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LAX FAMILIAL REPRESENTABILITY AND LAX GENERIC FACTORIZATIONS

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Abstract. A classical result due to Diers shows that a copresheaf $F: \mathcal{A} \to \mathbf{Set}$ on a category $\mathcal{A}$ is a coproduct of representables precisely when each connected component of F's category of elements has an initial object. Most often, this condition is imposed on a copresheaf of the form $\mathcal{B}(X, T-)$ for a functor $T: \mathcal{A} \to \mathcal{B}$, in which case this property says that T admits generic factorizations at X, or equivalently that T is familial at X.

Here we generalize these results to the two-dimensional setting, replacing $\mathcal{A}$ with an arbitrary bicategory $\mathscr{A}$, and $\mathbf{Set}$ with $\mathbf{Cat}$. In this two-dimensional setting, simply asking that a pseudofunctor $F: \mathscr{A} \to \mathbf{Cat}$ be a coproduct of representables is often too strong of a condition. Instead, we will only ask that F be a lax conical colimit of representables. This in turn allows for the weaker notion of lax generic factorizations (and lax familial representability) for pseudofunctors of bicategories $T: \mathscr{A} \to \mathscr{B}$.

We also compare our lax familial pseudofunctors to Weber's familial 2-functors, finding our description is more general (not requiring a terminal object in $\mathscr{A}$), though essentially equivalent when a terminal object does exist. Moreover, our description of lax generics allows for an equivalence between lax generic factorizations and lax familial representability.

Finally, we characterize our lax familial pseudofunctors as right lax F-adjoints followed by locally discrete fibrations of bicategories, which in turn yields a simple definition of parametric right adjoint pseudofunctors.

Contents

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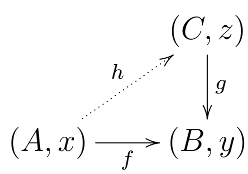
1. Introduction

This paper is concerned with the notion of *familial representability*, a condition first studied in detail by Diers [7] for 1-categories, and how the theory of familial representability can be generalized to the two-dimensional setting.

1.1. Familial representability. Given a category $\mathcal{A}$ and presheaf $F\colon \mathcal{A} \to \mathbf{Set}$ (actually a "copresheaf", we suppress the "co" for brevity), it is often useful to know whether this presheaf is a coproduct of representable presheaves; meaning

$$F \cong \sum_{m \in \mathfrak{M}} \mathcal{A}\left(P_m, -\right)$$

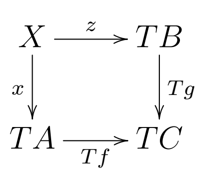
for some set $\mathfrak{M}$ and function $P_{(-)}\colon \mathfrak{M} \to \mathcal{A}$. Such presheaves have a straightforward characterization: a presheaf F is a coproduct of representables precisely when each connected component of its category of elements, denoted el F, has an initial object. Expressing this condition in more detail, this means that for any object (D,w) in el F there exists an object (A,x) and morphism $k\colon (A,x) \to (D,w)$ where (A,x) satisfies the following property (which defines initial objects in a connected component): for any diagram in el F as below

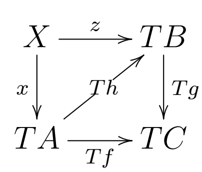


there exists a unique morphism $h\colon (A,x) \to (C,z)$, and consequently the above triangle commutes.

Of particular interest is the case where F is of the form $\mathcal{B}\left(X, T-\right)$ for a functor $T\colon \mathcal{A} \to \mathcal{B}$ between categories $\mathcal{A}$ and $\mathcal{B}$. This condition, first studied in detail by Diers [7], asks that we have an isomorphism $\mathcal{B}\left(X, T-\right) \cong \sum_{m \in \mathfrak{M}} \mathcal{A}\left(P_m, -\right)$ and generalizes T having a left adjoint. Thus such a T is often referred to as a functor having a left multiadjoint [7], however we will simply refer to such a T as *familial*. It is also worth noting that the functors T with this property may be seen as the admissible maps against the KZ pseudomonad [27] freely adding sums.[1]

If we specialize the above to this case, we see that asking $\mathcal{B}\left(X, T-\right)$ be familial amounts to asking that for any $w\colon X \to TD$ there exists an $x\colon X \to TA$ and $k\colon A \to D$ such that $w = Tk \cdot x$, and x is "generic" meaning that it satisfies the following property: given any commuting square as on the left below





[1] Under this characterization one would suitably replace categories with their opposites, as the condition given concerns *co*presheaves.

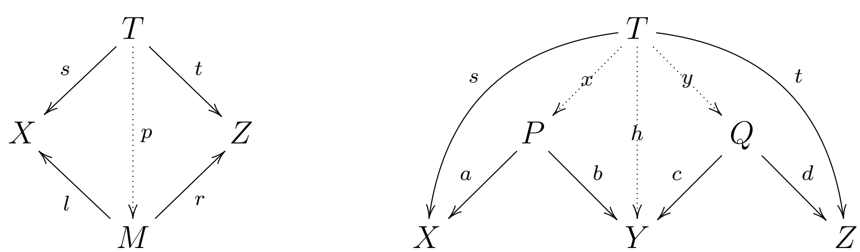
there exists a unique $h\colon A \to B$ such that $Th \cdot x = z$ (note that $g \cdot h = f$ can be shown as a consequence). Such a factorization $w = Tk \cdot x$ is called a generic factorization, and thus when this is true for all X, we say T *admits generic factorizations* [28].

There are a number of natural examples of familial functors (or equivalently functors which admit generic factorizations), with the author's favorite being composition of spans in a category $\mathcal{E}$ with pullbacks.

1.2. Example. *Given a category $\mathcal{E}$ with pullbacks, one may form the bicategory of spans in $\mathcal{E}$, typically denoted $\mathbf{Span}(\mathcal{E})$. For any triple of objects $X, Y, Z \in \mathcal{E}$ the composition functor*

$$c_{X,Y,Z}\colon \mathbf{Span}(\mathcal{E})(Y,Z) \times \mathbf{Span}(\mathcal{E})(X,Y) \to \mathbf{Span}(\mathcal{E})(X,Z)$$

is familial since for any three spans $(s,t): X \to Z$, $(a,b): X \to Y$ and $(c,d): Y \to Z$ the universal property of the limiting cone defining the composite of spans $(c,d) \circ (a,b)$ is a bijection $p \mapsto (x,h,y)$ as below



where (l,r) is the composite $(c,d) \circ (a,b)$. Written another way, this is a natural bijection between $\mathbf{Span}(\mathcal{E})(X,Z)[(s,t),(c,d) \circ (a,b)]$ and

$$\sum_{h\colon T \to Y} \mathbf{Span}(\mathcal{E})(X,Y)[(s,h),(a,b)] \times \mathbf{Span}(\mathcal{E})(Y,Z)[(h,t),(c,d)]$$

and thus we directly exhibit each presheaf

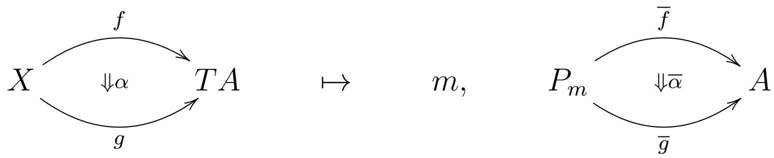
$$\mathbf{Span}(\mathcal{E})(X,Z)[(s,t), - \circ -] : \mathbf{Span}(\mathcal{E})(Y,Z) \times \mathbf{Span}(\mathcal{E})(X,Y) \to \mathbf{Set}$$

as a coproduct of representables, and thus exhibit $c_{X,Y,Z}$ as a familial functor. One thing to notice here is that $c_{X,Y,Z}$ is an example of a familial functor where the domain category does not have a terminal object; thus definitions of higher analogues of familial functors should also not require terminal objects.

1.3. The problem with pseudo familial representability. It is the purpose of this paper to generalize these notions of familial representability to the two-dimensional setting, replacing the category $\mathcal{A}$ with a bicategory $\mathscr{A}$, and replacing $\mathbf{Set}$ with $\mathbf{Cat}$. However, this is not a straightforward generalization, as asking that a pseudofunctor $F\colon \mathscr{A} \to \mathbf{Cat}$ be a coproduct of representables is often too strong of a condition. To see why, consider the case where a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$ is such that each $\mathscr{B}(X, T-)$ is a coproduct of representables, meaning we have an equivalence

$$\mathscr{B}(X, T-) \simeq \sum_{m \in \mathfrak{M}_X} \mathscr{A}(P_m, -)$$

for some set $\mathfrak{M}_X$ and function $P_{(-)}\colon \mathfrak{M}_X \to \mathscr{A}$. Such an equivalence must be defined by an assignation as below, which would send each 2-cell α as on the left



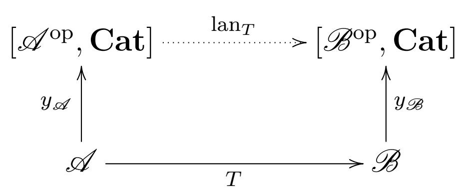
to an $\overline{\alpha}\colon \overline{f} \Rightarrow \overline{g}$ as on the right, where $f \cong T\overline{f}\cdot\delta$ and $g \cong T\overline{g}\cdot\delta$ for the same generic $\delta\colon X \to TP_m$ corresponding to the identity at P_m. This is an unreasonably strong condition: we should not in general expect two 1-cells to factor through the same generic just because there is a comparison map between them.[2] In general, this should only be expected when the comparison map is invertible. We will therefore need a weaker notion of familial representability in two dimensions.

1.4. Lax familial representability. To address the above problem, we weaken the condition on $\mathscr{B}(X, T-)$, now only asking that it be a *lax conical colimit of representables.*[3] We will use the convenient notation

$$\mathscr{B}(X, T-) \simeq \int_{\text{lax}}^{m\in\mathfrak{M}_X} \mathscr{A}(P_m, -)$$

which is justified as a lax conical colimit can be seen as an instance of a lax coend. We then define a pseudofunctor of bicategories $T\colon \mathscr{A} \to \mathscr{B}$ to be *lax familial* when each $\mathscr{B}(X, T-)$ is a lax conical colimit of representables (in a way which is natural in X in a suitable sense).

To see why being lax familial is a natural condition on a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$, consider the problem of calculating a left Kan extension as below



for a given pseudofunctor T (where $\mathscr{A}$ and $\mathscr{B}$ are small). In general this left extension should not be expected to have a nice form. However, if T is a pseudofunctor that is lax familial, so that each $\mathscr{B}(X, T-)$ is a lax conical colimit of representables, then this left

[2]Here "generic" means a morphism δ corresponding to the identity at some P_m, and the "generic" factorization is obtained from substituting an $f\colon X \to TA$ into the equivalence and applying naturality with respect to the induced $\overline{f}\colon P_m \to A$.

[3]A lax colimit is a weighted colimit in which the universal property of weighted colimits replaces pseudo natural transformations with lax natural transformations. A lax *conical* colimit is such a lax colimit where the weight $J = \mathbf{\Delta 1}$ is constant at the terminal category. We recall this notion in more detail in Subsection 2.11.

extension will have a simpler description. Said in more detail, such a left extension is generally computed as a bi-coend (whose construction generally requires formally adding in isomorphisms, hence the complexity), but in the case where T is lax familial the left extension may be more easily computed as a lax coend.

An important example of this situation (shown to be lax familial in Example 6.1) is given by taking T as the canonical inclusion of a small category $\mathcal{E}$ with pullbacks into its bicategory of spans $\mathbf{Span}(\mathcal{E})$

$$\begin{array}{ccc} [\mathcal{E}^{\mathrm{op}}, \mathbf{Cat}] & \xrightarrow{\ \mathrm{lan}_T\ } & [\mathbf{Span}(\mathcal{E})^{\mathrm{op}}, \mathbf{Cat}] \\ {\scriptstyle y_{\mathcal{E}}}\uparrow & & \uparrow{\scriptstyle y_{\mathbf{Span}(\mathcal{E})}} \\ \mathcal{E} & \xrightarrow[T]{} & \mathbf{Span}(\mathcal{E}) \end{array}$$

and forming the left extension lan_T as above, with right adjoint res_T given by restricting along T. Now, recognizing $[\mathbf{Span}(\mathcal{E})^{\mathrm{op}}, \mathbf{Cat}]$ as the 2-category of fibrations with sums (by the universal property of spans) [6], and noting that the extension-restriction adjunction is pseudomonadic (a consequence of T being bijective on objects) [18], the reader will recognize this left extension as the free functor for the pseudomonad $\Sigma_{\mathcal{E}}$ for fibrations over $\mathcal{E}$ with sums. In this way one can derive the pseudomonad for fibrations with sums (as the composite $\mathrm{res}_T \cdot \mathrm{lan}_T$), and understand why this pseudomonad has a simpler description that one would generally expect for pseudomonads arising in this way. Note the same can be done for fibrations with products, replacing $\mathbf{Span}(\mathcal{E})$ with $\mathbf{Span}(\mathcal{E})^{\mathrm{co}}$.

Also note that it will be shown in future work that the above is a special case of a more general result; for a category[4] $\mathcal{E}$ and bicategory $\mathscr{B}$, bijective on objects pseudofunctors $\mathcal{E} \to \mathscr{B}$ correspond with bi-cocontinuous pseudomonads on $[\mathcal{E}^{\mathrm{op}}, \mathbf{Cat}]$, in which case the pseudo-algebras of the pseudomonad form the 2-category $[\mathscr{B}^{\mathrm{op}}, \mathbf{Cat}]$.

Of course, whilst understanding the above situation is the author's motivation, there are other motivating examples of lax familial functors. For instance the results of Weber [30] (and the later shown equivalence with our definition) show that the composite 2-monads of [29] that describe symmetric and braided analogues of the ω-operads of [1] are examples of lax familial functors.

1.5. Main results and structure of the paper. The goal of this paper is to generalize all of the important results true for familial functors to the case of lax familial functors. The notions given in this paper concerning lax familial representability and its corresponding notions of genericity are all new. The only exception is to note that Weber's definitions of familial (2-)functors [30] and notions of genericity can be seen as instances of our definitions in the case where we have a terminal object (though this is not obvious, as Weber's definitions use fibrations and look considerably different at first glance).

[4]More generally, one can use a bicategory here. However, it suffices to use a 1-category in most of the interesting examples.

In Section 3 we generalize the basic result that a presheaf is a coproduct of representables if and only if each connected component of its category of elements has an initial object, now giving a description of when a **Cat**-valued presheaf is a lax conical colimit of represent-ables in Theorem 3.12.

In Section 4 we generalize the result that a functor is familial if and only if it admits generic factorizations, by defining a notion of lax-generic factorizations and showing this condition is equivalent to being lax familial in Theorem 4.10. Note that such an equivalence was not shown by Weber [30].

The main result of Section 5 is Theorem 5.8, which states that (a slightly stricter version of) our definition of lax familial pseudofunctors is equivalent to Weber's very different-looking definition. This is strong evidence that our definitions of familial pseudofunctors are the correct ones.

In Section 6 we provide a number of natural examples of lax familial pseudofunctors, further justifying our definitions.

In Section 7 we generalize Diers' result [7] that a functor is familial if and only if it factors as a right adjoint followed by a discrete fibration. This generalization is given in Theorem 7.17, which states that a pseudofunctor is lax familial if and only if it factors as an appropriate right lax F-adjoint (a special type of lax adjunction), followed by the two-dimensional version of a discrete fibration [17]. In the author's opinion this gives the most natural-looking characterization of lax familial pseudofunctors, as the other characterizations appear quite technical.

2. Background

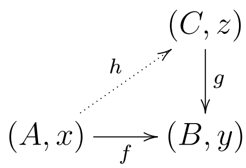
In this section we will recall the necessary background knowledge for this paper. We will first recall the basic theory of familial functors and generic factorizations in the one-dimensional case [28], and then go on to recall the basics of lax conical colimits [20] and the Grothendieck construction [21], which will replace the category of elements in the two-dimensional setting.

2.1. Generic factorizations in one dimension. We will first recall the basic fact that a presheaf is a coproduct of representables if and only if each connected component of its category of elements has an initial object. It is worth explaining this result in some more detail, as later on in the two-dimensional case simple conditions such as asking each connected component has an initial object will not suffice. Of course these are all well-known results of Diers [7, 8] (also see [28] for a more recent account).

2.2. Definition. *Given a presheaf* $F\colon \mathcal{A} \to \mathbf{Set}$*, recall the classical notion of the category of elements of* F *as the category with objects given by pairs* $(A \in \mathcal{A}, x \in FA)$ *and morphisms* $(A, x) \nrightarrow (B, y)$ *given by maps* $f\colon A \to B$ *such that* $Ff(x) = y$*. We denote this category* $\mathrm{el}\, F$*.*

The following definition is more complicated than it needs to be, in that the generics are precisely the initial objects in the connected components. The reason for stating it this way is that it more closely matches the definitions needed in two dimensions.

2.3. Definition. *Given a presheaf* $F\colon \mathcal{A} \to \mathbf{Set}$*, we say an object* $(A, x) \in \mathrm{el}\, F$ *is* el-generic *if for any given objects* (B, y), (C, z) *and morphisms* f *and* g *as below*



there exists a unique morphism $h\colon (A, x) \to (C, z)$*. It is then automatic that the above triangle commutes.*

Moreover, given two el-generic objects (A, x) *and* (D, w)*, an* el-generic morphism $(A, x) \to (D, w)$ *is any such morphism in* el F*.*[5]

2.4. Remark. *The reader will note that this is stronger than asking for the existence of a unique lifting* h*. In fact, asking that* h *be the unique morphism (and not just the unique lifting), is a condition which will turn out often to be too strong in dimension two. Nevertheless, this is the correct definition for dimension one, when one requires the indexing*[6] *to be a set.*

If h *was only required to be the unique lifting, our indexing would only be a groupoid in general, as any el-generic morphism* $(A, x) \to (D, w)$ *will still be invertible, but perhaps not unique. This weaker version of el-generic objects arises in the study of stable functors under the name of "candidate" generics [25], and also appears in the study of qualitative domains [16, 10]. It is interesting to note that our two-dimensional versions of familial representability will restrict to this weaker version in dimension one.*

The basic result describing when a presheaf is a coproduct of representables is then the following.

