\mathrm{Lari}(\mathcal{C}) \subset \mathsf{T}\text{-Emb}$. By definition of T-Emb as a pullback (see Definition 11.1) so it suffices to exhibit a commutative square

$$\begin{tikzcd} \mathrm{Lari}(\mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Lari}(\mathsf{T}\text{-Alg}) \arrow[d] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "(F^{\mathsf{T}})^{\mathbf{2}}"'] & \mathsf{T}\text{-Alg}^{\mathbf{2}} \end{tikzcd}$$

where the vertical arrows are the obvious inclusions. The **Ord**-functor $(F^{\mathsf{T}})^{\mathbf{2}}$ obviously induces another $\mathrm{Lari}(\mathcal{C}) \to \mathrm{Lari}(\mathsf{T}\text{-Alg})$ that makes the diagram commutative, since any **Ord**-functor preserves LARIs. ∎

13.3. Definition. We shall be interested in LOFS (L, R) that satisfy the following cancellation properties:

- If g and $g \cdot f$ are L-coalgebras, then f is an L-coalgebra.
- If, in the following diagram, g, g', $g \cdot f$ and $g' \cdot f'$ are L-coalgebras and (v, w) and (u, w) are morphisms of L-coalgebras, then (u, v) is a morphism of L-coalgebras.

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "f"'] & \cdot \arrow[d, "f'"] \\
\cdot \arrow[r, "v"] \arrow[d, "g"'] & \cdot \arrow[d, "g'"] \\
\cdot \arrow[r, "w"] & \cdot
\end{tikzcd}$$

We call these LOFSs *cancellative.*

13.4. Example. For LOFSs that are OFSs, or in other words, when both the comonad and the monad of the LOFSs are idempotent, the second condition of the definition above is superfluous. Therefore, cancellative OFSs are precisely the reflective OFSs, as shown in [7, Thm. 2.3]. This is the result that we will generalise in Theorem 13.6.

13.5. Lemma. *The* LOFS (E, M) *is cancellative.*

Proof. Recall that E-coalgebras are the same as LARIs. Suppose that f and g are composable morphisms and that $g \dashv r$ and $(g \cdot f) \dashv t$ are LARI structures. Defining $s = t \cdot g$, we have that $s \cdot f = t \cdot g \cdot f = 1$. It remains to prove that $f \cdot s = f \cdot t \cdot g \leqslant 1$, which is equivalent to $g \cdot f \cdot t \cdot g \leqslant g$, and this inequality holds since $g \cdot f \cdot t \leqslant 1$.

$$\begin{tikzcd}
\bullet \arrow[r, "u"] \arrow[d, "f"] & \bullet \arrow[d, "f'"'] \\
\bullet \arrow[r, "v"] \arrow[d, shift right, "g"'] \arrow[d, phantom, "\dashv"] & \bullet \arrow[d, shift left, "g'"] \arrow[d, phantom, "\vdash"] \\
\bullet \arrow[r, "w"] \arrow[u, shift right, "r"'] \arrow[uu, bend left=60, "t"] & \bullet \arrow[u, shift left, "r'"] \arrow[uu, bend right=60, "t'"']
\end{tikzcd}$$

Now suppose given morphisms of LARIs $(u, w) \colon g \cdot f \to g' \cdot f'$ and $(v, w) \colon g \to g'$, as depicted. We have to show that $(u, v) \colon f \to f'$ is a morphism of LARIs, i.e. that $u \cdot t \cdot g = t' \cdot g' \cdot v$, which holds by the following string of equalities

$$u \cdot t \cdot g = t' \cdot w \cdot g = t' \cdot g' \cdot v$$

completing the proof. ∎

13.6. Theorem. *A sub-*lari *lofs* (L,R) *on a finitely complete* **Ord***-category is* kz*-reflective if and only if it is cancellative.*

Proof. When L-Coalg is equal to T-Emb for some lax idempotent T, it always satisfies the cancellation properties of Definition 13.3 since laris do: if g and $g\cdot f$ are T-embeddings, i.e., if Tg and $T(g\cdot f)=Tg\cdot Tf$ are laris, then Tf is a lari, which is to say that f is a T-embedding; and similarly for morphisms. See Lemma 13.5.

Conversely, suppose that (L,R) is cancellative (Definition 13.3), that is, $\mathrm{Lari}(\mathcal{C})\subset$ L-Coalg. We shall show that the inclusion L-Coalg $\subset$ R_1-Emb of Corollary 11.8 is an identity, so $(\mathsf{L},\mathsf{R})\cong\Psi(\mathsf{R}_1)$ is reflective.

If $f\colon X\to Y$ is an R_1-embedding, then consider the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "L!"'] & Y \arrow[d, "L!"] \\
R_1X \arrow[r, "R_1f"] \arrow[d] & R_1Y \arrow[d] \\
1 \arrow[r, equal] & 1
\end{tikzcd}$$

The morphisms $L!$ are cofree L-coalgebras while R_1f is a lari and therefore an L-coalgebra. So, $L!\cdot f$ is an L-coalgebra and f is an L-coalgebra by the cancellation hypothesis. This means that each R_1-embedding is an L-coalgebra, and all that remains to prove is that morphisms of R_1-embeddings are morphisms of L-coalgebras.

Let $(u,v)\colon f\to f'$ be a morphism of R_1-embeddings, so $(R_1u,R_1v)\colon R_1f\to R_1f'$ is a morphism of laris, and, therefore, a morphism of L-coalgebras. It follows that (u,R_1v), depicted on the left below, is a morphism of L-coalgebras.

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "L!"'] & X' \arrow[d, "L!"] \\
R_1X \arrow[r, "R_1u"] \arrow[d, "R_1f"'] & R_1X' \arrow[d, "R_1f'"] \\
R_1Y \arrow[r, "R_1v"] & R_1Y'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\
Y \arrow[r, "v"] \arrow[d, "L!"'] & Y' \arrow[d, "L!"] \\
R_1Y \arrow[r, "R_1v"] & R_1Y'
\end{tikzcd}$$

On the other hand, (v,R_1v) is a morphism of L-coalgebras, being the image under L of the morphism $(v,1)\colon(Y\to 1)\to(Y'\to 1)$. By the second part of Definition 13.3, we deduce that (u,v) is a morphism of L-coalgebras, as required. ∎

14. Simple adjunctions

In §3 we saw that a reflection T on $\mathcal{C}$ is simple if and only if a certain copointed endofunctor (L,Φ) on $\mathcal{C}^{\mathbf{2}}$ underlies a comonad whose category of coalgebras is T-Iso $\subset\mathcal{C}^{\mathbf{2}}$. In this section we generalise that result in three directions. First, we work with **Ord**-enriched categories, **Ord**-enriched functors and so on. Secondly, the 2-dimensional aspect introduced by the enrichment over **Ord** allows us to substitute isomorphisms by laris and T-isomorphisms by T-embeddings. Thirdly, even though §3 speaks of reflections, the constructions therein only need an adjunction (not necessarily a reflection) and this is the framework we choose.

