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THE EXISTENTIAL COMPLETION

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ABSTRACT. We determine the existential completion of a primary doctrine, and we prove that the 2-monad obtained from it is lax-idempotent, and that the 2-category of existential doctrines is isomorphic to the 2-category of algebras for this 2-monad. We also show that the existential completion of an elementary doctrine is again elementary. Finally we extend the notion of exact completion of an elementary existential doctrine to an arbitrary elementary doctrine.

1. Introduction

In recent years, many relevant logical completions have been extensively studied in category theory. The main instance is the exact completion, see [Carboni, 1995; Carboni and Celia Magno, 1982; Carboni and Vitale, 1998], which is the universal extension of a category with finite limits to an exact category. In [Maietti and Rosolini, 2013a,b,c], Maietti and Rosolini introduce a categorical version of quotient for an equivalence relation, and they study that in a doctrine equipped with a sufficient logical structure to describe the notion of an equivalence relation. In [Maietti and Rosolini, 2013c] they show that both the exact completion of a regular category and the exact completion of a category with binary products, a weak terminal object and weak pullbacks can be seen as instances of a more general completion with respect to an elementary existential doctrine.

In this paper we present the existential completion of a primary doctrine, and we give an explicit description of the 2-monad $\mathrm{T}_e\colon \mathbf{PD} \longrightarrow \mathbf{PD}$ constructed from the 2-adjunction, where $\mathbf{PD}$ is the 2-category of primary doctrines.

It is well known that pseudo-monads can express uniformly and elegantly many algebraic structures; we refer the reader to [Tanaka and Power, 2006b,a; Kelly and Lack, 1997] for a detailed description of these topics.

Recall that an action of a 2-monad on a given object encodes a *structure* on that object. When the structure is uniquely determined to within unique isomorphism, to give an object with such a structure is just to give an object with a certain *property*. Those 2-monads for which the algebra structure is essentially unique, if it exists, are called *property-like*.

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In this paper we show that every existential doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ admits an action $a\colon \mathrm{T}_e(P) \longrightarrow P$ such that (P, a) is a T_e-algebra, and that if (R, b) is T_e-algebra then the doctrine is existential, and this gives an equivalence between the 2-category T_e-**Alg** and the 2-category **ED** whose objects are existential doctrines.

Here the action encodes the existential structure for a doctrine, and we prove that this structure is uniquely determined to within appropriate isomorphism, i.e. that the 2-monad T_e is lax-idempotent and hence property-like in the sense of [Kelly and Lack, 1997].

We also prove that the existential completion preserves the elementary structure of a doctrine, and then we generalize the bi-adjunction $\mathbf{EED} \to \mathbf{Xct}$ presented in [Maietti and Rosolini, 2013c; Maietti et al., 2017] to a bi-adjunction from the 2-category **ElD** of elementary doctrines to the 2-category of exact categories **Xct**.

In the sections 2 and 3 we recall definitions and results on 2-monads, and on primary and existential doctrines as needed for the rest of the paper.

In section 4 we describe the existential completion. We introduce a 2-functor from the 2-category of primary doctrines to the 2-category of existential doctrines $\mathrm{E}\colon \mathbf{PD} \longrightarrow \mathbf{ED}$, and we prove that it is a left 2-adjoint to the forgetful functor $\mathrm{U}\colon \mathbf{ED} \longrightarrow \mathbf{PD}$.

In section 5 we prove that the 2-monad T_e constructed from the 2-adjunction is lax-idempotent and that the 2-category T_e-**Alg** is 2-equivalent to the 2-category **ED** of existential doctrines.

In section 6 we show that the existential completion preserves the elementary structure, and we use this result to extend the notion of exact completion to elementary doctrines.

2. A brief recap of two-dimensional monad theory

This section is devoted to the formal definition of 2-monad on a 2-category and a characterization of the definitions. We use 2-categorical pasting notation freely, following the usual convention of the topic as used extensively in [Blackwell et al., 1989], [Tanaka and Power, 2006a] and [Tanaka and Power, 2006b].

You can find all the details of the main results of this section in the works of Kelly and Lack [Kelly and Lack, 1997]. For a more general and complete description of these topics, and a generalization for the case of pseudo-monad, you can see the Ph.D thesis of Tanaka [Tanaka, 2004], the articles of Marmolejo [Marmolejo and Wood, 2008], [Marmolejo, 1999] and the work of Kelly [Kelly and Street, 1974]. Moreover we refer to [Borceux, 1994] and [Leinster, 2003] for all the standard results and notions about 2-category theory.