2.5. Proposition. [Diers [7]] *Given a presheaf* $F\colon \mathcal{A} \to \mathbf{Set}$*, the following are equivalent:*

1. $F\colon \mathcal{A} \to \mathbf{Set}$ *is a coproduct of representables;*
2. *each connected component of* el F *has an initial object;*
3. *for any* $(B, y) \in \mathrm{el}\, F$ *there exists a generic object* (A, x) *and morphism* $f\colon (A, x) \to (B, y)$*.*[7]

2.6. Remark. *Of course (3) above is simply expanding (2) into more detail. This more detailed version will be more analogous to the characterizations we give in the two-dimensional case.*

We now consider the case of a functor $T\colon \mathcal{A} \to \mathcal{B}$, asking when $\mathcal{B}(X, T-)$ is a coproduct of representables.

[5] It is trivial to check that any such el-generic morphism is both invertible and unique.

[6] For a presheaf F, the "indexing" $\mathfrak{M}^F$ refers to the category of el-generics and el-generic morphisms between them.

[7] Clearly in this situation the generic object is necessarily unique.

2.7. Definition. *We say a functor* $T\colon \mathcal{A} \to \mathcal{B}$ *is* familial *if for every* $X \in \mathcal{B}$ *the presheaf* $\mathcal{B}(X, T-) : \mathcal{A} \to \mathbf{Set}$ *is a coproduct of representables.*

Specializing Definition 2.3 to this case, we recover the following definition of a "generic morphism" (also known as "diagonally universal morphism" in the work of Diers). This definition is originally due to Diers [7], but we follow the terminology of Weber [28] and [5].

2.8. Definition. *Given a functor* $T\colon \mathcal{A} \to \mathcal{B}$ *we say that a morphism* $x\colon X \to TA$ *for some* $X \in \mathcal{B}$ *and* $A \in \mathcal{A}$ *is* generic *if for any commuting square as on the left below*

$$\begin{array}{ccc} X & \xrightarrow{z} & TB \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle Tg} \\ TA & \xrightarrow[Tf]{} & TC \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{z} & TB \\ {\scriptstyle x}\downarrow & \nearrow{\scriptstyle Th} & \downarrow{\scriptstyle Tg} \\ TA & \xrightarrow[Tf]{} & TC \end{array}$$

there exists a unique $h\colon A \to B$ *such that* $Th \cdot x = z$*. That* $f = g \cdot h$ *follows as a consequence of this property.*

Applying Proposition 2.5 to presheaves of the form $\mathcal{B}(X, T-)$ for a given functor $T\colon \mathcal{A} \to \mathcal{B}$, we obtain the following characterization of familial functors in terms of these generic morphisms.

2.9. Proposition. [Diers [7]] *Given a functor* $T\colon \mathcal{A} \to \mathcal{B}$*, the following are equivalent:*

1. *the functor* T *is familial;*
2. *for every morphism* $f\colon X \to TW$ *there exists a generic morphism* $\delta\colon X \to TA$ *and morphism* $\overline{f}\colon A \to W$ *such that* $f = T\overline{f} \cdot \delta$*.*

Following Weber's terminology, condition (2) is often stated another way.

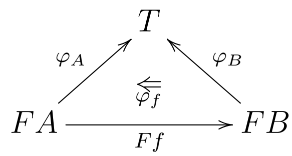
2.10. Definition. *[28] Given a functor* $T\colon \mathcal{A} \to \mathcal{B}$*, if condition (2) above is satisfied, we say that* T admits generic factorizations*.*

2.11. Lax conical colimits and the Grothendieck construction. Here we give the required background on lax conical colimits and the Grothendieck construction. In our convention, we specify the direction of 2-cells in a lax natural transformation $\alpha\colon F \Rightarrow G$ as below

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \alpha_A}\downarrow & \Uparrow{\scriptstyle \alpha_f} & \downarrow{\scriptstyle \alpha_B} \\ GA & \xrightarrow[Gf]{} & GB \end{array}$$

for any morphism $f\colon A \to B$ in the domain bicategory, with 2-cells in the opposite direction defining oplax natural transformations.

2.12. Definition. [lax conical colimits [20]] *Given a category $\mathcal{A}$, a bicategory $\mathscr{K}$, and pseudofunctor $F\colon \mathcal{A} \to \mathscr{K}$, the* lax conical colimit *of F consists of an object $T \in \mathscr{K}$, along with for every $A \in \mathcal{A}$ a map $\varphi_A\colon FA \to T$ and for every morphism $f\colon A \to B$ in $\mathcal{A}$ a 2-cell*



compatible with the binary and nullary constraints of F [20]. This data, which is also called a lax cocone, and may be seen as a lax natural transformation

$$\varphi\colon \Delta\mathbf{1} \Rightarrow \mathscr{K}(F-, T) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Cat},$$

is required to be universal in that

$$\begin{gathered}\mathscr{K}(T, S) \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}](\Delta\mathbf{1}, \mathscr{K}(F-, S)) \\ \alpha \mapsto \mathscr{K}(F-, \alpha) \cdot \varphi\end{gathered}$$

defines an equivalence (where $[\mathcal{A}^{\mathrm{op}}, \mathbf{Cat}]$ is the 2-category of pseudofunctors, lax natural transformations, and modifications). If one reverses the direction of the 2-cell φ_f, and replaces lax transformations with oplax transformations, one then has the notion of an oplax colimit*. The name* conical *refers to the fact that the weight of such a colimit is the terminal presheaf $\Delta\mathbf{1}$, so that the data takes the form of cone-shaped diagrams.*

2.13. Remark. *It is worth noting that the above definition can be used when $F\colon \mathcal{A} \to \mathscr{K}$ is only required to be a lax functor. Also, one may note that lax conical colimits can be seen as an instance of weighted bi-colimits (though we will not use this).*

When $\mathscr{K} = \mathbf{Cat}$, such a lax colimit can easily be calculated using the Grothendieck construction. We will proceed to describe this construction below (though we will be more general by replacing the category $\mathcal{A}$ with a bicategory $\mathscr{A}$).

2.14. Definition. [Grothendieck construction] *Given a bicategory $\mathscr{A}$ and pseudofunctor $F\colon \mathscr{A} \to \mathbf{Cat}$, the* (bi)category of elements *of F, denoted by* el F *or by*

$$\int_{\mathrm{lax}}^{A \in \mathscr{A}} FA$$

is the bicategory with:

Objects *An object is a pair of the form $(A \in \mathscr{A}, x \in FA)$;*

Morphisms *A morphism $(A, x) \nrightarrow (B, y)$ is a pair $f\colon A \to B$ in $\mathscr{A}$ and $\alpha\colon Ff(x) \to y$ in FB; we say such a morphism (f, α) is* opcartesian *if α is invertible;*

2-cells *A 2-cell* $(f,\alpha) \Rightarrow (g,\beta)\colon (A,x) \nrightarrow (B,y)$ *is a 2-cell* $\nu\colon f \Rightarrow g$ *in* $\mathscr{A}$ *rendering commutative*

$$Ff\,(x) \xrightarrow{(F\nu)_x} Fg\,(x) \xrightarrow{\beta} y\,, \qquad \alpha = \beta \circ (F\nu)_x\colon Ff\,(x) \to y\,.$$

It is also common to refer to the bicategory $\int_{\mathrm{lax}}^{A\in\mathscr{A}} FA$ *with its canonical projection to* $\mathscr{A}$ *as the* Grothendieck construction *of* F*.*

The "lax coend" notation as used above is defined as follows. However, we will not burden this paper with all the technical details of the definition.

2.15. Definition. *Let* $\mathscr{A}$ *be a bicategory and let* $F\colon \mathscr{A} \to \mathbf{Cat}$ *and* $G\colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Cat}$ *be pseudofunctors. We define the* lax-coend

$$\int_{\mathrm{lax}}^{A\in\mathscr{A}} FA \times GA$$

as the vertex of the universal diagram, over each morphism $f\colon A \to B$ *in* $\mathscr{A}$*,*

$$\begin{array}{ccccc} & & FA\times GB & & \\ & \swarrow^{\mathrm{id}\times Gf} & & \searrow^{Ff\times\mathrm{id}} & \\ FA\times GA & & \overset{\varepsilon_f}{\Rightarrow} & & FB\times GB \\ & \searrow_{\varepsilon_A} & & \swarrow_{\varepsilon_B} & \\ & & \int_{\mathrm{lax}}^{A\in\mathscr{A}} FA\times GA & & \end{array}$$

subject to the canonical nullary and binary coherence conditions with regards to the pseudofunctoriality of F *and* G *[19].*

2.16. Remark. *When* $\mathscr{A}$ *is a category, the notation* $\int_{\mathrm{lax}}^{A\in\mathscr{A}} FA$ *is justified, as the category of elements can be written as a lax colimit as in Definition 2.12. In the case where* $\mathscr{A}$ *is a bicategory,* el F *is an appropriate tri-colimit of* F*, and the notation is still justified (though in a more technical sense that we will not burden this paper with; see [4]).*

2.17. Remark. *To make clear the duality of covariance and contravariance in the above construction we note the following. For a pseudofunctor* $F\colon \mathscr{A} \to \mathbf{Cat}$*, its lax colimit is given by the lax coend* $\int_{\mathrm{lax}}^{A\in\mathscr{A}} FA$*. In the contravariant case of a pseudofunctor* $G\colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Cat}$*, the lax coend* $\int_{\mathrm{lax}}^{A\in\mathscr{A}} GA$ *coincides with the oplax colimit of* G*, which could also be written as the oplax coend* $\int_{\mathrm{oplax}}^{A\in\mathscr{A}^{\mathrm{op}}} GA$*. Typically given a contravariant* $G\colon \mathscr{A}^{\mathrm{op}} \to \mathbf{Cat}$*, one takes the oplax colimit of* G *as its category of elements (so that we may project from this category of elements into* $\mathscr{A}$*).*

Finally, we should recall the notion of a fibration and cartesian morphisms.

2.18. Definition. *[21] Let $p\colon \mathcal{F} \to \mathcal{E}$ be a functor. We say a morphism $\phi\colon W \to B$ in $\mathcal{F}$ is p-*cartesian *if for any $\psi\colon A \to B$ and $r\colon pA \to pW$ such that the right diagram below commutes*

$$\begin{array}{ccc} W & \xrightarrow{\phi} & B \\ {\scriptstyle \overline{r}}\uparrow & \nearrow_{\psi} & \\ A & & \end{array} \qquad\qquad \begin{array}{ccc} pW & \xrightarrow{p\phi} & pB \\ {\scriptstyle r}\uparrow & \nearrow_{p\psi} & \\ pA & & \end{array}$$

there exists a unique $\overline{r}\colon A \to W$ such that $p\overline{r} = r$ and the left diagram above commutes. If for any morphism $f\colon X \to pB$ in $\mathcal{E}$ there exists a p-cartesian morphism $\phi\colon f^*B \to B$ in $\mathcal{F}$ such that $p(\phi) = f$, we then say p is a fibration.

3. Lax generics in bicategories of elements

Generalizing the fact that a presheaf is a coproduct of representables if and only if each connected component of the category of elements has an initial object, our first main goal is to understand when a pseudofunctor $F\colon \mathscr{A} \to \mathbf{Cat}$ a lax conical colimit of representables, written

$$F \simeq \int_{\mathrm{lax}}^{m\in\mathfrak{M}} \mathscr{A}\left(P_m, -\right)$$

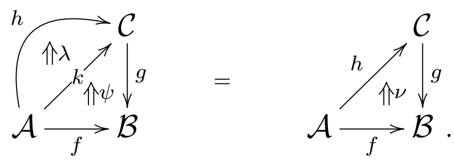
for some $\mathfrak{M} \in \mathbf{Cat}$ and pseudofunctor $P_{(-)}\colon \mathfrak{M} \to \mathscr{A}$, giving a characterization in terms of the category of elements of F (which is analogous to each connected component having an initial object in the one dimensional case). However, before we can describe lax el-generic[8] objects (the appropriate analogue of these initial objects) and morphisms in bicategories of elements, we will have to introduce the language needed to describe them. In particular, we introduce "mixed left liftings" which are similar to left liftings [14], except that the induced arrow's direction is reversed. Note that basic properties for left liftings, such as the pasting lemma, or the lifting through an identity being itself, do not hold in general for mixed left liftings [23].

3.1. Definition. [mixed left lifting property] *Let $\mathscr{C}$ be a bicategory. We say a diagram as on the left below*

$$\begin{array}{ccc} & & \mathcal{C} \\ & {}^{h}\nearrow \;\Uparrow\nu & \downarrow{\scriptstyle g} \\ \mathcal{A} & \xrightarrow[f]{} & \mathcal{B} \end{array} \qquad\qquad \begin{array}{ccc} & & \mathcal{C} \\ & {}^{k}\nearrow \;\Uparrow\psi & \downarrow{\scriptstyle g} \\ \mathcal{A} & \xrightarrow[f]{} & \mathcal{B} \end{array}$$

exhibits (h,ν) as the mixed left lifting *of f through g if for any diagram as on the right above, there exists a unique 2-cell $\lambda\colon k \Rightarrow h$ such that*

[8] We have the "el" here as this notion of genericity is used in bicategories of elements.



Moreover, we say such a lifting (h, ν) *is* strong *if* h *is subterminal*[9] *in* $\mathscr{C}(\mathcal{A}, \mathcal{C})$.

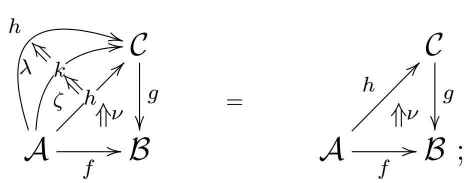
3.2. Remark. *It is not hard to see that strong mixed liftings are unique up to unique isomorphism. Indeed, it is this stronger notion that will be used though this section.*

The following lemma shows that an arrow h which arises as a strong mixed lifting has the property that the strong mixed lifting of h through the identity is itself.

3.3. Lemma. *Suppose the left diagram below*

exhibits (h, ν) *as the strong mixed lifting of* f *through* g. *Then the right diagram above exhibits* (h, id) *as the strong mixed lifting of* h *through* $1_\mathcal{C}$.

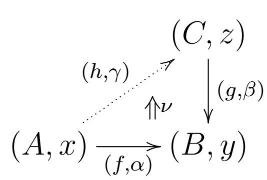
Proof. Given any $k \colon \mathcal{A} \to \mathcal{C}$ and $\zeta \colon h \Rightarrow k$ we have by universality of (h, ν) an induced $\lambda \colon k \Rightarrow h$ such that



that is, since h is subterminal, a unique induced $\lambda \colon k \Rightarrow h$ such that $\lambda\zeta$ is the identity. This proves the result. ∎

We now have the necessary background to introduce the notions of lax el-generic object and el-generic morphism in bicategories of elements.

3.4. Definition. [lax el-generic objects] *Let* $\mathscr{A}$ *be a bicategory and* $F \colon \mathscr{A} \to \mathbf{Cat}$ *be a pseudofunctor. We say that an object* (A, x) *in* el F *is a* lax el-generic object *if for any* (B, y), (C, z), (f, α) *and* (g, β) *as below with* β *invertible*



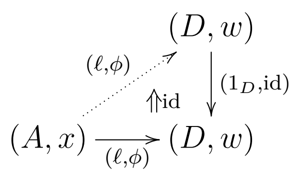
[9]A subterminal object I in a category is one where any morphism into it is unique if it exists. When the category in question has a terminal object, this is equivalent to the unique morphism $I \to \mathbf{1}$ being a monomorphism; hence the name subterminal.

1. *there exists a strong mixed left lifting* $(h,\gamma)\colon (A,x)\to (C,z)$ *exhibited by a 2-cell* $\nu\colon f\Rightarrow gh$*;*

2. *if* α *is invertible above, then both* γ *and* ν *are also invertible.*

3.5. Remark. *If we replace the isomorphism* β *with an identity above, the definition remains equivalent.*

3.6. Definition. [el-generic morphisms] *Let* $\mathscr{A}$ *be a bicategory and* $F\colon \mathscr{A}\to \mathbf{Cat}$ *be a pseudofunctor, and suppose that* (A,x) *is a lax-generic object in* el F*. We say that a morphism* $(\ell,\phi)\colon (A,x)\to (D,w)$ *out of* (A,x) *in* el F *is an* el-generic morphism *if the diagram below*

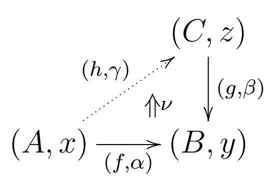


exhibits (ℓ,ϕ) *as the strong mixed left lifting of* (ℓ,ϕ) *through* $(1_D,\mathrm{id})$*.*

3.7. Remark. *It is clear from the second part of Definition 3.4 that for any opcartesian* (ℓ,ϕ) *(meaning* ϕ *is invertible) out of a lax el-generic* (A,x)*, the induced mixed lifting must be (isomorphic to)* (ℓ,ϕ)*. Thus opcartesian morphisms are always el-generic.*

3.8. Remark. *It is an easy consequence of the universal property that every 2-cell out of* (ℓ,ϕ) *is a section (in a unique way); and consequently that any 2-cell between el-generic morphisms is invertible. Moreover, as* (ℓ,ϕ) *is subterminal within its hom-category it follows that any isomorphism between el-generic morphisms is unique. It follows that if* (A,x) *and* (B,y) *are lax el-generic objects, then the category of el-generic morphisms* $(A,x)\to(B,y)$ *is equivalent to a discrete category (a set).*

3.9. Remark. *It is worth noting that for any lax el-generic object* (A,x) *and strong mixed lifting as below*



with β *invertible, the induced morphism* (h,γ) *is an el-generic morphism as a consequence of Lemma 3.3.*

The following is a step towards characterizing when an $F\colon \mathscr{A}\to \mathbf{Cat}$ is a lax conical colimit of representables, indexed by the following category of el-generic objects and morphisms. The reader will note the importance of the indexing being a 1-category (and thus the consideration of representatives of equivalence classes of morphisms in order to obtain a 1-category), just as it is important in the one-dimensional case that the indexing is a set.