14.1. Definition. Let $S \dashv G\colon \mathcal{B} \to \mathcal{C}$ be an adjunction between **Ord**-functors on **Ord**-categories, of which we require $\mathcal{C}$ to have pullbacks and $\mathcal{B}$ to have lax limits of morphisms. We can always construct a monad R on $\mathcal{C}^{\mathbf{2}}$ by considering the comma-object $Kf = GSf \downarrow \eta_Y$ and defining $Rf\colon Kf \to Y$ as the second projection. (The existence of this comma-object is explained in Remark 14.2, below.)

$$\begin{tikzcd}
X \arrow[dr, "Lf"] \arrow[drr, bend left, "\eta_X"] \arrow[ddr, bend right, "f"'] & & \\
& Kf \arrow[r, "q_f"] \arrow[d, "Rf"'] \arrow[dr, phantom, "\geqslant"] & GSX \arrow[d, "GSf"] \\
& Y \arrow[r, "\eta_Y"'] & GSY
\end{tikzcd}$$

The **Ord**-functorial factorisation $f = Rf \cdot Lf$ has an associated locally monotone co-pointed endofunctor $\Phi\colon L \Rightarrow 1$, where the component Φ_f is provided by the commutative square displayed.

$$\begin{tikzcd}
\cdot \arrow[r, equal] \arrow[d, "Lf"'] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "Rf"] & \cdot
\end{tikzcd}$$

We continue with the notation of previous sections, where (E, M) denotes the LOFS whose E-coalgebras are the LARIs.

14.2. Remark. The comma-square of Definition 14.1 can be obtained by pulling back along η_Y the image under G of the projection $M(Sf)\colon Sf \downarrow SY \to SY$.

$$\begin{tikzcd}
Kf \arrow[r] \arrow[d, "Rf"'] \arrow[dr, phantom, "\mathrm{pb}"] & G(Sf \downarrow SY) \arrow[r] \arrow[d, "G(MSf)"] \arrow[dr, phantom, "\geqslant"] & GSX \arrow[d, "GSf"] \\
Y \arrow[r, "\eta_Y"'] & GSY \arrow[r, equal] & GSY
\end{tikzcd}$$

14.3. Lemma. *There is a pullback square of locally monotone endofunctors of $\mathcal{C}^{\mathbf{2}}$, as depicted on the left. There is a pullback of* **Ord***-categories, as depicted on the right.*

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "\Phi"'] \arrow[dr, phantom, "\mathrm{pb}"] & G^{\mathbf{2}}ES^{\mathbf{2}} \arrow[d, "G^{\mathbf{2}}\Phi^E S^{\mathbf{2}}"] \\
1_{\mathcal{C}^{\mathbf{2}}} \arrow[r, "\eta^{\mathbf{2}}"] & G^{\mathbf{2}}S^{\mathbf{2}}
\end{tikzcd}
\qquad
\begin{tikzcd}
(L,\Phi)\text{-Coalg} \arrow[r] \arrow[d, "U"'] \arrow[dr, phantom, "\mathrm{pb}"] & (E,\Phi^E)\text{-Coalg} \arrow[d] \\
\mathcal{C}^{\mathbf{2}} \arrow[r, "S^{\mathbf{2}}"'] & \mathcal{B}^{\mathbf{2}}
\end{tikzcd}$$

Proof. In order to obtain a pullback square as on the left hand side of the statement, we need to give two pullback squares: one corresponding to the domain component and another corresponding to the codomain component. We define the domain component of $L \to G^{\mathbf{2}}ES^{\mathbf{2}}$ to be the unit $\eta\colon 1 \to GS$; this is possible since $\operatorname{dom} E = 1$. The resulting square has horizontal morphisms both equal to η and vertical morphisms equal to the identity, since $\operatorname{dom}\Phi^E = 1$. This square is manifestly a pullback. The codomain component we choose is the pullback square of Remark 14.2.

The fact that there is a pullback of **Ord**-functors as on the right hand side of the statement follows easily, and it is a well-known fact (see, eg, [19, Prop. 9.2]). ■

As a consequence of the previous lemma, the pullback square in (11.1) that defines S-Emb factors as two pullback squares, as depicted.

$$\begin{tikzcd}
S\text{-Emb} \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & \mathsf{E}\text{-Coalg} \arrow[d, equal] \\
(L,\Phi)\text{-Coalg} \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & (E,\Phi^E)\text{-Coalg} \arrow[d] \\
\mathcal{C}^{\mathbf{2}} \arrow[r, "S^{\mathbf{2}}"] & \mathcal{B}^{\mathbf{2}}
\end{tikzcd}$$

The equality E-Coalg $= (E,\Phi^E)$-Coalg was exhibited in Corollary 5.3. The **Ord**-functor S-Emb $\to (L,\Phi)$-Coalg is an isomorphism, being the pullback of an isomorphism. The remark that follows describes this functor and its inverse in more explicit terms.

14.4. Remark. Suppose that $f\colon X \to Y$ has a structure of (L,Φ)-coalgebra, given by $(1,s)\colon f \to Lf$, where $s\colon Y \to Kf$. This structure corresponds bijectively to an $r_f\colon SY \to SX$ in $\mathcal{B}$ with $r_f \cdot Sf = 1$ and $Sf \cdot r_f \leqslant 1$, in a way that can be explicitly described: $r_f\colon SY \to SX$ is the morphism whose transpose under the adjunction $S \dashv G$ is $q_f \cdot s\colon Y \to Kf \to GSX$, i.e.

$$r_f = \big(SY \xrightarrow{Ss} SKf \xrightarrow{Sq_f} SGSX \xrightarrow{\varepsilon_{SX}} SX\big).$$

and

$$Rf\cdot s = 1 \qquad q_f \cdot s = \big(Y \xrightarrow{\eta_Y} GSY \xrightarrow{Gr_f} GSX\big).$$

14.5. Definition. We say that the adjunction $S \dashv G$ is *simple* (or *simple with respect to* (E,M)) if, for each $f\colon X \to Y$ in $\mathcal{C}$, the morphism Lf has an S-embedding structure given by

$$\big(SX \xrightarrow{SLf} SKf\big) \dashv \big(SKf \xrightarrow{Sq_f} SGSX \xrightarrow{\varepsilon_{SX}} SX\big).$$

where ε is the counit of $S \dashv G$. This amounts to the existence of the inequality $SLf \cdot \varepsilon_{SX} \cdot Sq_f \leqslant 1$, or equivalently, the inequality $GS(Lf)\cdot q_f \leqslant \eta_{Kf}$.