A ***2-monad*** (T, μ, η) on a 2-category $\mathcal{A}$ is a 2-functor $\mathrm{T}\colon \mathcal{A} \longrightarrow \mathcal{A}$ together 2-natural transformations $\mu\colon \mathrm{T}^2 \longrightarrow \mathrm{T}$ and $\eta\colon 1_{\mathcal{A}} \longrightarrow \mathrm{T}$ such that the following diagrams

$$\begin{array}{ccc} \mathrm{T}^3 & \xrightarrow{\mathrm{T}\mu} & \mathrm{T}^2 \\ {\scriptstyle \mu\mathrm{T}}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathrm{T}^2 & \xrightarrow[\mu]{} & \mathrm{T} \end{array}$$

$$\begin{array}{ccccc} \mathrm{T} & \xrightarrow{\eta\mathrm{T}} & \mathrm{T}^2 & \xleftarrow{\mathrm{T}\eta} & \mathrm{T} \\ & {\scriptstyle id}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle id} & \\ & & \mathrm{T} & & \end{array}$$

commute.

Let (T, μ, η) be a 2-monad on a 2-category $\boldsymbol{\mathcal{A}}$. A T-***algebra*** is a pair (A, a) where, A is an object of $\mathcal{A}$ and $a\colon \mathrm{T}A \longrightarrow A$ is a 1-cell such that the following diagrams commute

$$\begin{array}{ccc} \mathrm{T}^2A & \xrightarrow{\mathrm{T}a} & \mathrm{T}A \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle a} \\ \mathrm{T}A & \xrightarrow[a]{} & A \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \mathrm{T}A \\ & {\scriptstyle 1_A}\searrow & \downarrow{\scriptstyle a} \\ & & A. \end{array}$$

A ***lax*** T-***morphism*** from a T-algebra (A, a) to a T-algebra (B, b) is a pair $(f, \overline{f})$ where f is a 1-cell $f\colon A \longrightarrow B$ and $\overline{f}$ is a 2-cell

$$\begin{array}{ccc} \mathrm{T}A & \xrightarrow{\mathrm{T}f} & \mathrm{T}B \\ {\scriptstyle a}\downarrow & \Downarrow\overline{f} & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

which satisfies the following ***coherence*** conditions

$$
\begin{array}{ccc}
\mathrm{T}^2A & \xrightarrow{\mathrm{T}^2f} & TB \\
{\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_B} \\
\mathrm{T}A & \xrightarrow{\mathrm{T}f} & \mathrm{T}B \\
{\scriptstyle a}\downarrow & \Downarrow\overline{f} & \downarrow{\scriptstyle b} \\
A & \xrightarrow[f]{} & B
\end{array}
\quad = \quad
\begin{array}{ccc}
\mathrm{T}^2A & \xrightarrow{\mathrm{T}^2f} & \mathrm{T}B \\
{\scriptstyle \mathrm{T}a}\downarrow & \Downarrow\mathrm{T}\overline{f} & \downarrow{\scriptstyle \mathrm{T}b} \\
\mathrm{T}A & \xrightarrow{\mathrm{T}f} & \mathrm{T}B \\
{\scriptstyle a}\downarrow & \Downarrow\overline{f} & \downarrow{\scriptstyle b} \\
A & \xrightarrow[f]{} & B
\end{array}
$$

and

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\
\mathrm{T}A & \xrightarrow{\mathrm{T}f} & \mathrm{T}B \\
{\scriptstyle a}\downarrow & \Downarrow\overline{f} & \downarrow{\scriptstyle b} \\
A & \xrightarrow[f]{} & B
\end{array}
\quad = \quad
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
{\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle 1_B} \\
A & \xrightarrow[f]{} & B.
\end{array}
$$

The regions in which no 2-cell is written always commute by the naturality of η and μ, and are deemed to contain the identity 2-cell.

A lax morphism $(f, \overline{f})$ in which $\overline{f}$ is invertible is said T-***morphism***. And it is ***strict*** when $\overline{f}$ is the identity.