3.10. Definition. *Let $\mathscr{A}$ be a bicategory and let $F\colon \mathscr{A} \to \mathbf{Cat}$ be a pseudofunctor. Suppose that el-generic morphisms between lax el-generic objects compose to el-generic morphisms. Define $\mathscr{A}_g^F$ as the sub-bicategory of* el F *consisting of lax el-generic objects and el-generic morphisms, and define $\mathfrak{M}^F$ as the 1-category consisting of lax el-generic objects in* el F *and chosen representatives of isomorphism classes of el-generic morphisms.*

Of course, in the above definition we will have an equivalence $\mathscr{A}_g^F \simeq \mathfrak{M}^F$.

3.11. Proposition. *Let $\mathscr{A}$ be a bicategory and let $F\colon \mathscr{A} \to \mathbf{Cat}$ be a pseudofunctor. Suppose that el-generic morphisms between lax el-generic objects compose to el-generic morphisms. Let $P_{(-)}\colon \mathfrak{M} \to \mathscr{A}$ be the assignment sending a lax el-generic object (A, x) to A and a representative el-generic morphism between el-generic objects $(s, \phi)\colon (A, x) \to (B, y)$ to $s\colon A \to B$. Then $P_{(-)}\colon \mathfrak{M} \to \mathscr{A}$ defines a pseudofunctor, and for every $T \in \mathscr{A}$ there exists fully faithful functors*

$$\Lambda_T\colon \int_{\mathrm{lax}}^{m\in\mathfrak{M}^F} \mathscr{A}\left(P_m, T\right) \to FT$$

pseudo-natural in $T \in \mathscr{A}$.

Proof. Firstly note that $P_{(-)}\colon \mathfrak{M} \to \mathscr{A}$ defines a pseudofunctor since it may be written as the composite $\mathfrak{M}^F \to \mathscr{A}_g^F \to \mathrm{el}\ F \to \mathscr{A}$. We may then define Λ_T on objects by the assignment $(A, x, f) \mapsto Ff(x)$, and on morphisms by the assignment (suppressing the pseudofunctoriality constraints of F)

$$\begin{array}{ccccc} (A,x,f\colon A\to T) & A & Fh(x) & A \xrightarrow{f} T & Ff(x) \\ \Big\downarrow{\scriptstyle (h,\gamma,\nu)} & \Big\downarrow{\scriptstyle h} & \Big\downarrow{\scriptstyle \gamma} & {\scriptstyle h}\downarrow \ \Downarrow\nu \ \nearrow{\scriptstyle g} & \Big\downarrow{\scriptstyle (F\nu)_x} \\ & & & & FgFh(x) \\ & & & \mapsto & \Big\downarrow{\scriptstyle Fg(\gamma)} \\ (B,y,g\colon B\to T) & B & y & B & Fg(y)\ . \end{array} \tag{3.1}$$

Observe that we have the following conditions satisfied.

Functoriality. Given another

$$\begin{array}{ccccc} (B,y,g\colon B\to T) & B & Fk(y) & B \xrightarrow{g} T & Fg(y) \\ \Big\downarrow{\scriptstyle (k,\zeta,\mu)} & \Big\downarrow{\scriptstyle k} & \Big\downarrow{\scriptstyle \zeta} & {\scriptstyle k}\downarrow \ \Downarrow\mu \ \nearrow{\scriptstyle q} & \Big\downarrow{\scriptstyle (F\mu)_y} \\ & & & & FqFk(y) \\ & & & \mapsto & \Big\downarrow{\scriptstyle Fq(\zeta)} \\ (C,z,q\colon A\to T) & C & z & C & Fq(z) \end{array}$$

the commutativity of

$$\begin{array}{ccccccccc} Ff(x) & \xrightarrow{(F\nu)_x} & FgFh(x) & \xrightarrow{Fg(\gamma)} & Fg(y) & \xrightarrow{(F\mu)_y} & FqFk(y) & \xrightarrow{Fq(\zeta)} & Fq(z) \\ & & & \searrow^{(F\mu)_{Fh(x)}} & & \nearrow_{FqFk(\gamma)} & & & \\ & & & & FqFkFh(x) & & & & \end{array}$$

by naturality of $F\mu$ shows Λ_T respects binary composition. It is trivial that identities are preserved.

Fullness. Given any objects $(A, x, f\colon A \to T)$ and $(B, y, g\colon B \to T)$ and morphism $\phi\colon Ff(x) \to Fg(y)$, we may construct the universal diagram

$$\begin{array}{ccc} & & (B, y) \\ & \overset{(h,\gamma)}{\nearrow} \;\; \Uparrow\nu & \Big\downarrow{\scriptstyle (g,\mathrm{id})} \\ (A, x) & \xrightarrow[(f,\phi)]{} & (B, Fg(y)) \end{array}$$

using lax el-genericity of (A, x). Now (h, γ) is el-generic by Lemma 3.3, and without loss of generality we can assume it is a representative el-generic (since it is necessarily isomorphic to one). Then $\Lambda_T(h, \gamma, \nu) = \phi$.

Faithfulness. Given another triple (k, ψ, ω) such that $\Lambda_T(k, \psi, \omega) = \phi$, we have the diagram

$$\begin{array}{ccc} & & (B, y) \\ & \overset{(k,\psi)}{\nearrow} \;\; \Uparrow\omega & \Big\downarrow{\scriptstyle (g,\mathrm{id})} \\ (A, x) & \xrightarrow[(f,\phi)]{} & (B, Fg(y)) \end{array}$$

But as (k, ψ) and (h, γ) are both el-generics, the induced $(k, \psi) \Rightarrow (h, \gamma)$ arising from universality of (h, γ) must be invertible. Also, as they are both representative, they must be equal. As the identity must then be the induced morphism we conclude $k = h$, $\psi = \gamma$ and $\omega = \nu$.

Pseudo-naturality. Given any 1-cell $\alpha\colon T \to S$ in $\mathscr{A}$ the squares

$$\begin{array}{ccc} (A, x, f\colon A \to T) & \xrightarrow{\alpha\cdot(-)} & (A, x, \alpha f\colon A \to S) \\ {\scriptstyle \Lambda_T}\Big\downarrow & & \Big\downarrow{\scriptstyle \Lambda_S} \\ Ff(x) & \xrightarrow[F\alpha\cdot(-)]{} & F(\alpha f)(x) \end{array}$$

commute up to pseudo-functoriality constraints of F, and the family of squares of the form above satisfy the required naturality, nullary and binary coherence conditions as a consequence of the corresponding pseudo-functoriality coherence conditions. ∎

We can now characterize precisely when a pseudofunctor $F\colon \mathscr{A} \to \mathbf{Cat}$ is a lax conical colimit of representables.

3.12. THEOREM. *Let $\mathscr{A}$ be a bicategory and $F\colon \mathscr{A} \to \mathbf{Cat}$ be a pseudofunctor. Then the following are equivalent:*

1. *the pseudofunctor $F\colon \mathscr{A} \to \mathbf{Cat}$ is a lax conical colimit of representables;*
2. *the following conditions hold:*
   - *(a) for every object (B, y) in $\mathrm{el}\ F$ there exists a lax el-generic object (A, x) and morphism $(f, \alpha) : (A, x) \nrightarrow (B, y)$ with α invertible;*
   - *(b) el-generic morphisms between lax el-generic objects compose to el-generic morphisms.*

PROOF. The direction (2) ⇒ (1) is clear from Proposition 3.11 as condition (a) means that for any $B \in \mathscr{A}$ and $y \in FB$ we have a lax el-generic (A, x) and morphism $(f, \alpha) : (A, x) \nrightarrow (B, y)$ in $\mathrm{el}\ F$ with α invertible, so that

$$\Lambda_B\,(A, x, f\colon A \to B) = Ff\,(x) \xrightarrow{\alpha} y$$

which witnesses the essential surjectivity of Λ_B at $y \in FB$.

For (1) ⇒ (2), suppose we are given a category $\mathfrak{M}$ and pseudofunctor $P_{(-)}\colon \mathfrak{M} \to \mathscr{A}$ (assuming without loss of generality that $P_{(-)}$ strictly preserves identities) such that $F \simeq \int_{\mathrm{lax}}^{m\in\mathfrak{M}} \mathscr{A}\,(P_m, -)$, and consequently

$$\mathrm{el}\ F \simeq \int_{\mathrm{lax}}^{m\in\mathfrak{M}^F} \mathrm{el}\ \mathscr{A}\,(P_m, -)\,.$$

This shows that $\mathrm{el}\ F$ is equivalent to the bicategory with:

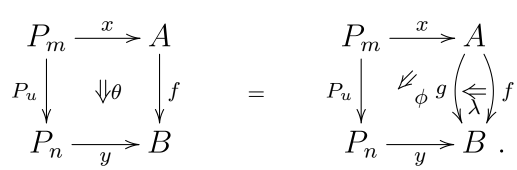
Objects An object is a triple of the form $(m \in \mathfrak{M}, A \in \mathscr{A}, x\colon P_m \to A)$;

Morphisms The morphisms $(m, A, x) \nrightarrow (n, B, y)$ are triples comprising a morphism $u\colon m \to n$ in $\mathfrak{M}^F$, a morphism $f\colon A \to B$ in $\mathscr{A}$ and a 2-cell

$$\begin{array}{ccc} P_m & \xrightarrow{x} & A \\ {\scriptstyle P_u}\downarrow & \Downarrow\theta & \downarrow{\scriptstyle f} \\ P_n & \xrightarrow[y]{} & B \end{array}$$

in $\mathscr{A}$;

2-cells A 2-cell $\lambda\colon (u, f, \theta) \Rightarrow (u, g, \phi) : (m, A, x) \nrightarrow (n, B, y)$ is a 2-cell $\lambda\colon f \Rightarrow g$ in $\mathscr{A}$ such that



Existence of lax el-generics. We first show that each

$$\Big(m \in \mathfrak{M}^F, P_m \in \mathscr{A}, \mathrm{id}\colon P_m \to P_m\Big)$$

in el F is lax el-generic. Consider a diagram

$$\begin{array}{ccc} & & (n,C,z) \\ & \overset{(u,h,\gamma)}{\nearrow} \quad \Uparrow\nu & \downarrow{\scriptstyle (\mathrm{id},g,\mathrm{id})} \\ (m,P_m,\mathrm{id}) & \xrightarrow[(u,f,\alpha)]{} & (n,B,y) \end{array}$$

where (u,f,α) and $(\mathrm{id},g,\mathrm{id})$ are respectively

$$\begin{array}{ccc} P_m & \xrightarrow{\mathrm{id}} & P_m \\ {\scriptstyle P_u}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle f} \\ P_n & \xrightarrow[y]{} & B \end{array} \qquad\qquad \begin{array}{ccc} P_n & \xrightarrow{z} & C \\ {\scriptstyle P_{\mathrm{id}}}\downarrow & \Downarrow\mathrm{id} & \downarrow{\scriptstyle g} \\ P_n & \xrightarrow[y]{} & B \end{array}$$

then we recover a canonical (u,h,γ) as

$$\begin{array}{ccc} P_m & \xrightarrow{\mathrm{id}} & P_m \\ {\scriptstyle P_u}\downarrow & \Downarrow\mathrm{id} & \downarrow{\scriptstyle z\cdot P_u} \\ P_n & \xrightarrow[z]{} & C \end{array} \tag{3.2}$$

with the 2-cell $\nu\colon f \Rightarrow gh = gzP_u = yP_u$ given as α. Now, for universality, suppose we have a (u,k,ϕ) given as

$$\begin{array}{ccc} P_m & \xrightarrow{\mathrm{id}} & P_m \\ {\scriptstyle P_u}\downarrow & \Downarrow\phi & \downarrow{\scriptstyle k} \\ P_n & \xrightarrow[z]{} & C \end{array}$$

with a 2-cell $\psi\colon f \Rightarrow gk$ such that

$$\begin{array}{ccc} P_m & \xrightarrow{\mathrm{id}} & P_m \\ {\scriptstyle P_u}\Big\downarrow & \Downarrow\alpha & \Big\downarrow{\scriptstyle f} \\ P_n & \xrightarrow[y]{} & B \end{array} \quad = \quad \begin{array}{cccc} P_m & \xrightarrow{\mathrm{id}} & P_m & \\ {\scriptstyle P_u}\downarrow & \Downarrow\phi & \downarrow{\scriptstyle k} & \\ P_n & \xrightarrow{z} & C \overset{\psi}{\Leftarrow} & {\scriptstyle f} \\ {\scriptstyle P_{\mathrm{id}}}\downarrow & \Downarrow\mathrm{id} & \downarrow{\scriptstyle g} & \\ P_n & \xrightarrow[y]{} & B & \end{array} \tag{3.3}$$

Then we can take our induced map $\lambda\colon k \Rightarrow h$ as $\phi\colon k \Rightarrow z \cdot P_u$. It is trivial that

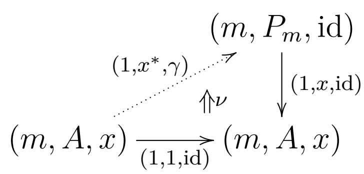
$$\begin{array}{ccc} P_m & \xrightarrow{\mathrm{id}} & P_m \\ {\scriptstyle P_u}\downarrow & \Downarrow\phi & \downarrow{\scriptstyle k} \\ P_n & \xrightarrow[z]{} & C \end{array} \quad = \quad \begin{array}{cccc} P_m & \xrightarrow{\mathrm{id}} & P_m & \\ {\scriptstyle P_u}\downarrow & \Downarrow\mathrm{id} & \downarrow{\scriptstyle z\cdot P_u} \overset{\lambda}{\Leftarrow} & {\scriptstyle k} \\ P_n & \xrightarrow[z]{} & C & \end{array} \tag{3.4}$$

so that λ is a 2-cell $(u,k,\phi) \Rightarrow (u,h,\gamma)$. Also, from (3.4) it is clear that $\lambda = \phi$ is the *only* 2-cell $(u,k,\phi) \Rightarrow (u,h,\gamma)$, meaning (u,h,γ) is subterminal within its hom-category. Moreover, (3.3) shows ψ pasted with $\lambda = \phi$ is $\alpha = \nu$.

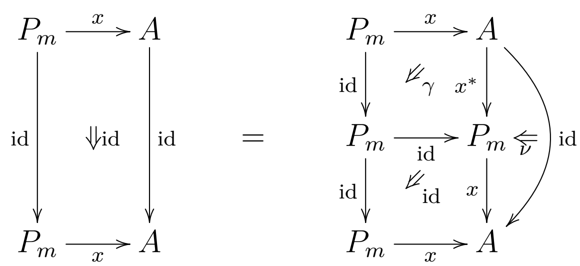
Classification of lax el-generics. We now show that an object

$$\left(m \in \mathfrak{M}^F, A \in \mathscr{A}, x\colon P_m \to A\right)$$

in el F is lax el-generic if and only if x is an equivalence. It is clear the above argument generalizes if one replaces (m, P_m, id) with (m, A, x) where x is an equivalence. Conversely, if (m, A, x) is a lax el-generic object then we may construct the universal diagram



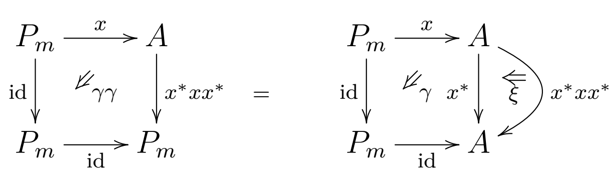
noting that ν and γ are both invertible. In fact, this gives an adjoint equivalence. That ν is a 2-cell says



which gives one triangle identity. For the other identity, note that 2-cells

$$\xi\colon\ (1, x^*xx^*, \gamma\gamma) \Rightarrow (1, x^*, \gamma)\,,$$

meaning 2-cells ξ such that



(3.5)

are unique, as $(1, x^*, \gamma)$ is subterminal within its hom-category. But we may take ξ to be

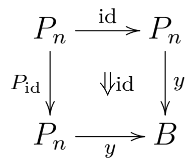
$$\gamma x^*\colon\ (1, x^*xx^*, \gamma\gamma) \Rightarrow (1, x^*, \gamma)$$

or

$$x^*\nu^{-1}\colon\ (1, x^*xx^*, \gamma\gamma) \Rightarrow (1, x^*, \gamma)$$

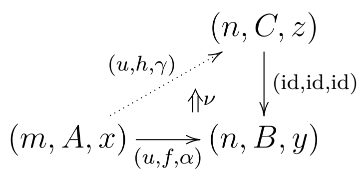
which both satisfy (3.5). Thus $\gamma x^* = x^*\nu^{-1}$ and so $\gamma x^* \cdot x^*\nu = \mathrm{id}$ giving the other triangle identity.

Existence of lax el-generic factorizations. Suppose we are given a $(n, B, y\colon P_n \to B)$ in el F. We have the map $(n, P_n, \mathrm{id}\colon P_n \to P_n) \nrightarrow (n, B, y\colon P_n \to B)$ given as



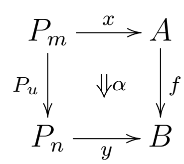
which is of the required form since the 2-cell involved is invertible.

El-generic morphisms form a category. Before showing that el-generic morphisms form a category, we will need a characterization of them. Now, specializing the earlier argument of "existence of expected lax el-generics" to the case when g is the identity (though generalizing the identity on P_m to an equivalence $x\colon P_m \to A$) we see that if (m, A, x) is el-generic (i.e. x is an equivalence)



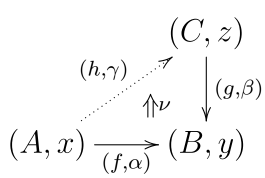
the lifting (u, h, γ) above, constructed as in (3.2), has γ invertible. It is also clear that if (u, f, α) is such that α is invertible, then the lifting (u, h, γ) through $(\mathrm{id}, \mathrm{id}, \mathrm{id})$ constructed as in (3.2) is given by (u, f, α).