The following theorem is a version of [8, Thm. 11.5] and a higher-dimensional analogue of the characterisation of simple reflections in §3.

14.6. Theorem. *The following statements are equivalent.*

1. *The adjunction $S \dashv G$ is simple.*
2. *The locally monotone forgetful functor $U\colon S\text{-Emb} \to \mathcal{C}^{\mathbf{2}}$ has a right adjoint and the induced comonad has underlying functor L and counit $\Phi\colon L \Rightarrow 1_{\mathcal{C}^{\mathbf{2}}}$.*
3. *The locally monotone copointed endofunctor $\Phi\colon L \Rightarrow 1_{\mathcal{C}^{\mathbf{2}}}$ admits a comultiplication $\Sigma\colon L \Rightarrow L^2$ making $\mathsf{L} = (\mathsf{L}, \Phi, \Sigma)$ into a comonad whose category of coalgebras is isomorphic to S-Emb over $\mathcal{C}^{\mathbf{2}}$.*

15. Simple monads

Our definition of simple adjunction, Definition 14.1, requires the existence of some **Ord**-enriched limits: pullbacks in one of the two **Ord**-categories involved, and lax limits of morphisms in the other. We shall now look at the situation when the adjunction is the Eilenberg-Moore adjunction of an **Ord**-monad T on $\mathcal{C}$. In this situation, all the completeness requirements can be burden on $\mathcal{C}$. Lax limits of morphisms can be constructed from pullbacks and a lax limits of the identity morphisms, i.e., cotensor products with the arrow poset **2**. These are the minimal requirements in the following definition, and ensure the existence of arbitrary comma-objects.

15.1. Definition. Let $\mathcal{C}$ be an **Ord**-category that admits pullbacks and cotensor products with **2**. A monad $\mathsf{T} = (T, \eta, \mu)$ on $\mathcal{C}$ whose functor part T is locally monotone (i.e., **Ord**-enriched) is *simple* if the free T-algebra adjunction is simple in the sense of Definition 14.5.

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F^{\mathsf{T}}"] \arrow[r, phantom, "\perp"] & \mathsf{T}\text{-Alg} \arrow[l, shift left, "U^{\mathsf{T}}"] \end{tikzcd}$$

Explicitly, T is simple when, for each $f\colon X \to Y$ in $\mathcal{C}$, the morphism $F^{\mathsf{T}}(Lf)$ is a right adjoint of $\varepsilon^{\mathsf{T}}_{F^{\mathsf{T}}X} \cdot F^{\mathsf{T}} q^{\mathsf{T}}_f$, with these morphisms defined by the following diagram, where the square is a comma-object.

$$\begin{tikzcd} X \arrow[dr, "Lf"] \arrow[drr, bend left, "\eta_X"] \arrow[ddr, bend right, "f"'] & & \\ & Kf \arrow[r, "q_f"] \arrow[d, "Rf"] \arrow[dr, phantom, "\geqslant"] & TX \arrow[d, "Tf"] \\ & Y \arrow[r, "\eta_Y"'] & TY \end{tikzcd} \tag{15.1}$$

15.2. Lemma. *An* **Ord***-monad T on $\mathcal{C}$ is simple if and only if there is an adjunction $T(Lf) \dashv \mu_X \cdot Tq_f$, where μ_X is the multiplication of T.*

Proof. The simplicity of T is the existence of an inequality $F^{\mathsf{T}}Lf \cdot \varepsilon_{F^{\mathsf{T}}X} \cdot F^{\mathsf{T}}q_f \leqslant 1$. The forgetful **Ord**-functor U^{T} reflects inequalities between morphisms. Then, the mentioned inequality is equivalent to $TLf \cdot \mu_X \cdot Tq_f \leqslant 1$, its image under U^{T}. ∎

15.3. Corollary. *An* **Ord***-monad* T *on* $\mathcal{C}$ *is simple if and only if* $TLf \cdot q_f \leqslant \eta_{Kf}$*.*

Proof. This is a reformulation of the observation at the end of Definition 14.5. ∎

Putting together Theorem 14.6 and Definition 12.1, we have:

15.4. Corollary. *Simple lax idempotent monads* T *on cocomplete* **Ord***-categories with terminal object, pullbacks and cotensor products with* **2** *are fibrantly* KZ*-generating. The induced* LOFS (L, R) *is isomorphic to* $\Psi(\mathsf{T})$*. In particular, this* LOFS *is reflective, and* $\mathsf{R}_1 \cong \mathsf{T}$*.*

Proof. The last is the only assertion that needs a proof, the first being just Proposition 12.2. (Existence of pullbacks and cotensors with **2** are required in Definition 15.1, while cocompletness is needed in Proposition 12.2.) If (L, R) is the LOFS induced by T, we have L-Coalg equal to T-Emb $\subset \mathcal{C}^{\mathbf{2}}$ (Theorem 14.6). Then, (L, R) is sub-LARI, as LARIs always are T-embeddings. It is easy to see that T-embeddings and their morphisms satisfy the conditions of Definition 13.3, so (L, R) is cancellative. By the characterisation of Theorem 13.6, (L, R) is a reflective LOFS, i.e., of the form $\Psi(\mathsf{R}_1)$ (Proposition 12.7). Since the left part of both $\Psi(\mathsf{R}_1)$ and $\Psi(\mathsf{T})$ is T-Emb, and Ψ is full and faithful (Theorem 12.5) we obtain $\mathsf{R}_1 \cong \mathsf{T}$. ∎

15.5. Remark. The existence of terminal object, pullbacks and lax limits of morphisms in the corollary above may be replaced by the existence of finite **Ord**-enriched limits. Indeed, terminal objects and pullbacks provide all finite conical limits, while the lax limits of the identity morphisms provide cotensor products with the arrow poset **2**. As explained in [18, p. 306] (in the case of **Cat**), this suffices to have finite **Ord**-enriched limits.