The category of T-algebras and lax T-morphisms becomes a 2-category T-$\mathbf{Alg}_l$, when provided with 2-cells the T-***transformations***. Recall from [Kelly and Lack, 1997] that a T-***transformation*** from $(f, \overline{f})\colon (A, a) \longrightarrow (B, b)$ to $(g, \overline{g})\colon (A, a) \longrightarrow (B, b)$ is a 2-cell $\alpha\colon f \Longrightarrow g$ in $\mathcal{A}$ which satisfies the following coherence condition

$$
\begin{array}{ccc}
\mathrm{T}A & \overset{\mathrm{T}f}{\underset{\mathrm{T}g}{\rightrightarrows}}\ {\scriptstyle \Downarrow \mathrm{T}\alpha} & \mathrm{T}B \\
{\scriptstyle a}\downarrow & \Downarrow\overline{g} & \downarrow{\scriptstyle b} \\
A & \xrightarrow[g]{} & B
\end{array}
\quad = \quad
\begin{array}{ccc}
\mathrm{T}A & \xrightarrow{\mathrm{T}f} & \mathrm{T}B \\
{\scriptstyle a}\downarrow & \Downarrow\overline{f} & \downarrow{\scriptstyle b} \\
A & \overset{f}{\underset{g}{\rightrightarrows}}\ {\scriptstyle \Downarrow \alpha} & B
\end{array}
$$

expressing compatibility of α with $\overline{f}$ and $\overline{g}$.

It is observed in [Kelly and Lack, 1997] that using this notion of T-morphism, one can express more precisely what it may mean that an action of a monad T on an object A is ***unique to within a unique isomorphism***. In our case it means that, given two action $a, a' \colon \mathrm{T}A \longrightarrow A$ there is a unique invertible 2-cell $\alpha \colon a \Rightarrow a'$ such that $(1_A, \alpha) \colon (A, a) \longrightarrow (A, a')$ is a morphism of T-algebras (in particular it is an isomorphism of T-algebras). In this case we will say that the T-***algebra structure is essentially unique***.

More precisely a 2-monad (T, μ, η) is said ***property-like***, if it satisfies the following conditions:

- for every T-algebra (A, a) and (B, b), and for every invertible 1-cell $f \colon A \longrightarrow B$ there exists a unique invertible 2-cell $\overline{f}$

$$\begin{array}{ccc} \mathrm{T}A & \xrightarrow{\mathrm{T}f} & \mathrm{T}B \\ {\scriptstyle a}\downarrow & \Downarrow \overline{f} & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

  such that $(f, \overline{f}) \colon (A, a) \longrightarrow (B, b)$ is a morphism of T-algebras;

- for every T-algebra (A, a) and (B, b), and for every 1-cell $f \colon A \longrightarrow B$ if there exists a 2-cell $\overline{f}$

$$\begin{array}{ccc} \mathrm{T}A & \xrightarrow{\mathrm{T}f} & \mathrm{T}B \\ {\scriptstyle a}\downarrow & \Downarrow \overline{f} & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

  such that $(f, \overline{f}) \colon (A, a) \longrightarrow (B, b)$ is a lax morphism of T-algebras, then it is the unique 2-cell with such property.

We conclude this section recalling a stronger property on a 2-monads (T, μ, η) on $\mathcal{A}$ which implies that T is property-like: a 2-monad (T, μ, η) is said ***lax-idempotent***, if for every T-algebras (A, a) and (B, b), and for every 1-cell $f \colon A \longrightarrow B$ there exists a unique 2-cell $\overline{f}$

$$\begin{array}{ccc} \mathrm{T}A & \xrightarrow{\mathrm{T}f} & \mathrm{T}B \\ {\scriptstyle a}\downarrow & \Downarrow \overline{f} & \downarrow{\scriptstyle b} \\ A & \xrightarrow[f]{} & B \end{array}$$

such that $(f, \overline{f}) \colon (A, a) \longrightarrow (B, b)$ is a lax morphism of T-algebras. In particular every lax-idempotent monad is property like. See [Kelly and Lack, 1997, Proposition 6.1].

3. Primary and existential doctrines

The notion of hyperdoctrine was introduced by F.W. Lawvere in a series of seminal papers [Lawvere, 1969, 1970]. We recall from [Maietti and Rosolini, 2013a] some definitions which will be useful in the following. The reader can find all the details about the theory of elementary and existential doctrines also in [Maietti and Rosolini, 2013a,b,c], and we refer to [Frey, 2014] for a detailed analysis of cocompletions of doctrines.

3.1. Definition. *Let $\mathcal{C}$ be a category with finite products. A* **primary doctrine** *is a functor $P\colon \mathcal{C}^{op} \longrightarrow$* **InfSL** *from the opposite of the category $\mathcal{C}$ to the category of inf-semilattices.*

3.2. Definition. *A primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow$* **InfSL** *is* **elementary** *if for every object A in $\mathcal{C}$ there exists an element δ_A in the fibre $P(A \times A)$ such that*

1. *the assignment*
$$\exists_{\langle id_A, id_A\rangle}(\alpha) := P_{pr_1}(\alpha) \wedge \delta_A$$
for α in the fibre $P(A)$ determines a left adjoint to $P_{\langle id_A, id_A\rangle}\colon P(A \times A) \longrightarrow P(A)$;

2. *for every morphism e of the form $\langle pr_1, pr_2, pr_2\rangle\colon X \times A \longrightarrow X \times A \times A$ in $\mathcal{C}$, the assignment*
$$\exists_e(\alpha) := P_{\langle pr_1, pr_2\rangle}(\alpha) \wedge P_{\langle pr_2, pr_3\rangle}(\delta_A)$$
for α in $P(X \times A)$ determines a left adjoint to $P_e\colon P(X \times A \times A) \longrightarrow P(X \times A)$.