This shows that the el-generic morphisms between lax el-generic objects are diagrams of the form



with α invertible, and it is clear that these are closed under composition and that identities are such diagrams. ∎

3.13. Remark. *When $F\colon \mathscr{A} \to \mathbf{Cat}$ is a lax conical colimit of representables, and from a lax el-generic object (A, x) we construct the universal diagram*



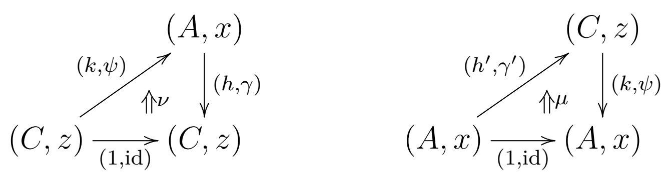
the 2-cell ν is the unique *2-cell $(f, \alpha) \Rightarrow (g, \beta) \cdot (h, \gamma)$. This is since for such an F, el-generic morphisms compose and any map (g, β) with β invertible is generic. Subterminality of $(g, \beta) \cdot (h, \gamma)$ then gives uniqueness.*

3.14. Remark. *When* $F\colon \mathscr{A} \to \mathbf{Cat}$ *is a lax conical colimit of representables, written* $F \simeq \int_{\mathrm{lax}}^{m\in\mathfrak{M}^F} \mathscr{A}(P_m,-)$, *then* $\mathfrak{M}^F$ *is equivalent to the category of strict*[10] *lax el-generic objects* (A,x) *and representative el-generic morphisms in* el F. *This is a consequence of the characterization of lax el-generic objects and morphisms given in the above proof of Theorem 3.12. Moreover, as Theorem 3.12 constructs* $\mathfrak{M}^F$ *as the the category of lax el-generic objects and morphisms, we conclude this non-strict choice of* $\mathfrak{M}^F$ *is also equivalent (to the category indexing* F *as a lax conical colimit of representables).*

The following lemma will not be used until the last section, but is expressed in terms of el-generic objects and so we give it here.

3.15. Lemma. *Let* $\mathscr{A}$ *be a bicategory and let* $F\colon \mathscr{A} \to \mathbf{Cat}$ *be a pseudofunctor. Then every opcartesian morphism between two lax el-generic objects* $(h,\gamma)\colon (A,x) \nrightarrow (C,z)$ *in* el F *is an equivalence.*

Proof. Given such a (h,γ) we may form a (k,ψ) as on the left below



and one can then form a (h',γ') as on the right above. As ν and μ have inverses

$$(h',\gamma') \cong (h,\gamma)\,(k,\psi)\,(h',\gamma') \cong (h,\gamma)$$

so (h,γ) has pseudo-inverse (k,ψ). ∎

4. Lax generic factorizations and lax familial pseudofunctors

Here we specialize the results of Section 3 to the case when $F\colon \mathscr{A} \to \mathbf{Cat}$ is of the form $\mathscr{B}(X,T-)$ for a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$. The following is a generalization of Diers' notion of familial functor (given in Definition 2.7) to the case of a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$.

4.1. Definition. *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be bicategories and let* $T\colon \mathscr{A} \to \mathscr{B}$ *be a pseudofunctor. We say that* T *is* lax familial *if there exists a pseudofunctor* $\mathfrak{M}_{(-)}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ *and a pseudofunctor* $\mathbf{P}\colon \int_{\mathrm{lax}}^{X\in\mathscr{B}} \mathfrak{M}_X \to \mathscr{A}$ *such that*

$$\mathscr{B}(X,T-) \simeq \int_{\mathrm{lax}}^{m\in\mathfrak{M}_X} \mathscr{A}\left(P_m^X,-\right)$$

for all $X \in \mathscr{B}$, *where each* $P_{(-)}^X\colon \mathfrak{M}_X \to \mathscr{A}$ *is obtained from* $\mathbf{P}$ *by composing with the inclusion* $\mathfrak{M}_X \to \int_{\mathrm{lax}}^{X\in\mathscr{B}} \mathfrak{M}_X$.

[10] Strict here means if both α and β are identities, then both ν and γ are identities.

4.2. Remark. *One might wonder why we did not simply define T to be lax familial when every*

$$\mathscr{B}(X, T-) : \mathscr{A} \to \mathbf{Cat}$$

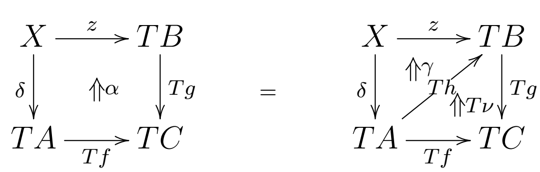
is a lax conical colimit of representables. The reason is that this condition would only be sufficient to force $\mathbf{P}$ *(which may be constructed from this condition) to be a normal lax functor.*

This should not be surprising. In dimension one, simply asking that each $\mathscr{B}(X, T-)$ *is a coproduct of representables is enough to define* T *being familial since the indexings* $\mathfrak{M}_X$ *are sets, and thus there are no naturality conditions to consider. However, when the indexing is a category we must account for these naturality conditions (equivalent to ensuring that* $\mathbf{P}$ *is a pseudofunctor), and so the definition of a lax familial pseudofunctor is slightly more complicated.*

4.3. Remark. *More abstractly, one can define familial functors as the admissible maps against the cocompletion under coproducts [27], sending a category* $\mathcal{A}$ *to* $\int_{\text{lax}}^{I \in \mathbf{Set}} \mathbf{CAT}(I, \mathcal{A})$*, and define lax familial functors as the admissible maps against the cocompletion under lax conical colimits, sending a bicategory* $\mathscr{A}$ *to* $\int_{\text{lax}}^{\mathcal{I} \in \mathbf{Cat}} \mathbf{BICAT}(\mathcal{I}, \mathscr{A})$*. However, there are a number of technicalities here, as one should consider opposite categories (as we are really using corepresentables), and the theory of "higher" versions of KZ pseudomonads [12] is not fully developed.*

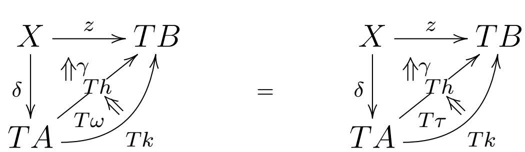
Before applying Theorem 3.12 to **Cat**-valued presheaves of the form $\mathscr{B}(X, T-)$, we will need the appropriate notions of genericity with respect to a pseudofunctor $T \colon \mathscr{A} \to \mathscr{B}$. The following definitions are recovered by specializing the definitions of genericity in the last section, namely Definitions 3.4 and 3.6, to the case when $F \colon \mathscr{A} \to \mathbf{Cat}$ is of the form $\mathscr{B}(X, T-)$ for a pseudofunctor $T \colon \mathscr{A} \to \mathscr{B}$.

4.4. Definition. *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be bicategories and let* $T \colon \mathscr{A} \to \mathscr{B}$ *be a pseudofunctor. Then a 1-cell* $\delta \colon X \to TA$ *is* lax-generic *if for any diagram and 2-cell* α *as on the left below*



there exists a diagram and 2-cells ν *and* γ *as on the right above*[11] *which is equal to* α*, such that:*

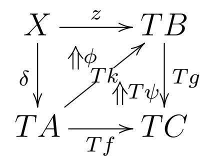
1. *given any 2-cells* $\omega, \tau \colon k \Rightarrow h$ *such that*



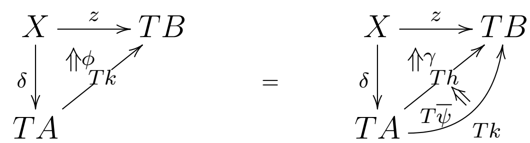
[11] We are suppressing the pseudofunctorality constraint $Tg \cdot Th \cong Tgh$.

we have $\omega = \tau$*;*

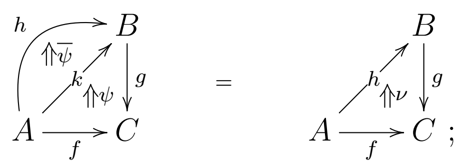
2. *given any other diagram*



equal to α*, there exists a (necessarily unique) 2-cell* $\overline{\psi}\colon k \Rightarrow h$ *such that*

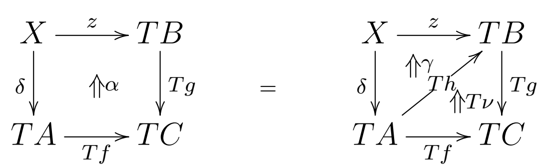


and



3. *if* α *is invertible, then both* γ *and* ν *are invertible.*

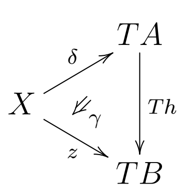
We call a factorization



the universal factorization *of* α *if both (1) and (2) are satisfied above.*

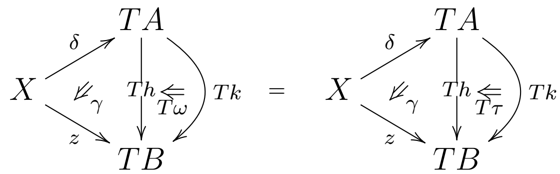
Earlier in Definition 3.6 we defined a 1-cell to be generic when it satisfied a certain strong mixed lifting property. Translating this definition into the context of a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$ results in the below definition.

4.5. Definition. *Let* $\mathscr{A}$ *and* $\mathscr{B}$ *be bicategories and let* $T\colon \mathscr{A} \to \mathscr{B}$ *be a pseudofunctor. Let* $\delta\colon X \to TA$ *be a generic 1-cell. Then a pair* (h, γ) *of the form*



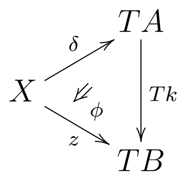
is a generic cell *if:*

1. *given any 2-cells* $\omega, \tau \colon k \Rightarrow h$ *such that*

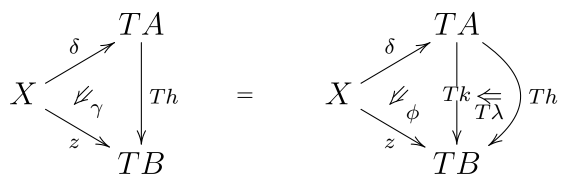


we have $\omega = \tau$;

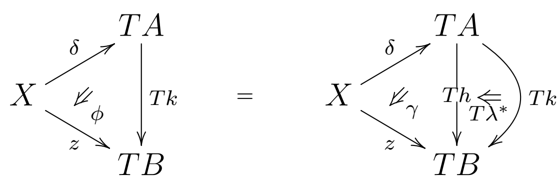
2. *given any other diagram*



and $\lambda \colon h \Rightarrow k$ *such that*



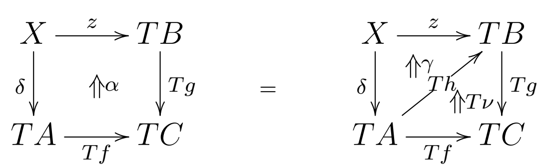
there exists a (necessarily unique) $\lambda^* \colon k \Rightarrow h$ *such that*



and $\lambda^* \lambda = \mathrm{id}_h$.

From this definition, the following is clear.

4.6. Corollary. *For any universal factorization*



it follows that (h, γ) *is a generic 2-cell.*

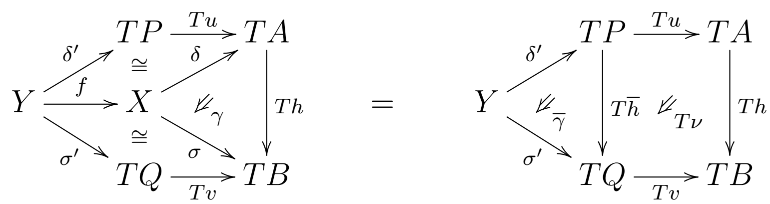
Before proving the main theorem of this section, it is worth defining the spectrum of a pseudofunctor. This is to be the two-dimensional analogue of Diers' definition of spectrum of a functor [8, Definition 3].

It turns out that for a lax familial functor, the reindexing **P** necessarily has domain given by the Grothendieck construction of the spectrum, hence why the spectrum appears in this section and in the proof of Theorem 4.10.

4.7. Definition. *Let $\mathscr{A}$ and $\mathscr{B}$ be bicategories and let $T\colon \mathscr{A} \to \mathscr{B}$ be a pseudofunctor such that $\mathscr{B}(X, T-)$ is a lax conical colimit of representables for every $X \in \mathscr{B}$.*[12] *For each $X \in \mathscr{B}$, define $\mathfrak{M}_X$ as the category with objects given by lax-generic morphisms out of X and morphisms given by representative generic cells between them. We define the* spectrum *of T to be the pseudofunctor*

$$\mathbf{Spec}_T\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$$

sending an object $X \in \mathscr{B}$ to $\mathfrak{M}_X$ and a morphism $f\colon Y \to X$ in $\mathscr{B}$ to the functor $\mathfrak{M}_f\colon \mathfrak{M}_X \to \mathfrak{M}_Y$ which takes a generic morphism $\delta\colon X \to TA$ to $\delta'\colon Y \to TP$ where $\delta \cdot f \cong Tu \cdot \delta'$ is a chosen generic factorization of $\delta \cdot f$, and takes a generic 2-cell $\gamma\colon Th \cdot \delta \Rightarrow \sigma$ as on the left below to the generic 2-cell $\overline{\gamma}\colon T\overline{h} \cdot \delta' \Rightarrow \sigma'$ as on the right below



constructed as the universal factorization of the left pasting above.

4.8. Remark. *When $\mathscr{A}$ has a terminal object the spectrum has an especially simple form, namely as the functor $\mathscr{B}(-, T1) : \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$.*

Later on we will need to use the the Grothendieck construction of the spectrum, which has the following relatively simple description.

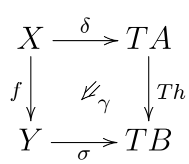
4.9. Lemma. *Let $\mathscr{A}$ and $\mathscr{B}$ be bicategories and let $T\colon \mathscr{A} \to \mathscr{B}$ be a pseudofunctor such that $\mathscr{B}(X, T-)$ is a lax conical colimit of representables for every $X \in \mathscr{B}$. Then the bicategory of elements of the spectrum $\mathbf{Spec}_T\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ is the bicategory*

$$\mathrm{el}\ \mathfrak{M}_{(-)} \cong \int_{\mathrm{lax}}^{X \in \mathscr{B}} \mathfrak{M}_X$$

consisting of:

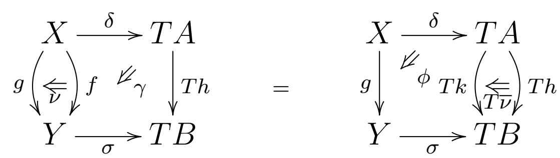
Objects *An object is a pair of the form $(X \in \mathscr{B}, \delta\colon X \to TA)$ where δ is a generic out of X;*

Morphisms *A morphism $(X \in \mathscr{B}, \delta\colon X \to TA) \nrightarrow (Y \in \mathscr{B}, \sigma\colon Y \to TB)$ is a morphism $f\colon X \to Y$ in $\mathscr{B}$ and a representative generic cell (h, γ) as below*



[12]It makes sense to define the spectrum with just this assumption. However, in most cases of interest T will satisfy the stronger condition of being lax familial.

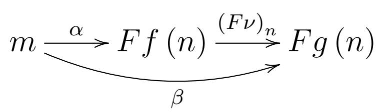
2-cells *A 2-cell* $(f,h,\gamma)\Rightarrow(g,k,\phi)\colon\ (X,\delta)\nrightarrow(Y,\sigma)$ *is a 2-cell* $\nu\colon f\Rightarrow g$ *in* $\mathscr{B}$ *such that*



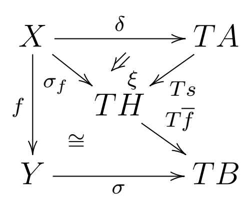
for some (necessarily unique) $\overline{\nu}\colon h\Rightarrow k$.

Moreover, the cartesian morphisms are precisely those (f,h,γ) *such that* γ *is invertible.*

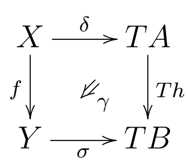
Proof. We know, using the formula of Definition 2.14 (adjusted to the contravariant case), that $\int_{\mathrm{lax}}^{X\in\mathscr{B}}\mathfrak{M}_{(-)}$ is the bicategory with objects pairs $(X\in\mathscr{B},m\in\mathfrak{M}_X)$, morphisms $(X\in\mathscr{B},m\in\mathfrak{M}_X)\nrightarrow(Y\in\mathscr{B},n\in\mathfrak{M}_Y)$ given by a 1-cell $f\colon X\to Y$ and morphism $\alpha\colon m\to Ff\,(n)$ in $\mathfrak{M}_X$, and 2-cells $\nu\colon\ (f,\alpha)\Rightarrow(g,\beta)$ those 2-cells $\nu\colon f\Rightarrow g$ such that



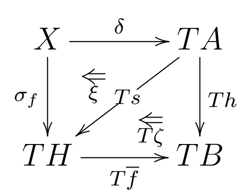
commutes. The objects are clearly as desired. Thus a morphism $(X\in\mathscr{B},\delta\colon X\to TA)\nrightarrow(Y\in\mathscr{B},\sigma\colon Y\to TB)$ consists of an $f\colon X\to Y$ and an $\alpha\colon\delta\to\mathfrak{M}_f\,(\sigma)$ in $\mathfrak{M}_X$. Hence a morphism is a pair $(f,(s,\xi))$ as below



where (s,ξ) is a representative generic cell, and $T\overline{f}\cdot\sigma_f$ is the chosen generic factorization of $\sigma\cdot f$. Using that generic cells (s,ξ) remain generic when composed with opcartesian cells $\left(\overline{f},\cong\right)$ (because opcartesian cells are themselves generic), the above diagram is itself a generic cell, isomorphic to a unique representative generic cell



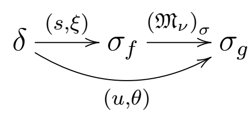
Conversely, one may form the representative generic factorization of γ



to recover (s, ξ) (note that ζ is invertible as genericity of (s, ξ) is preserved by $\left(\overline{f}, \mathrm{id}\right)$ and γ is generic). That the assignment $(s, \xi) \mapsto (h, \gamma)$ defines a bijection is a consequence of the fact that any 2-cells factors through a unique representative generic 2-cell (once a choice of a generic 1-cell factoring each general 1-cell has been made).