Corollary 15.4 means that, if $\mathcal{C}$ is finitely complete, each simple lax idempotent monad T induces a LOFS (L, R) with L-Coalg isomorphic to T-Emb over $\mathcal{C}^{\mathbf{2}}$.

15.6. Proposition. *The monad* P *on* **Ord** *described in Example 2.3 is simple.*

Even though a direct proof of the proposition is easy, we will not include it here since it is a particular instance of the more general [8, Thm. 13.5] (for categories enriched in the symmetric monoidal category **2**).

In [8] a couple of criteria where introduced to discern whether a lax idempotent monad is simple. In §18 we shall make use of one of these criteria, which we proceed to explain.

For each morphism $f\colon X \to Y$ there is a "comparison" morphism

$$\kappa\colon T(Tf \downarrow \eta_Y) \longrightarrow T^2 f \downarrow T\eta_Y$$

induced by the universal property of comma-objects. More explicitly, κ is a morphism, as displayed in the diagram below, unique with the property of making the triangles (A) and (B) commutative.

$$\begin{tikzcd}
TKf \arrow[rr, "Tq_f"] \arrow[d, "TRf"'] \arrow[dr, "\kappa"] \arrow[dr, phantom, "\text{(A)}", shift right=6] & \text{(B)} & T^2X \arrow[d, "T^2f"] \\
TY \arrow[rr, "\eta_{TY}"'] & T^2f \downarrow T\eta_Y \arrow[l] \arrow[ur] \arrow[dr, phantom, "\geqslant"] & T^2Y
\end{tikzcd}$$

We may now state a version of [8, Prop. 12.6].

15.7. Proposition. *A lax idempotent* **Ord**-*monad* T *is simple provided that, for every* f *and* $u : Kf \to TKf$, $u \leqslant \eta_{Kf}$ *whenever* $\kappa \cdot u \leqslant \kappa \cdot \eta_{Kf}$, *where* κ *is the comparison morphism* $TKf \to T^2f \downarrow T\eta_Y$.

16. Submonads of simple monads

The aim of the present section is to provide easy criteria that will allow us to recognise simple submonads of simple lax idempotent monads. These results will be later used in Corollary 18.2. A morphism in an **Ord**-category is full if it is representably so, i.e., if postcomposing with this morphism is a full morphism of posets. For example, LARIs are full.

16.1. Lemma. *Let* T *be a lax idempotent* **Ord**-*monad. Then,* T-*embeddings are full if and only if the components of the unit* $X \to TX$ *are full.*

Proof. By definition of lax idempotent monad, the unit components $\eta_X \colon X \to TX$ are T-embeddings, and, hence, they are full provided that T-embeddings are full.

Conversely, suppose that $f \colon X \to Y$ is a T-embedding. Then, $\eta_Y \cdot f = Tf \cdot \eta_X$ is full, being a composition of the LARI Tf and the full morphism η_X. Therefore, f is full. ∎

16.2. Proposition. *Suppose that* $\varphi \colon \mathsf{S} \to \mathsf{T}$ *is a morphism of* **Ord**-*monads whose components* φ_X *are* T-*embeddings. If* T *is lax idempotent and the components of the unit* $\eta_X : X \to TX$ *are full, then* S *is lax idempotent, with full unit components* $e_X : X \to SX$.

Proof. That S is lax idempotent follows from the following calculations and fullness of $T\varphi_X \cdot \varphi_{SX} = \varphi_{TX} \cdot S\varphi_X$:

$$\varphi_{TX} \cdot S\varphi_X \cdot Se_X = T\eta_X \cdot \varphi_X \leqslant \eta_{TX} \cdot \varphi_X = \varphi_{TX} \cdot e_{TX} \cdot \varphi_X = \varphi_{TX} \cdot S\varphi_X \cdot e_{SX}.$$

Moreover, with $\eta_X = \varphi_X \cdot e_X$ full, also e_X is full. ∎

We say that a morphism $f\colon X \to Y$ is a *pullback-stable* T-*embedding* if the pullback of f along any morphism into Y is a T-embedding.

16.3. Theorem. *Suppose that $\varphi\colon \mathsf{S} \to \mathsf{T}$ is a monad morphism between* **Ord**-*monads on an* **Ord**-*category with pullbacks and cotensors with* **2**. *Assume that the components of φ are pullback-stable T-embeddings, and that T-embeddings are full. If T is lax idempotent, then S is simple whenever T is so.*

Proof. Let us denote the unit of S by $e\colon 1 \Rightarrow S$, and the **Ord**-functorial factorisations obtained from S and T following the construction of the comma-object (15.1), respectively, by

$$\big(X \xrightarrow{L_Sf} K_Sf \xrightarrow{R_Sf} Y\big) = \big(X \xrightarrow{f} Y\big) = \big(X \xrightarrow{L_Tf} K_Tf \xrightarrow{R_Tf} Y\big)$$

Consider the following diagram where $K_Tf = Tf \downarrow \eta_Y$, $K_Sf = Sf \downarrow e_Y$, and $L_Tf = \overline{\varphi}_f \cdot L_Sf$, and note that $(*)$ is a pullback.

$$\begin{tikzcd}
X \arrow[rrr, "e_X"] \arrow[dr, "L_Sf"] \arrow[ddd, "f"'] & & & SX \arrow[r, "\varphi_X"] \arrow[d, phantom, "(*)"] & TX \arrow[ddd, "Tf"] \\
& K_Sf \arrow[urr, "t_f"'] \arrow[dr, "\overline{\varphi}_f"] \arrow[ddl, "R_Sf"] & & {} & \\
& & K_Tf \arrow[uurr, "q_f"'] \arrow[dll, "R_Tf"] & {} & \\
Y \arrow[rrr, "e_Y"'] & & & SY \arrow[r, "\varphi_Y"'] \arrow[u, phantom, "\geqslant"] & TY
\end{tikzcd}$$

By Corollary 15.3, to conclude that S is simple, it is enough to show that $SL_Sf \cdot t_f \leqslant e_{K_Sf}$. And this inequality follows from the following calculations, using the fullness of $T\overline{\varphi_f} \cdot \varphi_{K_Sf}$.