3.3. Definition. *A primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow$* **InfSL** *is* **existential** *if, for every object A_1 and A_2 in $\mathcal{C}$, for any projection $pr_i\colon A_1 \times A_2 \longrightarrow A_i$, $i = 1, 2$, the functor*

$$P_{pr_i}\colon P(A_i) \longrightarrow P(A_1 \times A_2)$$

has a left adjoint $\exists_{pr_i}$, and these satisfy:

1. **Beck-Chevalley condition:** *for any pullback diagram*
$$\begin{array}{ccc} X' & \xrightarrow{pr'} & A' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[pr]{} & A \end{array}$$
with pr and pr' projections, for any β in $P(X)$ the canonical arrow
$$\exists_{pr'} P_{f'}(\beta) \leq P_f \exists_{pr}(\beta)$$
is an isomorphism;

2. **Frobenius reciprocity:** *for any projection $pr\colon X \longrightarrow A$, for any object α in $P(A)$ and β in $P(X)$, the canonical arrow*
$$\exists_{pr}(P_{pr}(\alpha) \wedge \beta) \leq \alpha \wedge \exists_{pr}(\beta)$$
in $P(A)$ is an isomorphism.

3.4. Remark. In an existential elementary doctrine, for every map $f: A \longrightarrow B$ in $\mathcal{C}$ the functor P_f has a left adjoint $\exists_f$ that can be computed as

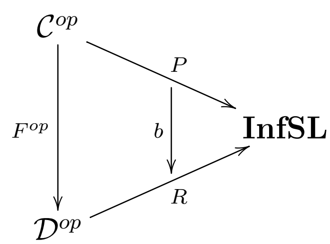
$$\exists_{pr_2}(P_{f \times id_B}(\delta_B) \wedge P_{pr_1}(\alpha))$$

for α in $P(A)$, where pr_1 and pr_2 are the projections from $A \times B$.

Observe that primary doctrines, elementary doctrines, and existential doctrines have a 2-categorical structure given as follow. We refer to [Maietti and Rosolini, 2013a,b,c] for more details.

3.5. Definition. *The class of primary doctrines* **PD** *is a 2-category, where:*

- **0-cells** *are primary doctrines;*
- **1-cells** *are pairs of the form* (F, b)



  such that $F: \mathcal{C} \longrightarrow \mathcal{D}$ *is a functor preserving products, and* $b: P \longrightarrow R \circ F^{op}$ *is a natural transformation such that the functor* $b_A: P(A) \longrightarrow RF(A)$ *preserves all the structure for every object* A *in* $\mathcal{C}$*, i.e.* b_A *preserves finite meets;*

- **2-cells** $\theta: (F, b) \Longrightarrow (G, c)$ *are natural transformations* $\theta: F \longrightarrow G$ *such that for every object* A *in* $\mathcal{C}$ *and for every* α *in* $P(A)$*, we have*

$$b_A(\alpha) \leq R_{\theta_A}(c_A(\alpha)).$$

Similarly we can define two 2-full 2-subcategories of **PD**: the 2-category of existential doctrines **ED**, and the 2-category of elementary doctrines **ElD**. In these cases one should require that the 1-cells preserve the appropriate structures, in particular 1-cells of **ED** are those pairs (F, b) such that b preserves the left adjoints along projections. The 1-cells of **ElD** are those pairs $(F, b): P \longrightarrow R$ such that for every object A in $\mathcal{C}$ we have

$$b_{A \times A}(\delta_A) = R_{\langle F\, pr_1, F\, pr_2 \rangle}(\delta_{FA})$$

where $\delta_A = \exists_{\Delta_A}(\top_A)$. See [Maietti and Rosolini, 2013a,b,c] for more details.