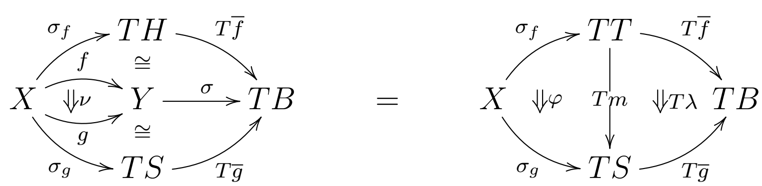
It is also worth noting that the opcartesian morphisms, corresponding to the case where (s, ξ) is an equivalence (meaning s is an equivalence and ξ is invertible), are those squares where γ is invertible.

Finally, a 2-cell $\nu\colon (f, s, \xi) \Rightarrow (g, u, \theta)$ consists of a 2-cell $\nu\colon f \Rightarrow g$ such that

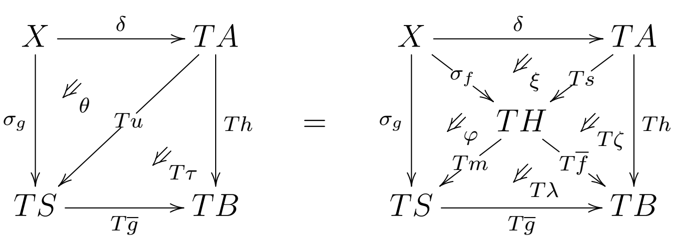


(4.1)

commutes, where $(\mathfrak{M}_\nu)_\sigma$ is given by the representative generic factorization, denoted by the pair (m, φ), as in the diagram below



Hence given such a ν we have



for some (necessarily unique) $\tau\colon h \Rightarrow \overline{g} \cdot u$. Moreover, given a diagram as above we can take the representative generic factorization to recover (4.1). ∎

We can now apply Theorem 3.12 to the case where $F\colon \mathscr{A} \to \mathbf{Cat}$ is of the form $\mathscr{B}(X, T-)$ for a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$ to prove the following theorem.

4.10. Theorem. *Let $\mathscr{A}$ and $\mathscr{B}$ be bicategories and let $T\colon \mathscr{A} \to \mathscr{B}$ be a pseudofunctor. Then the following are equivalent:*

1. *the pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$ is lax familial;*
2. *the following conditions hold:*
   - (a) *for every object $X \in \mathscr{A}$ and 1-cell $y\colon X \to TC$ in $\mathscr{B}$, there exists a lax-generic morphism $\delta\colon X \to TA$ and 1-cell $f\colon A \to C$ such that $Tf \cdot \delta \cong y$;*

(b) for any triple of lax-generic morphisms δ, σ and ω, and pair of generic cells (h,θ) and (k,ϕ) as below

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle\delta}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle\sigma} & \overset{\phi}{\nearrow} & \downarrow{\scriptstyle\omega} \\ TA & \xrightarrow[Th]{} & TB & \xrightarrow[Tk]{} & TC \end{array} \tag{4.2}$$

the above pasting $(kh, \phi f \cdot \theta)$ is a generic cell.[13]

PROOF. (1) $\Rightarrow$ (2) : Supposing that T is lax familial, it follows that each $\mathscr{B}(X, T-)$ is a lax conical colimit of representables. By Theorem 3.12, we have (2)(a), as well as 2(b) when f and g are both the identity at X. To get the full version of (2)(b) we use that

$$\mathbf{P}\colon \int_{\text{lax}}^{X\in\mathscr{B}} \mathfrak{M}_X \to \mathscr{A}$$

is a pseudofunctor, where we have assumed without loss of generality that each $\mathfrak{M}_X$ is the category of generic morphisms out of X and representative cells, using Remark 3.14. Indeed, $\int_{\text{lax}}^{X\in\mathscr{B}} \mathfrak{M}_X$ is the bicategory with objects pairs $(X, \delta\colon X \to TA)$ and morphisms $(X, \delta\colon X \to TA) \nrightarrow (Y, \sigma\colon Y \to TB)$ given by triples (f, h, θ) as below

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\delta}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle\sigma} \\ TA & \xrightarrow[Th]{} & TB \end{array}$$

such that (h, θ) is a generic cell. As the lax functoriality constraints of $\mathbf{P}$ are given by factoring diagrams such as (4.2) though a generic, the invertibility of these lax constraints of $\mathbf{P}$ forces (2)(b).

(2) $\Rightarrow$ (1) : Applying Theorem 3.12 to the conditions 2(a) and 2(b) (only needing the case when f and g are identities at X), it follows that we may write

$$\mathscr{B}(X, T-) \simeq \int_{\text{lax}}^{m\in\mathfrak{M}_X} \mathscr{A}\left(P_m^X, -\right)$$

where $\mathfrak{M}_X$ is the category of generic morphisms out of X and representative generic cells between them. From this, we recover the spectrum $\mathbf{Spec}_T\colon \mathscr{B}^{\text{op}} \to \mathbf{Cat}$ taking each X to $\mathfrak{M}_X$. Also, we again have the canonical normal lax functor

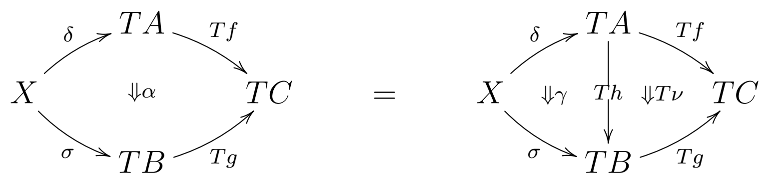
$$\mathbf{P}\colon \int_{\text{lax}}^{X\in\mathscr{B}} \mathfrak{M}_X \to \mathscr{A}$$

defined as in the previous implication (1) $\Rightarrow$ (2). Using the full version of (2)(b), meaning we are no longer just using the case when f and g are identities at X, forces this to be a pseudofunctor (not just a normal lax functor) as required. ∎

[13] Suppressing pseudofunctoriality constraints of T.

Under the conditions of this theorem, we also have a notion of generic factorizations on 2-cells, in a sense we now describe.

4.11. Remark. *Suppose T is lax familial, δ and σ are generic objects, and consider a 2-cell $\alpha\colon Tf \cdot \delta \Rightarrow Tg \cdot \sigma$. Then α has a lax generic factorization*



Also note that any map $k\colon X \to TC$ can be factored as $T\overline{k} \cdot \xi$ for some generic ξ and morphism $\overline{k}$, and so when T is surjective on objects we have a generic factorization of every 1-cell and 2-cell in the bicategory $\mathscr{B}$.

5. Comparing to Weber's familial 2-functors

The purpose of this section is to compare our definition of a lax familial 2-functor $T\colon \mathscr{A} \to \mathscr{B}$ between 2-categories (meaning Definition 4.1 specialized to 2-categories and 2-functors), with Weber's definition of familial 2-functor (which requires that $\mathscr{A}$ has a terminal object). It turns out that these two definitions are essentially equivalent. Note also that Weber's definition assumes some "strictness conditions" (such as identity 2-cells factoring into identity 2-cells) which are natural conditions on 2-functors, but arguably less natural in the case of pseudofunctors.

In one dimension, a functor $T\colon \mathcal{A} \to \mathcal{B}$ (where $\mathcal{A}$ has a terminal object) is said to be a parametric right adjoint (or a local right adjoint) when the canonical functor $T_1\colon \mathcal{A} \cong \mathcal{A}/\mathbf{1} \to \mathcal{B}/T\mathbf{1}$ is a right adjoint [22]. The following is what Weber refers to as the "naive" 2-categorical analogue of parametric right adjoint [30].

5.1. Definition. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are 2-categories, and that $\mathscr{A}$ has a terminal object. We say a 2-functor $T\colon \mathscr{A} \to \mathscr{B}$ is a* naive parametric right adjoint *if every canonical functor (on the 2-slices) $T_1\colon \mathscr{A} \cong \mathscr{A}/\mathbf{1} \to \mathscr{B}/T\mathbf{1}$ is a right 2-adjoint.*

We now recall the notion of generic morphism corresponding to this "naive" 2-categorical analogue of parametric right adjoints [30].

5.2. Definition. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are 2-categories. Given a 2-functor $T\colon \mathscr{A} \to \mathscr{B}$ we say a morphism $x\colon X \to TA$ is* naive-generic *if:*

1. *for any commuting square as on the left below*

there exists a unique $h\colon A \to B$ such that $Th \cdot x = z$ and $f = gh$;

2. *for two commuting diagrams*

the 2-cells $\theta\colon z_1 \Rightarrow z_2$ *such that* $Tg{\cdot}\theta = \mathrm{id}$ *bijectively correspond to 2-cells* $\overline{\theta}\colon h_1 \Rightarrow h_2$ *such that* $T\left(\overline{\theta}\right) \cdot x = \theta$ *and* $g \cdot \overline{\theta} = \mathrm{id}$.

From this one can prove the following expected result [30].

5.3. Proposition. *[30] Suppose $\mathscr{A}$ and $\mathscr{B}$ are 2-categories, and that $\mathscr{A}$ has a terminal object. Then a 2-functor $T\colon \mathscr{A} \to \mathscr{B}$ is a naive parametric right adjoint if and only if every $f\colon X \to TA$ factors as $T\overline{f} \cdot x$ for a naive-generic morphism x.*

Weber's actual definition of familial 2-functors (which we will soon recall) requires certain maps in a 2-category to be fibrations [30]. Thus we will need to recall the definition of fibration in a 2-category $\mathscr{B}$. Note that when $\mathscr{B}$ is finitely complete there are other equivalent characterizations of fibrations [21].

5.4. Definition. *We say a morphism $p\colon E \to B$ in a 2-category $\mathscr{B}$ is a* fibration *if:*

1. *for every $X \in \mathscr{B}$, the functor $\mathscr{B}(X,p) : \mathscr{B}(X,E) \to \mathscr{B}(X,B)$ is a fibration;*

2. *for every $f\colon X \to Y$ in $\mathscr{B}$, the functor $\mathscr{B}(f,E) : \mathscr{B}(Y,E) \to \mathscr{B}(X,E)$ preserves cartesian morphisms.*

If we have a choice of cartesian lifts which strictly respects composition and identities we say the fibration splits.

We now have the required background to define familial 2-functors in the sense of Weber.

5.5. Definition. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are 2-categories and that $\mathscr{A}$ has a terminal object. We say a 2-functor $T\colon \mathscr{A} \to \mathscr{B}$ is* Weber-familial *if*

1. *T is a naive parametric right adjoint;*

2. *for every $A \in \mathscr{A}$, and unique $t_A\colon A \to 1$ in $\mathscr{A}$, the morphism $Tt_A\colon TA \to T1$ is a split fibration in $\mathscr{B}$.*

The following is Weber's analogue of lax-generic morphisms.

5.6. Definition. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are 2-categories, and that $\mathscr{A}$ has a terminal object. Given a 2-functor $T\colon \mathscr{A} \to \mathscr{B}$ for which each $Tt_A\colon TA \to T1$ is a split fibration, we say a morphism $x\colon X \to TA$ is* Weber-lax-generic *if for any 2-cell α as on the left below,*

$$\begin{array}{ccc} X & \xrightarrow{z} & TB \\ {\scriptstyle x}\downarrow & \Uparrow\alpha & \downarrow{\scriptstyle Tg} \\ TA & \xrightarrow[Tf]{} & TC \end{array} \quad = \quad \begin{array}{ccc} X & \xrightarrow{z} & TB \\ {\scriptstyle x}\downarrow & \Uparrow\gamma \nearrow^{Th} \Uparrow T\nu & \downarrow{\scriptstyle Tg} \\ TA & \xrightarrow[Tf]{} & TC \end{array}$$

there exists a unique factorization (h, γ, ν) as above such that (h, γ) is chosen[14] *$Tt_B\colon TB \to T1$ cartesian.*[15]

The following lemma shows that for Weber-familial 2-functors T, the lax-generics of both our sense and Weber's coincide, and our generic 2-cells can equivalently be characterized as certain cartesian morphisms.

5.7. Lemma. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are 2-categories and that $\mathscr{A}$ has a terminal object. Let $T\colon \mathscr{A} \to \mathscr{B}$ be a Weber-familial 2-functor. Define $\mathfrak{M}$ as the category with objects given by chosen naive-generics $\delta\colon X \to TA$,*[16] *and morphisms given by pairs (h, γ)*

$$\begin{array}{ccc} & & TA \\ & \nearrow^{\delta} & \\ X & \Swarrow\gamma & \downarrow{\scriptstyle Th} \\ & \searrow_{\sigma} & \\ & & TB \end{array} \tag{5.1}$$

where γ is chosen $Tt_B\colon TB \to T1$ cartesian. Then:

1. *for every $X \in \mathscr{B}$ we have isomorphisms*
$$\mathscr{B}(X, T-) \cong \int_{\text{lax}}^{m \in \mathfrak{M}} \mathscr{A}(P_m, -)\,;$$

2. *a map $\delta\colon X \to TA$ in $\mathscr{B}$ is naive-generic if and only if it is strict*[17] *lax-generic;*

3. *a 2-cell in $\mathscr{B}$ as below*
$$\begin{array}{ccc} & & TA \\ & \nearrow^{\delta} & \\ X & \Swarrow\gamma & \downarrow{\scriptstyle Th} \\ & \searrow_{z} & \\ & & TB \end{array}$$

[14] Recall part of the data of a split fibration is a choice of coherent cartesian lifts.

[15] This definition of lax-generics has the downside that it assumes some of the conditions for a 2-functor being familial for it to make sense (namely that each $Tt_A\colon TA \to T1$ is a split fibration), thus not allowing for a theorem describing an equivalence between a 2-functor being familial and admitting lax-generic factorizations.

[16] Here "chosen" means that it is to be identified with another naive-generic $\sigma\colon X \to TB$ if there exists a pair (h, γ) as in (5.1) with h invertible and γ an identity; thus it is a choice of a representative of an equivalence class of naive-generics.

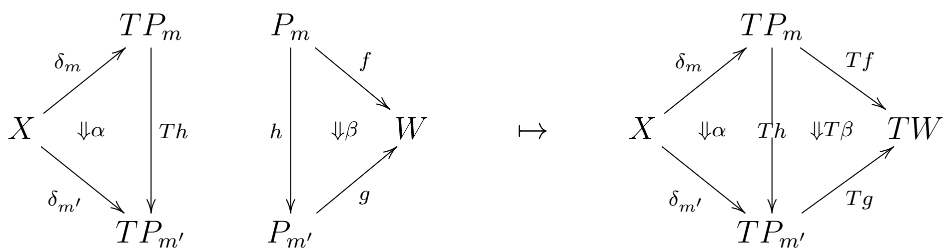
[17] By strict we mean identity 2-cells universally factor into identity 2-cells.

is generic if and only if it is $Tt_B \colon TB \to T1$ cartesian.

Proof. (1) : It suffices to check that the functors

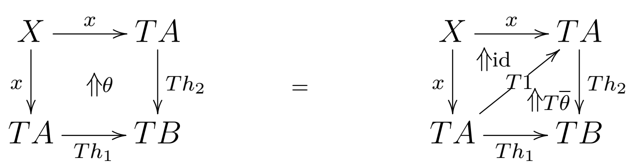
$$\int_{\mathrm{lax}}^{m \in \mathfrak{M}} \mathscr{A}(P_m, W) \to \mathscr{B}(X, TW)$$

are isomorphisms. That this assignment is bijective on objects is a consequence of the well-known one-dimensional case (for instance, see [26, Prop. 7]). That the assignment on morphisms



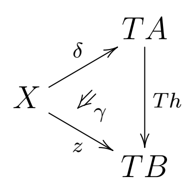
is bijective follows from the fact each naive-generic is Weber-lax generic [30, Lemma 5.8]. Naturality is also an easy consequence of this fact.

(2) : If δ is naive-generic, and thus isomorphic to a representative naive-generic, then δ is lax-generic by (1). If δ is strict lax-generic, then from a $\theta \colon z_1 \Rightarrow z_2$ we have a universal factorization

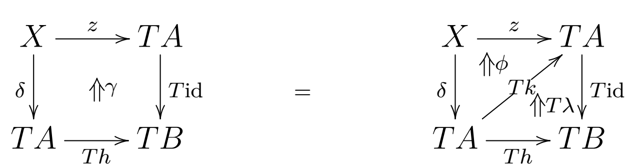


where we have used that $Tg \cdot \theta$ is an identity to see the top right triangle above can be taken as an identity. In this way, we recover the bijection required of a naive-generic.

(3) : Consider a 2-cell

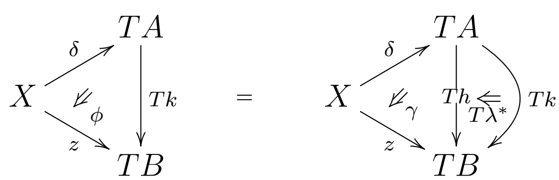


If this 2-cell is generic, then we have a factorization



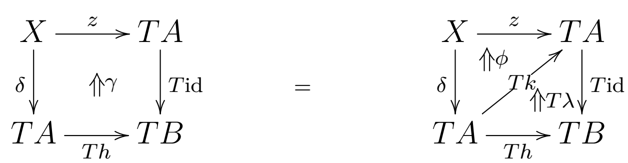
(5.2)

where ϕ is chosen cartesian. By genericity of γ, we have an $\lambda^*\colon k \Rightarrow h$ such that



(5.3)

and $\lambda^*\lambda = \mathrm{id}_h$. Substituting (5.2) into (5.3) and using that δ is Weber-lax-generic gives $\lambda\lambda^* = \mathrm{id}_k$. Conversely, if this 2-cell is cartesian we then have a factorization



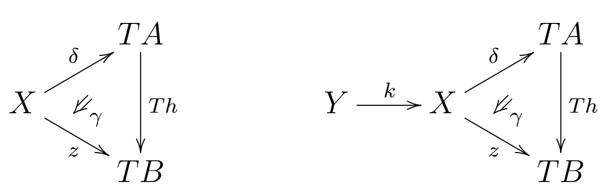
where (k, ϕ) is a generic 2-cell (which must also be cartesian by the above argument). Since ϕ and γ are cartesian, and thus isomorphic to chosen cartesian morphisms, it follows that the comparison λ is invertible. ∎

Finally, we give the main result of this section, showing that for 2-functors $T\colon \mathscr{A} \to \mathscr{B}$ our notion of a lax familial 2-functor is essentially equivalent to Weber's definition.