$$\begin{aligned}
T\overline{\varphi}_f \cdot \varphi_{K_Sf} \cdot SL_Sf \cdot t_f = T\overline{\varphi}_f \cdot TL_Sf \cdot \varphi_X \cdot t_f = T\overline{\varphi}_f \cdot TL_Sf \cdot q_f \cdot \overline{\varphi}_f \leqslant \\
\leqslant \eta_{Kf} \cdot \overline{\varphi}_f = T\overline{\varphi}_f \cdot \varphi_{K_Sf} \cdot e_{K_Sf}
\end{aligned}$$

■

16.4. Corollary. *Suppose that $\varphi\colon \mathsf{S} \to \mathsf{T}$ is a monad morphism between* **Ord**-*monads on an* **Ord**-*category with pullbacks and cotensors with* **2**. *Assume that the components of φ are T-embeddings, and that T is lax idempotent and simple, with full unit components $X \to TX$. Then:*

1. *S is lax idempotent and simple, with full unit components $X \to SX$;*
2. *every S-embedding is a T-embedding;*
3. *S-embeddings are full.*

Proof. (1) follows from Proposition 16.2, while (3) follows directly from (2) and our assumptions. Finally, (2) is an instance of Corollary 11.5. ■

17. Frames

Recall that a *frame* is a complete lattice that satisfies the infinite distributive law $(\bigvee_i s_i) \wedge a = \bigvee_i (s_i \wedge a)$. If B is a distributive lattice, let us denote by $P(B)$ the frame of down-closed subsets of B, and by $\mathrm{Idl}(B)$ the frame of ideals of B. An ideal is a subset I that is down-closed and closed under finite joins. There is a functor Idl sending a morphism of lattices $f\colon A \to B$ to the morphism $f_*\colon \mathrm{Idl}(A) \to \mathrm{Idl}(B)$ that sends an ideal I to the smallest ideal that contains $f(I)$. The morphism $\mathrm{Idl}(f)$ always has a right adjoint which is the restriction of $f^{-1}\colon P(B) \to P(A)$ to ideals.

17.1. Proposition. *The monad* Idl *on* **DLat** *is lax idempotent and simple.*

Proof. It is well known that a distributive lattice A is a frame precisely when each ideal I of A has a supremum, i.e., when the inclusion $A \to \mathrm{Idl}(A)$ has a left adjoint $\vee\colon \mathrm{Idl}(A) \to A$. This is to say, by Definition 2.1, that the monad Idl is lax idempotent.

We will use Corollary 15.3 to prove the simplicity. Let $f\colon A \to B$ in **DLat**. The comma-object $Kf = f_* \downarrow B$ is the distributive lattice $\{(I, b) \in \mathrm{Idl}(A) \times B : f_*(I) \leqslant {\downarrow}(b)\}$. The projection $q_f\colon Kf \to \mathrm{Idl}(A)$ is simply given by $(I, b) \mapsto I$, and the morphism $Lf\colon A \to Kf$ is given by $Lf(a) = ({\downarrow}(a), f(a))$. We have to show $(Lf)_* \cdot q_f \leqslant \eta_{Kf}$, which, evaluating at $(I, b) \in Kf$, is

$$(Lf)_*(I) \leqslant {\downarrow}(I, b).$$

This is equivalent to saying that $Lf(a) = ({\downarrow}(a), f(a)) \leqslant (I, b)$ for all $a \in I$, which holds since $f(I) \leqslant f_*(I) \leqslant {\downarrow}(b)$. ∎

17.2. Corollary. *There is a* lofs *on the* **Ord***-category* **DLat** *of distributive lattices such that:*

1. *Its left morphisms are injective (equivalently, full) morphisms of lattices $f\colon A \to B$ that satisfy: for all $b, b' \in B$ and $a \in A$, if $f(a) \leqslant b \vee b'$, then there exist $x, x' \in A$ such that $f(x) \leqslant b$, $f(x') \leqslant b'$ and $a \leqslant x \vee x'$.*
2. *Frames are precisely the distributive lattices that are injective with respect the morphisms described above.*

Proof. The existence of the lofs, say (L, R), follows from Proposition 17.1 and Corollary 15.4. The L-coalgebras coincide with the Idl-embeddings, so we have to show that these coincide with the full morphisms described in the first part of the statement.

First we show that a morphism of distributive lattices $f\colon A \to B$ is full if and only if $\mathrm{Idl}(f)$ is full. That full morphisms between lattices are the same as injective morphisms is easy to see ($f(x) \leqslant f(y)$ means $f(x \wedge y) = f(x) \wedge f(y) = f(x)$). If $\mathrm{Idl}(f)$ is injective, then, f, being the restriction of $\mathrm{Idl}(f)$ to principal ideals, has to be injective, i.e., full.

Conversely, suppose that f is full. Since there always is an adjunction $\mathrm{Idl}(f) \dashv f^{-1}$, we have to show that $f^{-1} \cdot \mathrm{Idl}(f) = 1$. Up to isomorphism, f is the inclusion of a full sublattice

$C \subseteq B$. The right adjoint f^{-1} becomes identified with $(- \cap C)\colon \mathrm{Idl}(B) \to \mathrm{Idl}(C)$. Recall that $\mathrm{Idl}(f)(I)$ is the ideal of B generated by I, and since $\mathrm{Idl}(f)$ preserves suprema,

$$\mathrm{Idl}(f)(I) = \mathrm{Idl}(f)\Big(\bigvee_{a\in I} \downarrow(a)\Big) = \bigvee_{a\in I} \mathrm{Idl}(f)(\downarrow(a)) = \bigvee_{a\in I}(\downarrow(f(a))) = \bigcup_{a\in I}(\downarrow(f(a))),$$

where the last equality holds because filtered suprema of ideals coincide with the set-theoretical union.

We can show that $\mathrm{Idl}(f)$ is full too, as follows. Given $\mathrm{Idl}(f)(I) \subseteq \mathrm{Idl}(f)(J)$, for all $a \in I$ there exists $b \in J$ such that $f(a) \leqslant f(b)$. By the fullness of f, then $a \leqslant b$, so $a \in J$. Thus, $I \subseteq J$.

As mentioned above, the morphism $\mathrm{Idl}(f)$ has always a right adjoint f^{-1} from $\mathrm{Idl}(B)$ to $\mathrm{Idl}(A)$ given by taking preimage under f. When f is full, the unit of the adjunction is an equality $1 = f^{-1} \cdot \mathrm{Idl}(f)$, since $\mathrm{Idl}(f)$ is full (and faithful). We would have proven that f is an $\mathrm{Idl}(f)$-embedding when we show that f^{-1} is a morphism of frames. Being a right adjoint, it automatically preserves meets, so we need to show that it preserves suprema. Furthermore, f^{-1} also preserves directed suprema of ideals, since these are just set-theoretical unions, and preserves the bottom element. All that remains to have preservation of suprema is the preservation of binary suprema. In other words, that for $I, J \in \mathrm{Idl}(B)$, we should have

$$f^{-1}(I) \vee f^{-1}(J) \supseteq f^{-1}(I \vee J). \tag{17.1}$$

Since $I \vee J = \cup\{\downarrow(b \vee b') : b \in I, b' \in J\}$, and f^{-1} preserves unions, it suffices to show that $f^{-1}(\downarrow(b \vee b')) \in f^{-1}(I) \vee f^{-1}(J)$ for $b \in I$ and $b' \in J$. This clearly follows from the condition of f in the first part of the statement. Conversely, (17.1) for $I = \downarrow(b)$ and $J = \downarrow(b')$ yields the condition of the statement.

The second part of the statement is a consequence of Corollary 15.4. ∎

17.3. Example. We point out that the extra condition on the full morphism f in the first part of the corollary above is not redundant. To see this, first observe that the ideals of a finite distributive lattice B are necessarily principal ideals, or in other words, $B \to \mathrm{Idl}(B)$ is an isomorphism. If $f\colon A \to B$ is a full inclusion of finite distributive lattices, we have that f is an Idl-embedding if and only if f has a right adjoint. This condition is not always satisfied.

18. Filter monads

In this section we exhibit AWFSs on the **Ord**-category of T_0 topological spaces arising from simple lax idempotent **Ord**-monads. These factorisations were constructed in [6].

Each T_0 topological space X carries an order given by

$$x \leqslant y \text{ if and only if } y \in \overline{\{x\}} \tag{18.1}$$

– this is the opposite of what is usually called the *specialisation order*. This induces an order structure on each hom-set $\mathbf{Top}_0(X,Y)$ by defining $f \leqslant g$ if $f(x) \leqslant g(x)$, for all $x \in X$, making $\mathbf{Top}_0$ into an $\mathbf{Ord}$-enriched category.

A comma-object $f \downarrow g$ in $\mathbf{Top}_0$ can be described as the subspace of $X \times Y$ defined by the subset $\{(x,y) \in X \times Y \colon f(x) \leqslant g(y)\}$.

$$\begin{tikzcd} f \downarrow g \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\leqslant"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

Denote by $\mathsf{F}\colon \mathbf{Top}_0 \to \mathbf{Top}_0$ the filter monad. If X is a T_0 space, FX is the set of filters of open sets of X, with topology generated by the subsets $U^{\#} = \{\varphi \in FX : U \in \varphi\}$, where $U \in \mathcal{O}(X)$. The (opposite of the) specialisation order on FX results in the opposite of the inclusion of filters. In particular, FX is a poset. If $f\colon X \to Y$ is continuous, then Ff is defined by $Ff(\varphi) = \{V \in \mathcal{O}(Y) : f^{-1}(V) \in \varphi\}$. The unit of the monad has components $\eta_X\colon X \to FX$, where $\eta_X(x)$ is the *principal filter* generated by x, that is $\eta_X(x) = \{U \in \mathcal{O}(X) : x \in U\}$. The multiplication of the monad has components $\mu_X\colon F^2X \to FX$, given by $\mu_X(\Theta) = \{U \in \mathcal{O}(X) : U^{\#} \in \Theta\}$.

Observe that η_X is a full morphism. It is in fact an *embedding* meaning a topological embedding, in the usual sense: a continuous function that is an homeomorphism onto its image, where the latter is equipped with the subspace topology.

It was shown in [10] that the category of algebras for this monad is isomorphic to the category whose objects are continuous lattices [29] and morphisms poset maps that preserve directed sups and arbitrary infs. Our choice of the (opposite of the) specialisation order on spaces, which is the opposite of the order used in [10], grants a few comments as a way of avoiding confusion. A space $X \in \mathbf{Top}_0$ has an F-algebra structure precisely when the opposite of the poset $(X,\leqslant)$ is a continuous lattice, where $\leqslant$ is the order (18.1). The topology of the space X can be recovered as the Scott topology of the continuous lattice $(X,\leqslant)^{\mathrm{op}}$. A morphism of F-algebras $f\colon X \to Y$ is a continuous function that preserves arbitrary suprema, as a poset map $(X,\leqslant) \to (Y,\leqslant)$ [10, Thm. 4.4].

The filter monad F was shown to be lax idempotent in [13], where it is also proved that a continuous function f between T_0 spaces is an embedding if and only if Ff is a LARI. In other words, F-embeddings are precisely the topological embeddings.

18.1. THEOREM. *The* $\mathbf{Ord}$*-monad* F *is simple.*

PROOF. We verify the hypothesis of Proposition 15.7. For any pair of continuous maps $f\colon X \to Z$ and $g\colon Y \to Z$, the comparison morphism

$$\kappa\colon F(f \downarrow g) \longrightarrow Ff \downarrow Fg \subset FX \times FY$$

sends a filter φ on $f \downarrow g$ to the pair of filters (ψ_0, ψ_1)

$$\psi_0 = \{U \in \mathcal{O}(X) : d_0^{-1}(U) \in \varphi\} \qquad \psi_1 = \{V \in \mathcal{O}(Y) : d_1^{-1}(V) \in \varphi\}$$

where d_0 and d_1 are the projections from $f \downarrow g$ to X and Y, respectively. Given $(x, y) \in f \downarrow g$, recall that its image under the unit is

$$\eta_{f\downarrow g}(x, y) = \{W \in \mathcal{O}(f \downarrow g) : (x, y) \in W\}.$$

We have $(Fd_0)\eta_{f\downarrow g}(x, y) = \eta_X d_0(x, y) = \eta_X(x)$, and similarly, $(Fd_1)\eta_{f\downarrow g}(x, y) = \eta_Y(y)$.