5.8. Theorem. *Suppose $\mathscr{A}$ and $\mathscr{B}$ are 2-categories and that $\mathscr{A}$ has a terminal object. Then for a 2-functor $T\colon \mathscr{A} \to \mathscr{B}$ the following are equivalent:*

1. *T is Weber-familial;*
2. *T is strictly[18] lax familial.*

Proof. (1) ⇒ (2) : Supposing $T\colon \mathscr{A} \to \mathscr{B}$ is Weber-familial, we have that each $\mathscr{B}(X, T-)$ is a lax conical colimit of representables by Lemma 5.7 part (1). Also, as the generic 2-cells may be identified with the cartesian 2-cells, we know since the fibration $Tt_B\colon TB \to T1$ respects precomposition (meaning it satisfies part 2 of Definition 5.4) we have the following property: for any generic 2-cell out of an $X \in \mathscr{B}$ as on the left below



(5.4)

and map $k\colon Y \to X$ in $\mathscr{B}$, the right diagram is a generic 2-cell. It is this property (along with closure of generic cells under composition) which gives (2)(b) of Theorem 4.10.

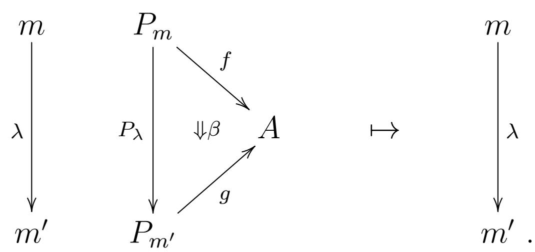
[18]By strict we mean isomorphic to a lax conical colimit of representables in place of equivalent, and that the reindexings $P^X_{(-)}$ are 2-functors instead of pseudofunctors.

(2) $\Rightarrow$ (1) : Suppose $T\colon \mathscr{A} \to \mathscr{B}$ is strictly lax familial. Then T is a naive parametric right adjoint since T has strict lax generic factorizations, and lax-generic implies naive generic (shown in part (2) of Lemma 5.7).

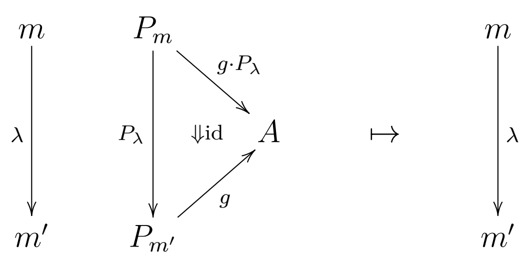
It remains to check that each $Tt_A\colon TA \to T1$ is a split fibration. To see this, note that for each $X \in \mathscr{B}$ the functor $\mathscr{B}(X, TA) \to \mathscr{B}(X, T1)$ may be written as the functor

$$\int_{\mathrm{lax}}^{m \in \mathfrak{M}} \mathscr{A}(P_m, A) \to \int_{\mathrm{lax}}^{m \in \mathfrak{M}} \mathscr{A}(P_m, 1) \cong \mathfrak{M}$$

defined by the assignment



It is straightforward to verify that for each $(m', g\colon P'_m \to A)$ and $\lambda\colon m \to m'$ we recover a cartesian lift as on the left below



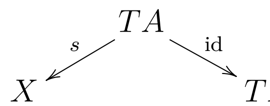
and it is clear the canonical choice of cartesian lifts given above splits. The cartesian morphisms are diagrams as above (if the identity 2-cell is replaced by an isomorphism it is still cartesian), and these correspond to generic cells in $\mathscr{B}(X, TA)$. That for each $k\colon Y \to X$ the functor $\mathscr{B}(k, TA) : \mathscr{B}(X, TA) \to \mathscr{B}(Y, TA)$ preserves cartesian morphisms then follows from condition (2)(b) of Theorem 4.10. ■

6. Examples of familial pseudofunctors

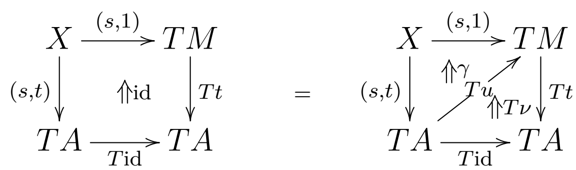
We will first consider some simple examples of lax familial pseudofunctors which concern pseudofunctors $T\colon \mathscr{A} \to \mathscr{B}$ where $\mathscr{A}$ is a 1-category. Our first and simplest examples of such pseudofunctors $T\colon \mathscr{A} \to \mathscr{B}$ concern the canonical embeddings into bicategories of spans and polynomials.

The reader will also recall that in this setting where $\mathscr{A}$ is a 1-category, $\mathrm{el}\, F \cong \mathrm{el}\, \mathscr{B}(X, T-)$ is a 1-category for each $X \in \mathscr{B}$, and so the mixed lifting properties become the usual lifting properties. Indeed, it is clear that Definition 4.5 becomes trivial in this case, so that every pair (h, γ) out of a lax-generic δ is a generic cell.

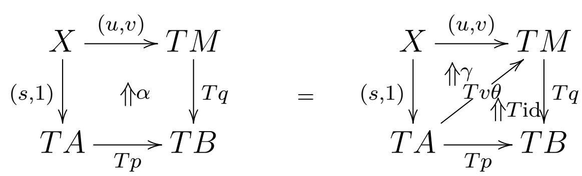
6.1. Example. *The bicategory of spans* $\mathbf{Span}(\mathcal{E})$ *in a category* $\mathcal{E}$ *with pullbacks was introduced by Bénabou [2], and admits a canonical embedding* $T\colon \mathcal{E} \to \mathbf{Span}(\mathcal{E})$ *sending a morphism* f *in* $\mathcal{E}$ *to the span* $(1, f)$. *It is interesting to note that this pseudofunctor is lax familial. To see this, first observe that a span* $X \nrightarrow TA$ *is generic if it is isomorphic to a span of the form*



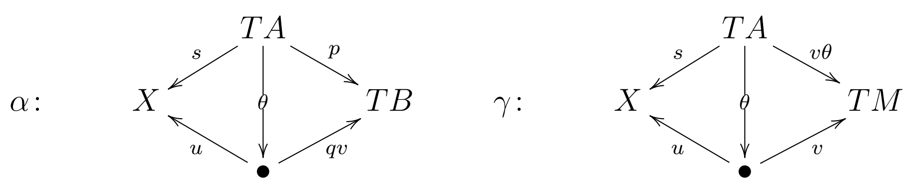
This is since if a span (s, t) *is generic we can then factor the diagram on the left below*



as on the right above, where ν *is necessarily an identity (because the domain of* T *is a 1-category) and* γ *invertible. Hence* $tu = \mathrm{id}$ *and* ut *is invertible, showing that* t *is invertible. Conversely, to see such a span* $(s, 1)$ *is generic, note that any diagram as on the left below*



universally factors as on the right above, where α *and* γ *are the respective morphisms of spans*



As all cells between generic morphisms are generic, it follows that the category $\mathfrak{M}_X$ *of generics out of* X *is the slice* $\mathcal{E}/X$, *and so for any* $X \in \mathcal{E}$ *we may take* $P_{(-)}$ *as the functor* $\mathrm{dom}\colon \mathcal{E}/X \to \mathcal{E}$, *giving*

$$\mathbf{Span}(\mathcal{E})(X, T-) \cong \int_{\mathrm{lax}}^{m \in \mathcal{E}/X} \mathcal{E}(P_m, -)$$

Dual to the above, we see that $T\colon \mathcal{E} \to \mathbf{Span}(\mathcal{E})^{\mathrm{co}}$ *admits "oplax-generic factorizations" (meaning the same as lax-generic factorizations except the direction of the 2-cells are reversed in Definition 4.4); indeed we may write*

$$\mathbf{Span}(\mathcal{E})^{\mathrm{co}}(X, T-) \cong \int_{\mathrm{oplax}}^{m \in \mathcal{E}/X} \mathcal{E}(P_m, -)$$

Moreover, the pseudofunctor $T\colon \mathcal{E} \to \mathbf{Span}_{\text{iso}}(\mathcal{E})$ *admits both lax and oplax generic factorizations, as we may write*

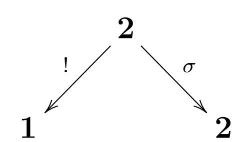
$$\mathbf{Span}_{\text{iso}}(\mathcal{E})(X, T-) \cong \int_{\text{lax}}^{m\in(\mathcal{E}/X)_{\text{iso}}} \mathcal{E}(P_m, -) \cong \int_{\text{oplax}}^{m\in(\mathcal{E}/X)_{\text{iso}}} \mathcal{E}(P_m, -)$$

where $(\mathcal{E}/X)_{\text{iso}}$ *contains the objects of* $\mathcal{E}/X$ *and only those morphisms which are invertible. The reader will also note that we do not have* $\mathbf{Span}_{\text{iso}}(\mathcal{E})(X, T-) \simeq \sum_{\text{ob}\ \mathcal{E}/X} \mathcal{E}(P_m, -)$ *as for each* $T \in \mathcal{E}$*, the right above is a discrete category, but isomorphisms of spans are not unique (and so the canonical functor* $\mathbf{Span}_{\text{iso}}(\mathcal{E})(X, T-) \to \sum_{\text{ob}\ \mathcal{E}/X} \mathcal{E}(P_m, -)$ *is not fully faithful).*

The case of spans is also interesting as it gives a simple example in which generic factorizations are not unique in the sense that one might initially expect. That is to say, given two generic factorizations

(meaning isomorphisms α and β as above), there is not necessarily a coherent comparison 2-cell $\overline{f} \Rightarrow \overline{g}$.

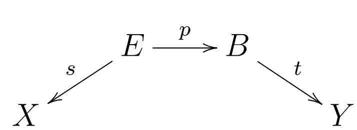
6.2. Example. *Consider a span*



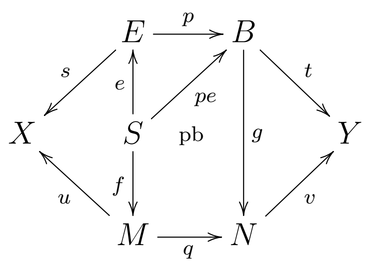
where σ *is the swap map. Here we have the two distinct generic factorizations*

In the following examples we will omit the verification that the generic morphisms and cells are classified correctly, as these calculations involving polynomial functors are quite technical.

6.3. Example. *For a locally cartesian closed category* $\mathcal{E}$*, one may form the bicategory of polynomials in* $\mathcal{E}$*, denoted* $\mathbf{Poly}(\mathcal{E})$*, whose objects are those of* $\mathcal{E}$*, morphisms are triples* $(s, p, t) : X \to Y$ *as below*

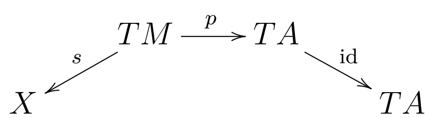


called polynomials, *and a (general) 2-cell of polynomials is a diagram as below*

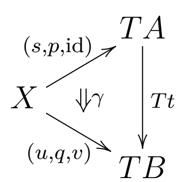


where the indicated square is a pullback[19] *[9, 31]. If e is invertible (so that the 2-cell is just a pullback) the 2-cell is said to be* cartesian *[9, 31].*

Similar to the case of spans, we see that the canonical pseudofunctor $T\colon \mathcal{E} \to \mathbf{Poly}(\mathcal{E})$ sending a morphism f to $(1, 1, f)$ is lax familial. Indeed, one can verify that a polynomial $X \nrightarrow TA$ is generic precisely when it is isomorphic to the form



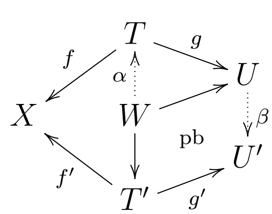
and it is trivial that any general 2-cell of polynomials as below



is generic (as the domain of T is a 1-category). Consequently, denoting by $\Pi_{\mathcal{E}}$ the pseudomonad freely adding products to a fibration [24], we may take $P_{(-)}$ as the functor $\mathrm{pr}\colon \Pi_{\mathcal{E}}(\mathcal{E}/X) \to \mathcal{E}$ *where $\Pi_{\mathcal{E}}(\mathcal{E}/X)$ is the category with objects given by spans*

$$X \xleftarrow{f} T \xrightarrow{g} U$$

out of X, and morphisms of spans from $(f, g) \nrightarrow (f', g')$ given by a pair $\beta\colon U \to U'$ and $\alpha\colon W \to T$ (where W is the chosen pullback of β and g') rendering commutative the diagram



[19]Actually a 2-cell is an equivalence class of such diagrams; or such a diagram where the pullback is chosen.

As a consequence we have the formula

$$\mathbf{Poly}\left(\mathcal{E}\right)\left(X,T-\right)\cong\int_{\text{lax}}^{m\in\Pi_{\mathcal{E}}\left(\mathcal{E}/X\right)}\mathcal{E}\left(P_m,-\right)$$

for all $X\in\mathbf{Poly}\left(\mathcal{E}\right)$.

6.4. Remark. *From Example 6.3, we see that the usual inclusion* $\mathbf{Span}\left(\mathcal{E}\right)\to\mathbf{Poly}\left(\mathcal{E}\right)$ *can be seen as coming from the unit components* $u_{\mathcal{E}/X}\colon\mathcal{E}/X\to\Pi_{\mathcal{E}}\left(\mathcal{E}/X\right)$ *of the pseudo-monad* $\Pi_{\mathcal{E}}$ *for fibrations with products [24]. Indeed, the family of functors*

$$\mathbf{Span}\left(\mathcal{E}\right)\left(X,Y\right)\to\mathbf{Poly}\left(\mathcal{E}\right)\left(X,Y\right)$$

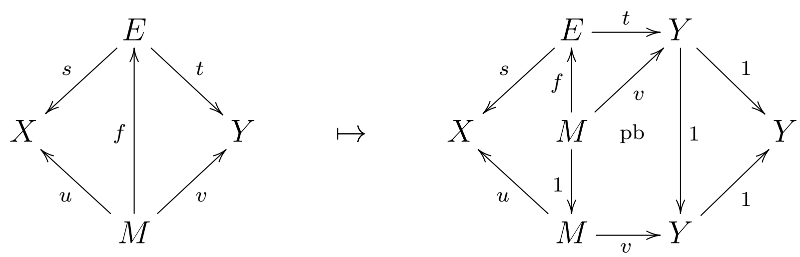
may be written as the functors

$$\int_{\text{lax}}^{m\in\mathcal{E}/X}\mathcal{E}\left(P_m,Y\right)\to\int_{\text{lax}}^{m\in\Pi_{\mathcal{E}}\left(\mathcal{E}/X\right)}\mathcal{E}\left(P_m,Y\right)$$

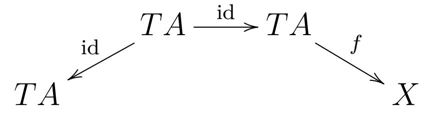
resulting from this reindexing.

We now give a more complicated example, where $\mathscr{A}$ is not a 1-category. In this situation the mixed lifting properties are necessary (unlike the earlier examples where usual liftings would suffice), and so it is no longer the case that every (h,γ) out of a generic morphism is a generic 2-cell.

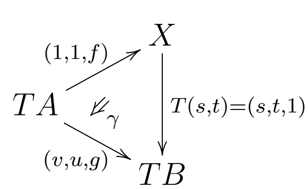
6.5. Example. *The canonical pseudofunctor* $T\colon\mathbf{Span}\left(\mathcal{E}\right)^{\text{co}}\to\mathbf{Poly}\left(\mathcal{E}\right)$*, given by*



is such that T^{op} *is lax familial. Here a polynomial* $TA\nrightarrow X$ *is opgeneric (meaning the opposite morphism in* $\mathbf{Poly}\left(\mathcal{E}\right)^{\text{op}}$ *is lax generic with respect to* T^{op}*) if it is isomorphic to the form*

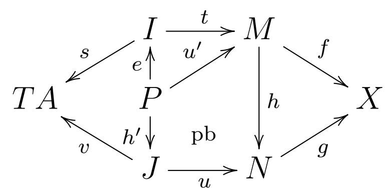


and a pair $\left(\left(s,t\right),\gamma\right)$ *out of a opgeneric as below*

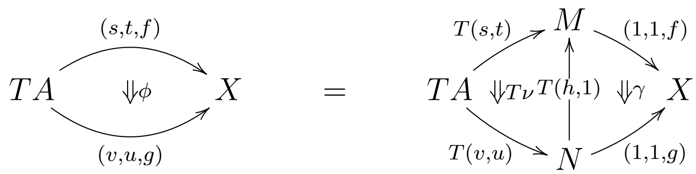


is generic when $\gamma\colon (s,t,f) \Rightarrow (v,u,g)$ *is a cartesian 2-cell of polynomials. We note also that cartesian morphisms of polynomials are closed under vertical composition as well as precomposition by another polynomial.*

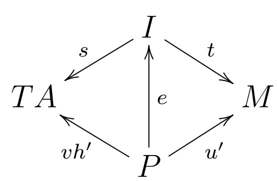
Given a general morphism of polynomials $\phi\colon (s,t,f) \Rightarrow (v,u,g)$ *as given by the diagram*

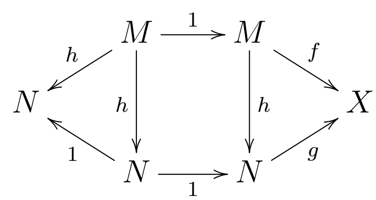


the opgeneric factorization of ϕ *is given by*



where ν *is the reversed morphism of spans on the left below*





and γ *is the cartesian morphism of polynomials on the right above. It follows that for any* $X \in \mathcal{E}$ *we may take* $P_{(-)}$ *as the functor*

$$\mathcal{E}/X \xrightarrow{\text{dom}} \mathcal{E} \xrightarrow{\iota} \mathbf{Span}\,(\mathcal{E})^{\text{coop}}$$

where ι *sends each morphism* $h\colon A \to B$ *to* $(h, 1_A) \in \mathbf{Span}\,(\mathcal{E})^{\text{coop}}$*, and get*

$$\mathbf{Poly}\,(\mathcal{E})^{\text{op}}\,(X, T-) \cong \int_{\text{lax}}^{m\in\mathcal{E}/X} \mathbf{Span}\,(\mathcal{E})^{\text{coop}}\,(P_m, -)\,.$$

We now give a natural example which does not come from a pseudofunctor of bicategories $T\colon \mathscr{A} \to \mathscr{B}$. Indeed, the following may be seen as the main motivating example for this paper.