The hypothesis of Proposition 15.7 will be satisfied if we show that $\kappa \cdot u \leqslant \kappa \cdot \eta_{f\downarrow g}$ implies $u \leqslant \eta_{f\downarrow g}$; or, in terms of filters, if we show that, given $\varphi \in F(f \downarrow g)$, $(x, y) \in f \downarrow g$ as above, the inequalities $\psi_0 \leqslant \eta_X(x)$ and $\psi_1 \leqslant \eta_Y(y)$ imply $\varphi \leqslant \eta_{f\downarrow g}(x, y)$. By definition of the (opposite) specialisation order, we need to show the two inclusions

$$\{U \in \mathcal{O}(X) : d_0^{-1}(U) \in \varphi\} \supseteq \{U \in \mathcal{O}(X) : x \in U\}$$
$$\{V \in \mathcal{O}(Y) : d_1^{-1}(V) \in \varphi\} \supseteq \{V \in \mathcal{O}(Y) : y \in V\}$$

imply $\varphi \supseteq \{W \in \mathcal{O}(f \downarrow g) : (x, y) \in W\}$. Given $x \in U \in \mathcal{O}(X)$, $y \in V \in \mathcal{O}(Y)$, then

$$(U \times V) \cap (f \downarrow g) = d_0^{-1}(U) \cap d_1^{-1}(V) \in \varphi.$$

But any neighbourhood W of (x, y) contains another of the form $(U \times V) \cap (f \downarrow g)$, so $W \in \varphi$, completing the proof. ∎

Since every principal filter is completely prime, and so in particular prime and proper, and $\mu_X(\Theta)$ is completely prime (resp. prime, proper) whenever Θ is so, the functors F_1, F_ω and F_Ω that assign to each space X the space of proper (resp. prime, completely prime) filters are the functor part of submonads F_1, F_ω and F_Ω of the filter monad, with the monad morphisms defined pointwise by the corresponding embeddings. Hence, using Corollary 16.4, we can immediately conclude:

18.2. Corollary. *The* **Ord***-monads of proper filters, of prime filters and of completely prime filters are lax idempotent and simple.*

Therefore these monads induce LOFSs $(\mathsf{L}_\alpha, \mathsf{R}_\alpha)$, with $\alpha = 0, 1, \omega, \Omega$ (denoting F by F_0), with associated weak factorisation systems $(\mathcal{L}_\alpha, \mathcal{R}_\alpha)$, where $\mathcal{L}_0$ is the class of embeddings, $\mathcal{L}_1$ is the class of dense embeddings, $\mathcal{L}_\omega$ is the class of flat embeddings, and $\mathcal{L}_\Omega$ is the class of completely flat embeddings [12, 13, 6]. Moreover, $\mathcal{R}_\alpha$ is the class of morphisms which are injective with respect to $\mathcal{L}_\alpha$ (see [6] for details).

19. Metric spaces

It is an insight of Bill Lawvere [26, 25] that metric spaces can be regarded as enriched categories and that, from this point of view, completeness can be interpreted in terms of "modules." The necessary base of enrichment is the category of *extended non-negative real numbers* $\bar{\mathbb{R}}_+$.

The category $\bar{\mathbb{R}}_+$ has objects the real non-negative numbers plus an extra object ∞, and has one morphism $\alpha \to \beta$ if and only if $\alpha \geqslant \beta$; ∞ is an initial object and 0 a terminal object. One can use the addition of real numbers to define a symmetric monoidal structure on $\bar{\mathbb{R}}_+$, with the convention that adding ∞ always produces ∞. The unit object of this tensor product is 0. Furthermore, $\bar{\mathbb{R}}_+$ is closed, with internal hom $[\alpha, \beta]$ equal to $\beta - \alpha$ if this difference is non-negative, and equal to zero otherwise, with the convention that $[\alpha, \infty] = \infty$, $[\infty, \infty] = 0$ and $[\infty, \alpha] = 0$.

A small $\bar{\mathbb{R}}_+$-category can be described as a set A with a distance function $A(-,-)\colon A \times A \to \bar{\mathbb{R}}_+$ that satisfies $A(a,a) = 0$ for all $a \in A$ and the triangular inequality. In general, it may very well happen that $A(a,b) = 0$ even if $a \neq b$; the distance may not be symmetric, i.e., $A(a,b) \neq A(b,a)$, and the distance between two points may be ∞. We regard $\bar{\mathbb{R}}_+$-categories as generalised metric spaces and think of $A(a,b) \in \bar{\mathbb{R}}_+$ as the “distance” from a to b.

For example, $\bar{\mathbb{R}}_+$ itself is a generalised metric space with distance from α to β given by $[\alpha, \beta]$.

Each generalised metric space A has an *opposite* A^{op} with the same points and distance $A^{\mathrm{op}}(a,b) = A(b,a)$. We will concentrate on *skeletal* generalised metric spaces, i.e., those spaces A for which $A(a,b) = 0 = A(b,a)$ implies $a = b$. For example, $\bar{\mathbb{R}}_+$ is skeletal.

$\bar{\mathbb{R}}_+$-enriched functors $f\colon A \to B$ are identified with functions $A \to B$ that are non-expansive: $A(a,b) \geqslant B(f(a), f(b))$. It is easy to verify that there exists a unique $\bar{\mathbb{R}}_+$-natural transformation $f \Rightarrow g\colon A \to B$ if and only if $0 = B(f(a), g(a))$ for all $a \in A$. In this way we obtain an **Ord**-category $\mathbf{Met}_{\mathrm{sk}}$ of skeletal generalised metric spaces, with objects the skeletal $\bar{\mathbb{R}}_+$-categories, morphisms the $\bar{\mathbb{R}}_+$-functors and inequality $f \leqslant g$ between two of them given by the existence of a $\bar{\mathbb{R}}_+$-natural transformation $f \Rightarrow g$. Observe that $\mathbf{Met}_{\mathrm{sk}}(A,B)$ is not only a preorder but a poset, because B is skeletal.

There is a notion of colimit suited to enriched categories, known as *weighted colimit* (or *indexed colimit* in older texts); see [20, 21] for a standard reference. Each family of weights induces a lax idempotent **Ord**-monad on $\mathbf{Met}_{\mathrm{sk}}$ whose algebras are the skeletal generalised metric spaces that admit colimits with weights in the family (see [23, Theorems 6.1 and 6.3]). This monad is in fact simple (§15), as shown in the more general context in [8, §12]. It follows from the theory developed herein that there is a LOFS on $\mathbf{Met}_{\mathrm{sk}}$ whose left morphisms are the embeddings with respect to that monad and whose fibrant objects are the skeletal generalised metric spaces that admit all Φ-colimits (see Proposition 12.2 and Corollary 15.4). The rest of the section is occupied by the example of a particular class of colimits that admit an explicit description.