6.6. Example. *Following Carboni and Johnstone [5], we consider the pseudofunctor* $\mathbf{Fam}\colon \mathbf{CAT} \to \mathbf{CAT}$ *sending a category* $\mathcal{C}$ *to the category* $\mathbf{Fam}\,(\mathcal{C})$ *with objects given by families of objects of* $\mathcal{C}$ *denoted* $(A_i \in \mathcal{C}\colon i \in I)$*, and morphisms* $(A_i \in \mathcal{C}\colon i \in I) \nrightarrow (B_j \in \mathcal{C}\colon j \in J)$ *given by a reindexing* $\varphi\colon I \to J$ *with comparison maps* $A_i \to B_{\varphi(i)}$ *for each* $i \in I$*.*

A trivial case of such a family is when $\mathcal{C}$ is any set I, and $(i : i \in I)$ is such a family of objects (set-elements) of I indexed by the set I. Noting this, one can show the el-generic objects of el **Fam** *are those elements of the form $(I, (i : i \in I))$ for a set I. And it is clear that for any general element $(\mathcal{C}, (B_j : j \in J))$ of* el **Fam** *that we have the "generic factorization" (that is an opcartesian map from a generic)*

$$(J, (j : j \in J)) \xrightarrow{\left(B_{(-)}, \mathrm{id}\right)} (\mathcal{C}, (B_j : j \in J))$$

Also, a general morphism out of an el-generic object

$$(I, (i : i \in I)) \xrightarrow{\left(H_{(-)}, (\varphi, \gamma)\right)} (\mathcal{C}, (B_j : j \in J))$$

consists of a functor $H_{(-)} : I \to \mathcal{C}$, a function $\varphi : I \to J$, and morphisms $\gamma_i : H_i \to B_{\varphi(i)}$ indexed over $i \in I$. Such a morphism is generic precisely when every γ_i is invertible.

It is then clear that the category of generic objects and generic morphisms between them (note $H_{(-)}$ is uniquely determined by φ in this case) is isomorphic to **Set**. *It follows that the* **Fam** *construction is given by*[20]

$$\mathbf{Fam}\,(\mathcal{C}) = \int_{\mathrm{lax}}^{X \in \mathbf{Set}} \mathcal{C}^X, \qquad \mathcal{C} \in \mathbf{CAT}$$

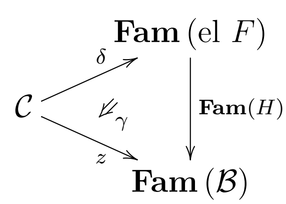
It is worth noting that restricting to the category of finite sets $\mathbf{Set}_{\mathrm{fin}}$, *yields the finite families construction* $\mathbf{Fam}_f$, *and restricting further the category of finite sets and bijections $\mathbb{P}$ yields the free symmetric (strict) monoidal category construction.*

The above shows that **Fam** is a lax conical colimit of representables. However, it is also interesting that **Fam** is lax familial in the sense of Definition 4.1.

6.7. Example. *The pseudofunctor* $\mathbf{Fam} : \mathbf{CAT} \to \mathbf{CAT}$ *is lax familial. Here the generic morphisms are those functors of the form*

$$\delta_F : \mathcal{C} \to \mathbf{Fam}\,(\mathrm{el}\ F) : X \mapsto ((X, x) \in \mathrm{el}\ F : x \in FX)$$

for a presheaf $F : \mathcal{C} \to$ **Set** *(Weber refers to these as "functors endowing $\mathcal{C}$ with elements" [30, Definition 5.10]). A cell out of such a generic morphism*



[20] Whilst this example involves large categories, the indexing **Set** is locally small, and our results still apply. We do not wish to burden this paper with a discussion of size conditions.

is generic when the comparison maps (not necessarily the reindexing maps) comprising each γ_X for $X \in \mathcal{C}$ are required invertible. It follows that the lax familial representatibly of **Fam** *is shown by the formula*

$$\mathbf{CAT}\,(\mathcal{C}, \mathbf{Fam}\,(-)) \cong \int_{\text{lax}}^{F\colon \mathcal{C}\to\mathbf{Set}} \mathbf{CAT}\,(\text{el } F, -)$$

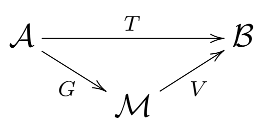
for each $\mathcal{C} \in \mathbf{CAT}$.

7. The spectrum factorization of a lax familial pseudofunctor

In the simpler dimension-one case, Diers [7] showed that familial functors (as defined in Definition 2.7) have the following simple characterization:

7.1. Theorem. [Diers] *Let $T\colon \mathcal{A} \to \mathcal{B}$ be a functor of categories. Then the following are equivalent:*

1. *the functor T is familial;*
2. *there exists a factorization*



such that:

(a) *V is a discrete fibration;*

(b) *G has a left adjoint.*

When $\mathcal{A}$ has a terminal object, it is not hard to see that $\mathcal{M} \simeq \mathcal{B}/T\mathbf{1}$. This gives the following simple consequence:

7.2. Corollary. *Let $T\colon \mathcal{A} \to \mathcal{B}$ be a functor of categories, and assume $\mathcal{A}$ has a terminal object. Then T is familial (is a parametric right adjoint) if and only if the canonical projection*

$$T_1\colon \mathcal{A}/\mathbf{1} \to \mathcal{B}/T\mathbf{1}$$

has a left adjoint.

It is the purpose of this section to find an analogue of these results in the dimension two case. However, as we will see, this is much more complicated than simply asking for a left bi-adjoint. Instead we will require certain types of “lax” adjunctions (or adjunctions up to adjunction).

7.3. Lax F-adjunctions. In the setting of an adjunction of functors $F \dashv G \colon \mathcal{A} \to \mathcal{M}$ we have natural hom-set isomorphisms $\mathcal{A}(F_m, A) \cong \mathcal{M}(m, GA)$. More generally, one can talk about bi-adjunctions of pseudofunctors $F \dashv G \colon \mathscr{A} \to \mathscr{M}$ where we only ask for natural hom-category equivalences $\mathscr{A}(F_m, A) \simeq \mathscr{M}(m, GA)$ [11]. However, even this notion is often too strong.

Central to the theory of lax familial pseudofunctors is the theory of lax adjunctions [3], where one only asks that we have adjoint pairs

$$L_{m,A} \colon \mathscr{A}(F_m, A) \to \mathscr{M}(m, GA), \qquad R_{m,A} \colon \mathscr{M}(m, GA) \to \mathscr{A}(F_m, A)$$

pseudonatural (or even lax natural) in $A \in \mathscr{A}$ and $m \in \mathscr{M}$.

The following type of lax adjunctions, called *lax* F*-adjunctions*, appear when studying familial pseudofunctors.[21] These are the lax adjunctions which naturally restrict to biadjunctions on a class of "tight" maps. A simple example of tight maps are the pseudo-commuting triangles (pseudo slice) of the lax slice category of a pseudofunctor. Before defining lax F-adjunctions, we must first define F-bicategories and see how they assemble into a tricategory F-**Bicat**. It is not hard to verify this data forms a tricategory given that bicategories, pseudofunctors, pseudo-natural transformations and modifications do [13].

7.4. Definition. *The following notions below:*

- *an* F-bicategory *is a bicategory $\mathscr{A}$ equipped with an identity on objects, injective on 1-cells, locally fully faithful functor $\mathscr{A}_{\mathsf{T}} \to \mathscr{A}$. The 1-cells of $\mathscr{A}_{\mathsf{T}}$ are called the* tight *1-cells of $\mathscr{A}$ and are required to be closed under invertible 2-cells;*
- *an* F-pseudofunctor $(\mathscr{A}, \mathscr{A}_T) \nrightarrow (\mathscr{B}, \mathscr{B}_T)$ *is a pseudofunctor $F \colon \mathscr{A} \to \mathscr{B}$ which restricts to a pseudofunctor $F_T \colon \mathscr{A}_T \to \mathscr{B}_T$;*
- *a* lax F-natural transformation $\alpha \colon F \Rightarrow G \colon (\mathscr{A}, \mathscr{A}_T) \to (\mathscr{B}, \mathscr{B}_T)$ *is a lax natural transformation $\alpha \colon F \Rightarrow G$ such that both:*
    1. *for all $X \in \mathscr{A}$, $\alpha_X \colon FX \to GX$ is tight;*
    2. *for all $f \colon X \to Y$ tight, $\alpha_f \colon Gf \cdot \alpha_X \Rightarrow \alpha_Y \cdot Ff$ is invertible.*

define the tricategory F-**Bicat** *of* F*-bicategories,* F*-pseudofunctors, lax* F*-natural transformations, and modifications.*

The above allows for a particularly simple definition of lax F-adjunctions.

[21] Here the F denotes the category whose objects are fully faithful functors and morphisms are pseudo-commuting squares. Moreover, these concepts arise from considering F-enriched (bi)categories; though we will not use this enrichment perspective [15].

7.5. Definition. [Lax F-adjunction] *A* lax F-adjunction *of* F*-pseudofunctors*

$$(\mathscr{A},\mathscr{A}_T) \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathscr{B},\mathscr{B}_T)$$

is a biadjunction [11] in the tricategory F-**Bicat**.

7.6. Remark. *It is worth noting that this definition immediately tells us that lax* F*-adjunctions enjoy nice properties such as uniqueness of adjoints up to equivalence.*

Whilst the above definition is conceptually informative, for our purposes it will be more useful to define these adjunctions in terms of universal arrows. This is due to the connection between the universal arrow definition and notions of genericity.

7.7. Remark. *From now on we will regard the right adjoint G as a* F*-pseudofunctor* $G\colon (\mathscr{A},\mathscr{A}_T) \to (\mathscr{M},\mathscr{M}_T)$ *to more closely match the notation we will use use later on.*

The characterization of lax F-adjunctions by universal arrows is slightly technical, so we will break it into parts.

7.8. Definition. [Universal pair] *Given an pseudofunctor $G\colon \mathscr{A} \to \mathscr{M}$, object F_m in $\mathscr{M}$, and a diagram*

$$\begin{tikzcd} m \arrow[rr, "f"] \arrow[dr, "\eta_m"'] & {\Uparrow \gamma_f} & GA \\ & GF_m \arrow[ur, "G\overline{f}"'] & \end{tikzcd}$$

we say the pair $\left(\overline{f},\gamma_f\right)$ is universal *if for any $\overline{g}\colon F_m \to A$ and 2-cell β as below*

$$\begin{tikzcd} m \arrow[rr, "f"] \arrow[dr, "\eta_m"'] & {\Uparrow \beta} & GA \\ & GF_m \arrow[ur, "G\overline{g}"'] & \end{tikzcd} \quad = \quad \begin{tikzcd} m \arrow[rr, "f"] \arrow[dr, "\eta_m"'] & {\Uparrow \gamma_f} & GA \\ & GF_m \arrow[ur, "G\overline{f}"] \arrow[ur, bend right, "G\overline{g}"'] & {\scriptstyle G\widetilde{\beta}\,\Uparrow} \end{tikzcd}$$

there exists a unique $\widetilde{\beta}\colon \overline{g} \Rightarrow \overline{f}$ such that the above equality holds.

The following defines what one should think of as the "universal arrows" of a lax F-adjunction.

7.9. Definition. [F-universal arrows] *Given an* F*-pseudofunctor* $G\colon (\mathscr{A},\mathscr{A}_T) \to (\mathscr{M},\mathscr{M}_T)$, *we say a morphism $\eta_m\colon m \to GF_m$ (where F_m is some object of $\mathscr{M}$) is* universal *if for any 1-cell $f\colon m \to GA$ there exists a $\overline{f}\colon F_m \to A$ and a 2-cell*

$$\begin{tikzcd} m \arrow[rr, "f"] \arrow[dr, "\eta_m"'] & {\Uparrow \gamma_f} & GA \\ & GF_m \arrow[ur, "G\overline{f}"'] & \end{tikzcd}$$

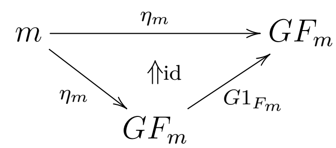
such that the pair $\left(\overline{f},\gamma_f\right)$ is universal. We say that η_m is F*-universal if in addition*[22]

[22]The reader will of course notice that such a η_m is unique up to equivalence.

(i) the 1-cell η_m *is tight;*

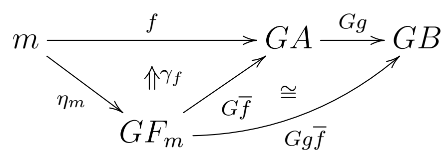
(ii) for every tight 1-cell $f\colon m \to GA$ *in* $\mathscr{M}$*, the 2-cell* γ_f *is invertible and* $\overline{f}\colon F_m \to A$ *is tight;*

(iii) the diagram



exhibits $(1_{F_m}, \mathrm{id})$ *as a universal pair;*

(iv) for any universal pair $\left(\overline{f}, \gamma_f\right)$*, the G-whiskering by a tight* $g\colon A \to B$



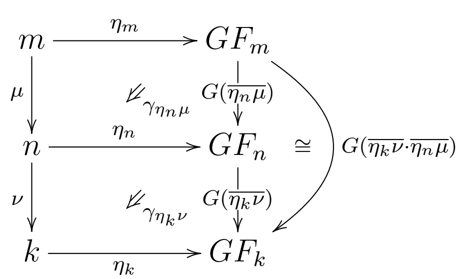
exhibits $\left(g\overline{f}, Gg \cdot \gamma_f\right)$ *as a universal pair.*

The following proposition makes precise the characterization of a lax F-adjunction in terms of universal arrows.

7.10. Proposition. *Given an* F*-pseudofunctor* $G\colon (\mathscr{A}, \mathscr{A}_T) \to (\mathscr{M}, \mathscr{M}_T)$*, G has a left lax* F*-adjoint as in Definition 7.5 if and only if both:*

1. *for every object m in* $\mathscr{M}$*, there exists a* F*-universal 1-cell* $\eta_m\colon m \to GA$*;*

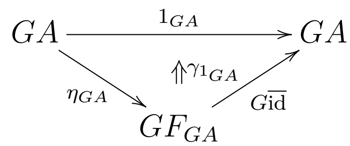
2. *for all 1-cells* μ *and* ν *as below,* $\overline{\eta_k\nu} \cdot \overline{\eta_n\mu}$ *equipped with the 2-cell*



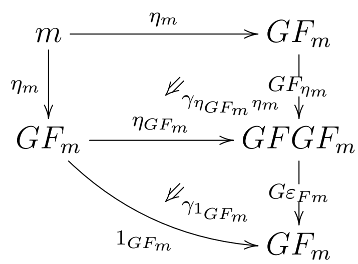
is universal.

We will not give all the technical details of the proof, but the following remark should convince the reader of this characterization.

7.11. Remark. *It comes for free that for all $A \in \mathscr{A}$, the universal pair*



has the 2-cell component $\gamma_{1_{GA}}$ invertible (as identity 1-cells are necessarily tight). This is one of the triangle identities. The other triangle identity which asks for the composite of F_{η_m} and ε_{F_m} constructed as below



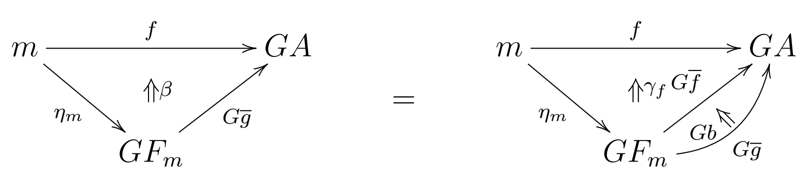
to be isomorphic to the identity, is equivalent to (iii) in the presence of (iv). Pseudofunctoriality of F is clear from (2) and (iii).

The reader will also recognize that $L_{m,A}$ and $R_{m,A}$ are pseudonatural in $A \in \mathscr{A}$ and $m \in \mathscr{M}$ respectively; and also pseudonatural in $m \in \mathscr{M}_T$ and $A \in \mathscr{A}_T$ respectively. Indeed, $L_{m,A} \colon \mathscr{A}(F_m, A) \to \mathscr{M}(m, GA)$ is defined by applying G and composing with η_m, and $R_{m,A} \colon \mathscr{M}(m, GA) \to \mathscr{A}(F_m, A)$ is defined by applying F and composing with ε_A. Also, it is not hard to see that η and ε become lax F-natural transformations given the universal arrow viewpoint. Finally, it is worth noting that each γ is invertible if and only if the unit η is pseudonatural.

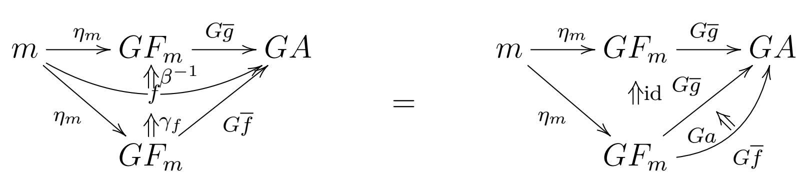
The following property of lax F-adjunctions, that the operations $\widetilde{(-)}$ respect isomorphisms, will be useful later in this section.

7.12. Lemma. *Given a pseudofunctor $G \colon \mathscr{A} \to \mathscr{M}$ with a left lax F-adjoint (F, η, γ), the operation $\beta \mapsto \widetilde{\beta}$ respects isomorphisms on tight maps.*

Proof. Suppose we have an equality as below where $\overline{g} \colon F_m \to A$ is tight



and suppose further that β has an inverse, so that we have the equality

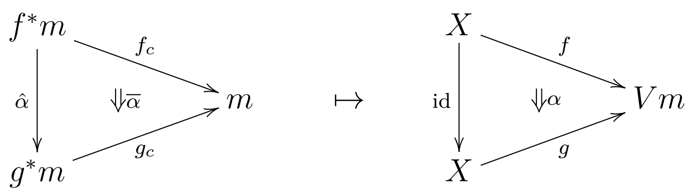


where we have used axioms (iii) and (iv) to realize the pair consisting of the identity 2-cell (given on the right above) and $\overline{g}$ as universal. It is then straightforward to verify a is inverse to b. ∎

7.13. Remark. *It is not hard to see that in the presence of axiom (iv), the above lemma is equivalent to (iii).*

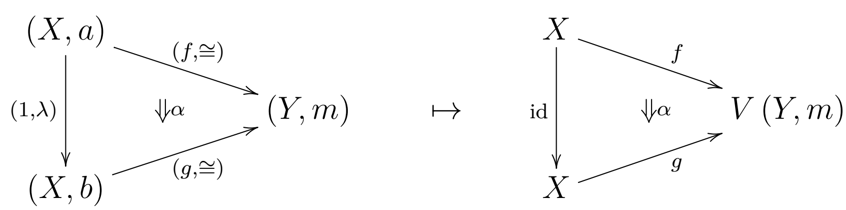
7.14. Factoring through the spectrum. We now have the necessary background on lax adjunctions, and can move towards understanding how a lax familial pseudofunctor factors through the spectrum. This will only require the following simple lemma.

7.15. Lemma. *Suppose $V\colon \mathscr{M} \to \mathscr{B}$ is a locally discrete fibration of bicategories. Then given any 2-cell $\alpha\colon f \Rightarrow g\colon X \to Vm$ as on the right below*



with cartesian lifts f_c and g_c of f and g, there exists a unique pair $(\hat{\alpha}, \overline{\alpha})$ as on the left above which is sent to α by V. Moreover, if α is invertible then both $\hat{\alpha}$ and $\overline{\alpha}$ are.

Proof. Suppose without loss of generality that V is the projection $\int_{\mathrm{lax}}^{B\in\mathscr{B}} FB \to \mathscr{B}$ for a pseudofunctor $F\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$. Then we may construct a diagram as on the left below



where λ is the unique map such that

$$a \xrightarrow{\cong} Ff(m) \xrightarrow{(F\alpha)_m} Fg(m) \qquad = \qquad a \xrightarrow{\lambda} b \xrightarrow{\cong} Fg(m)$$

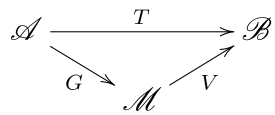
holds. It is clear this is the only choice of λ, and that if α is invertible then so is λ. ∎

7.16. Remark. *There should be an analogue of the above without assuming V to be locally discrete, so that V is the projection $\int_{\mathrm{lax}}^{B\in\mathscr{B}} FB \to \mathscr{B}$ for a trifunctor $F\colon \mathscr{B}^{\mathrm{op}} \to$ **Bicat**. However, this is beyond the scope of this paper.*

We can now prove the main result of this section, which provides a conceptually nice description of lax familial pseudofunctors. Recall also that the tight maps of a bicategory are something we must specify, and not part of the data of the original bicategory.

7.17. THEOREM. [Spectrum factorization] *Let $T\colon \mathscr{A} \to \mathscr{B}$ be a pseudofunctor of bicategories. Then the following are equivalent:*

1. *the pseudofunctor T is lax familial;*
2. *there exists a factorization*

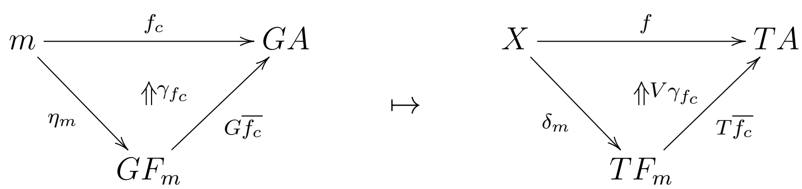


   such that:

   (a) V is a locally discrete fibration of bicategories;

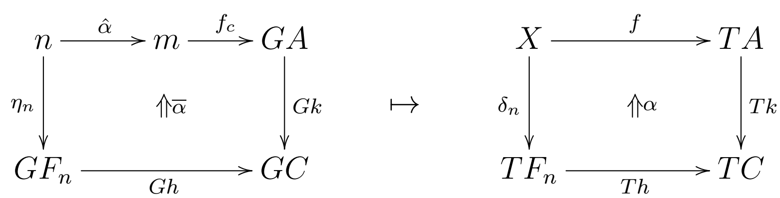
   (b) G has a left lax F-adjoint (where all 1-cells in $\mathscr{A}$ are tight and the V-cartesian 1-cells of $\mathscr{M}$ are tight).

PROOF. (2) ⇒ (1) : We first note that for any $f\colon X \to TA$ in $\mathscr{B}$, we have a cartesian lift $f_c\colon m \to GA$ in $\mathscr{M}$. We thus have an assignment

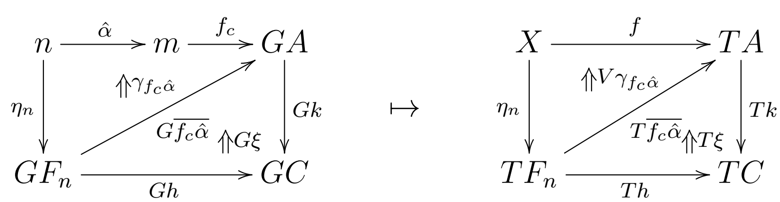


and as γ is invertible on cartesian maps, this is a factorization of f. We thus need only check that each δ_m is lax-generic, and that generic 2-cells compose.

Consider now a 2-cell α as on the right below

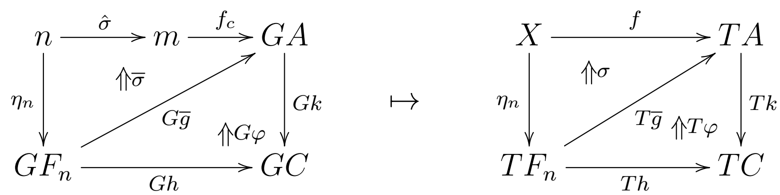


and its unique preimage as on the left above given by Lemma 7.15. This $\overline{\alpha}$ in turn has a factorization as on the left below



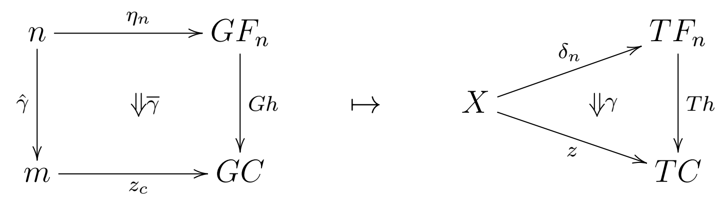
since universality of $(f_c\hat{\alpha}, \gamma_{f_c\hat{\alpha}})$ is preserved by Gk, thus giving a factorization of α as on the right above. Note that if α, and hence $\hat{\alpha}$ and $\overline{\alpha}$ are invertible, then $\gamma_{f_c\hat{\alpha}}$ is invertible (as it is on all cartesian 1-cells), and ξ is invertible by Lemma 7.12.

Given another factorization as on the right below, we can lift σ by Lemma 7.15

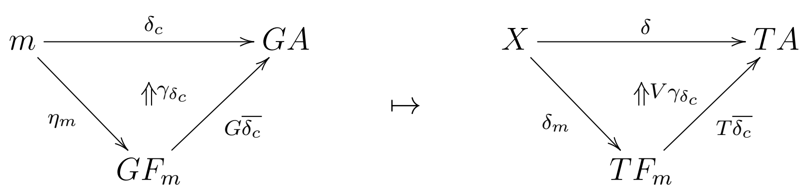


giving the left above. Noting that $\hat{\sigma} = \hat{\alpha}$ and that the left pasting above is $\overline{\alpha}$ by uniqueness, we can then factor $\overline{\sigma}$ through $\gamma_{f_c\hat{\alpha}}$ recovering a comparison map $\psi\colon \overline{g} \Rightarrow \overline{f_c\hat{\alpha}}$ satisfying the required conditions. The subterminality of each $V\gamma_{f_c\hat{\alpha}}$ stems from the uniqueness of factorizations through $\gamma_{f_c\hat{\alpha}}$.

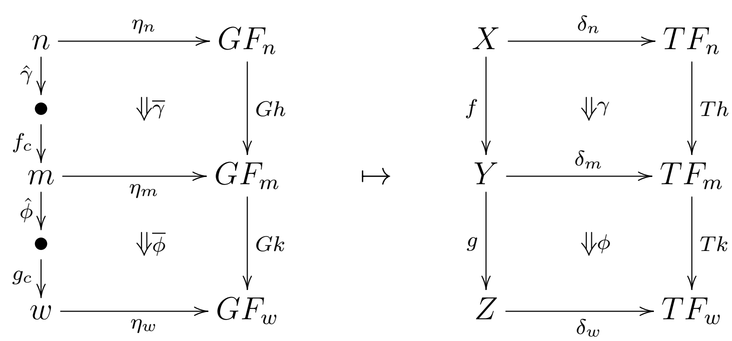
Finally, to see that generic cells compose, observe that a cell as on the right below



is generic precisely when its lift as on the left above, given by Lemma 7.15, exhibits $(h, \overline{\gamma})$ as a universal pair. Also observe that every generic is of the form δ_n, since given any generic δ and cartesian lift δ_c we have an isomorphism



and we know that $\left(\overline{\delta_c}, V\gamma_{\delta_c}\right)$ is an equivalence by Lemma 3.15. It follows that two generic cells as on the right below



compose to a generic, as the composite on the left above is universal.

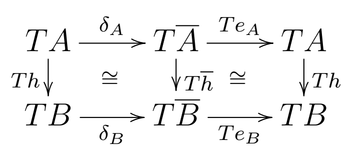
(1) ⇒ (2) : Supposing that $T\colon \mathscr{A} \to \mathscr{B}$ is lax familial, we may construct the spectrum $\mathfrak{M}_{(-)}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ as in Lemma 4.9 and factor T as

$$\mathscr{A} \xrightarrow{G} \int_{\mathrm{lax}}^{X\in\mathscr{B}} \mathfrak{M}_{(-)} \xrightarrow{V} \mathscr{B}$$

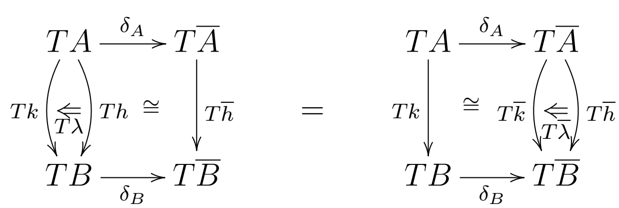
where G sends each $A \in \mathscr{A}$ to $TA \in \mathscr{B}$ with the generic morphism $\delta_A\colon TA \to T\overline{A}$ being part of the generic factorization

$$TA \xrightarrow{\delta_A} T\overline{A} \xrightarrow{Te_A} TA$$

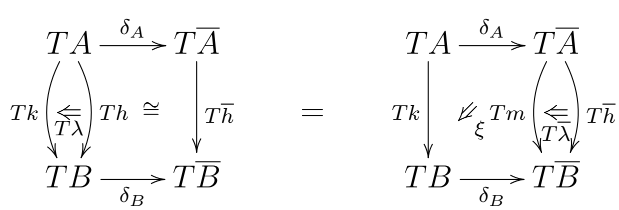
of the identity. We choose all morphisms of $\mathscr{A}$ to be tight, and the cartesian morphisms against the projection V to be tight. A 1-cell $h\colon A \to B$ in $\mathscr{A}$ is sent to Th with the pair $\left(\overline{h}, \cong\right)$ comprising the left side



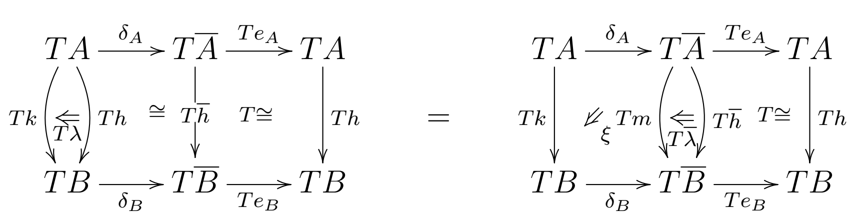
of the generic factorization above. A given 2-cell $\lambda\colon h \Rightarrow k$ is sent to $T\lambda\colon Th \Rightarrow Tk$, which satisfies



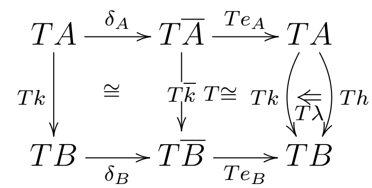
for some (necessarily unique) $\overline{\lambda}\colon \overline{h} \Rightarrow \overline{k}$. To see this, note that the left diagram has a generic factorization



and thus the left diagram below has the generic factorization

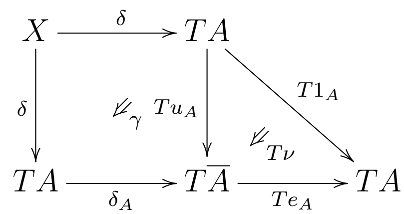


But this is also the generic factorization of the diagram

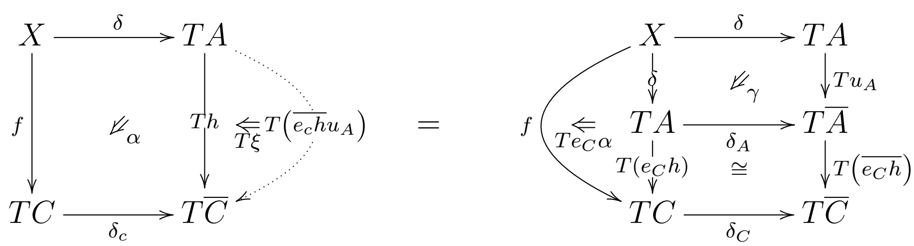


which has already been factored. By uniqueness of representative generic factorizations we have $(m,\xi) = \left(\overline{k}, \cong\right)$ as required.

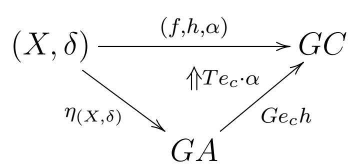
Now, we have the pseudofunctor $\mathbf{P}\colon \int_{\text{lax}}^{X\in\mathscr{B}} \mathfrak{M}_{(-)} \to \mathscr{A}$, and will sketch why $\mathbf{P}$ is a left lax F-adjoint to G. To do this, we take our universal 1-cell $\eta_{(X,\delta)}\colon (X,\delta) \to GF(X,\delta)$ at an object $(X, \delta\colon X \to LA)$ to be the pair (u_A, γ) as below.



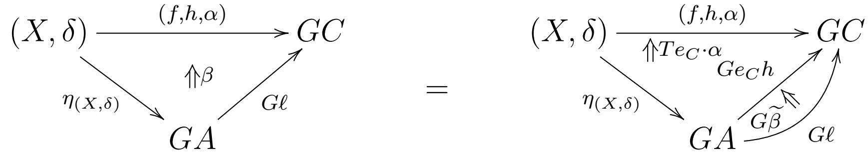
Moreover, for a given 1-cell $(f,h,\alpha)\colon (X,\delta) \to GC$ as on the left below, we have



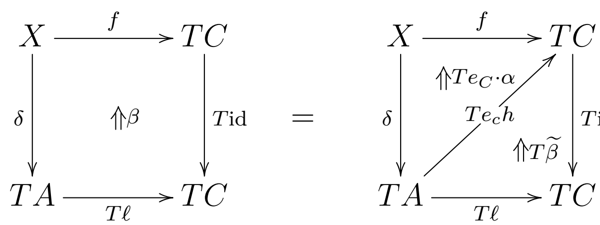
where ξ is the unique map induced from the fact that the RHS whiskered by Te_C is $Te_C \cdot \alpha$. This defines the universal 2-cell



where we have a bijection $\beta \mapsto \widetilde{\beta}$ as below



or equivalently, a bijection



as genericity of (h, α) is respected by composition with Te_C. The verification that this bijection satisfies the required axioms (with the tight maps being defined as above) is left for the reader. ■

Finally, the following provides what is perhaps a more natural definition of parametric right adjoint (local right adjoint) pseudofunctors, obtained by applying the above theorem in the setting where $\mathscr{A}$ has a terminal object. In more detail, this is obtained by noting the reduced form of the spectrum in the presence of a terminal object, namely $\mathbf{Spec}_T(X) = \mathscr{B}(X, T\mathbf{1})$, and applying the spectrum factorization.

7.18. Corollary. [Parametric right adjoints] *Suppose $\mathscr{A}$ is a bicategory with a terminal object. Then a pseudofunctor $T\colon \mathscr{A} \to \mathscr{B}$ is lax familial if and only if the canonical projection on the oplax slice,*[23] *where the underlying pseudo slice defines the tight maps,*[24]

$$T_1\colon \mathscr{A} /\!\!/ 1 \to \mathscr{B} /\!\!/ T1$$

has a left lax F*-adjoint.*

7.19. Remark. *There are of course four variants of the above, concerning the case when $T/T^{\mathrm{op}}/T^{\mathrm{co}}/T^{\mathrm{coop}}$ is familial.*

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[23] In our convention a morphism $f \nrightarrow g$ in the oplax slice $\mathscr{B} /\!\!/ T1$ is a morphism $m\colon \mathrm{dom} f \to \mathrm{dom} g$ and a 2-cell $\alpha\colon f \Rightarrow gm$ in $\mathscr{B}$. The tight morphisms are those for which α is invertible.

[24] In the case of $\mathscr{A} /\!\!/ 1$, the underlying pseudo-slice coincides with the lax slice. Thus, just as in Theorem 7.17, every morphism in the domain category is tight.

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