The class of absolute colimits, i.e., the weights whose associated colimits are preserved by any $\bar{\mathbb{R}}_+$-functor whatsoever, generates a simple lax idempotent monad Q on $\mathbf{Met}_{\mathrm{sk}}$. Putting together [26] and [30] one can give a description of Q in terms of Cauchy sequences.

Cauchy sequences in a skeletal generalised metric space A are defined in the same way as for classical metric spaces. Two Cauchy sequences (a_n) and (b_n) are *equivalent* if both $A(a_n, b_n)$ and $A(b_n, a_n)$ have limit 0. Denote by QA the set of equivalence classes of Cauchy sequences in A with distance $QA([a_n],[b_n]) = \lim_n A(a_n, b_n)$. It is not hard to see that QA is a skeletal generalised metric space.

The assignment $A \mapsto QA$ is part of an **Ord**-monad Q on $\mathbf{Met}_{\mathrm{sk}}$, with unit $A \to QA$

the map that sends $a \in A$ to the constant sequence on a, that we denote by c_a.

The convergence of a sequence (x_n) to a point a in a generalised metric space A differs from ordinary convergence in metric spaces only in that we have to require that both $A(a, x_n)$ and $A(x_n, a)$ converge to 0 in $\bar{\mathbb{R}}_+$. The following assertions are equivalent for a skeletal generalised metric space A: it is an algebra for Q; the canonical isometry $A \to QA$ has a left adjoint; A is a retract of a space of the form QB; every Cauchy sequence in A converges. Spaces that satisfy these equivalent properties are known as *Cauchy-complete*.

If $(\mathsf{L}_Q, \mathsf{R}_Q)$ is the KZ-reflective LOFS on $\mathbf{Met}_{\mathrm{sk}}$ generated by Q, the L_Q-coalgebras, or left maps of the factorisation, are the Q-embeddings and can be characterised as follows.

19.1. PROPOSITION. *A non-expansive map $f\colon A \to B$ between skeletal generalised spaces is a Q-embedding if and only if it is an isometry and for each $b \in B$ the non-expansive function $B(f-, b)\colon A^{\mathrm{op}} \to B$ can be written as $B(f-, b) = \lim_n A(-, x_n)$ for a Cauchy sequence (x_n) in A.*

PROOF. First, if Qf has a retract r, then Qf is an isometry and thus f is an isometry; for, $B(f(a), f(a')) = QB(c_{f(a)}, c_{f(a')}) = QB(Qf(c_a), Qf(c'_a)) = QA(c_a, c_{a'}) = A(a, a')$.

If r is moreover a right adjoint of Qf, and, for a given $b \in B$, $r(c_b)$ has an associated Cauchy sequence (x_n) in A, we must have

$$B(f(a), b) = QB\big(c_{f(a)}, c_b\big) = QB\big(Qf(c_a), c_b\big) = QA\big(c_a, r(c_b)\big) = \lim_n A(a, x_n)$$

for all $a \in A$.

Conversely, suppose that f is an isometry and $B(f-, b) = \lim_n A(-, x_n)$. We must define an equivalence class of Cauchy sequences $r[b_n] \in QA$ for each $[b_n] \in QB$ in a way such that $QB([f(a_n)], [b_n]) = QA([a_n], r[b_n])$. Since any Cauchy sequence is a limit of constant sequences (eg, $b_n = \lim_n c_{b_n}$), it suffices to define r and to verify this equality for constant sequences; i.e., we have to give $r[c_b] \in QA$ such that $B(f(a), b) = QA(c_a, r[c_b])$. Since we know that $B(f-, b) = \lim_n A(-, x_n)$, we may set $r[c_b] = [x_n]$ and the equality holds. In this way we prove that there is an adjunction $Qf \dashv r\colon QB \to QA$. It remains to prove that $r \cdot Qf = 1$, but f is an isometry, which implies that Qf is an isometry and therefore one-to-one, so the equality follows from the adjunction triangle equation $Qf \cdot r \cdot Qf = Qf$. ∎

It follows from the general theory that, given a Q-embedding $f\colon A \to B$ and a non-expansive function $h\colon A \to C$ into a Cauchy-complete skeletal generalised metric space C, there is an extension d.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \\ B \arrow[ur, dotted, "d"'] & \end{tikzcd}$$

Furthermore, Cauchy-complete skeletal generalised metric spaces are precisely those injective with respect to the Q-embeddings. In terms of sequences, the extension d is given by $d(b) = \lim_n h(x_n)$, where (x_n) is a Cauchy sequence in A such that $B(f-, b) = \lim_n A(-, x_n)$.

19.2. Corollary. *Let $f\colon A \to B$ be a non-expansive function between skeletal generalised metric spaces, and assume that B is a metric space. Then, f is a Q-embedding if and only if it is a dense isometry.*

Proof. If f is a Q-embedding and $b \in B$, there is a Cauchy sequence (x_n) in A such that $\lim_n A(-, x_n) = \lim_n B(f-, b)$. Given $\varepsilon > 0$, there is an n_0 such that $A(x_n, x_m) < \varepsilon/2$ if $n, m \geqslant n_0$. Thus, for $m \geqslant n_0$ we have

$$B(f(x_m), b) = \lim_n B(f(x_m), f(x_n)) = \lim_n A(x_m, x_n) \leqslant \varepsilon/2 < \varepsilon.$$

It follows that $(f(x_m))$ converges to b, and f is dense. Observe that we have used that the distance of B is symmetric.

Conversely, if f is a dense isometry, any $b \in B$ is $\lim_n f(x_n)$ for some sequence (x_n) in A, which is Cauchy since f preserves distances and $(f(x_n))$ converges. Then $B(f(a), b) = \lim_n A(a, x_n)$ for all $a \in A$, and Proposition 19.1 applies. ∎

The definition of QA given in terms of Cauchy sequences immediately tells us that if A is a metric space then QA is a metric space too; i.e., its distance function is symmetric. We deduce:

19.3. Corollary. *The* lofs $(\mathsf{L}_Q, \mathsf{R}_Q)$ *restricts to an* ofs *on the category of metric spaces. Its left maps are the dense isometries.*

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