3.6. Examples. The following examples are discussed in [Lawvere, 1969].

1. Let $\mathcal{C}$ be a category with finite limits. The functor

$$\mathsf{Sub}_{\mathcal{C}}\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$$

assigns to an object A in $\mathcal{C}$ the poset $\mathsf{Sub}_{\mathcal{C}}(A)$ of subobjects of A in $\mathcal{C}$ and, for an arrow $B \xrightarrow{f} A$ the morphism $\mathsf{Sub}_{\mathcal{C}}(f)\colon \mathsf{Sub}_{\mathcal{C}}(A) \longrightarrow \mathsf{Sub}_{\mathcal{C}}(B)$ is given by pulling a subobject back along f. The fiber equalities are the diagonal arrows, so this is an elementary doctrine. Moreover it is existential if and only if the category $\mathcal{C}$ is regular. See [Hughes and Jacobs, 2003].

2. Consider a category $\mathcal{D}$ with finite products and weak pullbacks: the doctrine is given by the functor of weak subobjects

$$\Psi_{\mathcal{D}}\colon \mathcal{D}^{op} \longrightarrow \mathbf{InfSL}$$

where $\Psi_{\mathcal{D}}(A)$ is the poset reflection of the slice category $\mathcal{D}/A$, and for an arrow $B \xrightarrow{f} A$, the homomorphism $\Psi_{\mathcal{D}}(f)\colon \Psi_{\mathcal{D}}(A) \longrightarrow \Psi_{\mathcal{D}}(B)$ is given by a weak pullback of an arrow $X \xrightarrow{g} A$ with f. This doctrine is existential, and the existential left adjoint are given by the post-composition.

3. Let $\mathcal{H}$ be a theory in a first order language $\mathcal{L}$. We define a primary doctrine

$$LT_{\mathcal{H}}\colon \mathcal{C}_{\mathcal{H}}^{op} \longrightarrow \mathbf{InfSL}$$

where $\mathcal{C}_{\mathcal{H}}$ is the category of lists of variables and term substitutions:

- ***objects*** of $\mathcal{C}_{\mathcal{H}}$ are finite lists of variables $\vec{x} := (x_1, \dots, x_n)$, and we include the empty list $()$;
- a ***morphism*** from $(x_1, \dots, x_n)$ to $(y_1, \dots, y_m)$ is a substitution $[t_1/y_1, \dots, t_m/y_m]$ where the terms t_i are built in $\mathcal{L}$ on the variable $x_1, \dots, x_n$;
- the ***composition*** of two morphisms $[\vec{t}/\vec{y}]\colon \vec{x} \longrightarrow \vec{y}$ and $[\vec{s}/\vec{z}]\colon \vec{y} \longrightarrow \vec{z}$ is given by the substitution

$$[s_1[\vec{t}/\vec{y}]/z_k, \dots, s_k[\vec{t}/\vec{y}]/z_k]\colon \vec{x} \longrightarrow \vec{z}\,.$$

The functor $LT_{\mathcal{H}}\colon \mathcal{C}_{\mathcal{H}}^{op} \longrightarrow \mathbf{InfSL}$ sends $(x_1, \dots, x_n)$ in the class $LT_{\mathcal{H}}(x_1, \dots, x_n)$ of all well formed formulas in the context $(x_1, \dots, x_n)$. We say that $\psi \leq \phi$ where $\phi, \psi \in LT_{\mathcal{H}}(x_1, \dots, x_n)$ if $\psi \vdash_{\mathcal{H}} \phi$, and then we quotient in the usual way to obtain a partial order on $LT_{\mathcal{H}}(x_1, \dots, x_n)$. Given a morphism of $\mathcal{C}_{\mathcal{H}}$

$$[t_1/y_1, \dots, t_m/y_m]\colon (x_1, \dots, x_n) \longrightarrow (y_1, \dots, y_m)$$

the functor $LT_{\mathcal{H}[\vec{t}/\vec{y}]}$ acts as the substitution $LT_{\mathcal{H}[\vec{t}/\vec{y}]}(\psi(y_1,\dots,y_m)) = \psi[\vec{t}/\vec{y}]$.

The doctrine $LT_{\mathcal{H}}\colon \mathcal{C}_{\mathcal{H}}^{op} \longrightarrow \mathbf{InfSL}$ is elementary exactly when $\mathcal{H}$ has an equality predicate. For all the detail we refer to [Maietti and Rosolini, 2013b], and for the case of a many sorted first order theory we refer to [Pitts, 1995].

4. Existential completion

In this section we construct an existential doctrine $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$, starting from a primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$.

Let $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ be a fixed primary doctrine for the rest of the section, and let $\Lambda \subset \mathcal{C}_1$ be a subset of morphisms closed under pullbacks, compositions and such that it contains the identity morphisms.

For every object A of $\mathcal{C}$ consider the following preorder:

- the objects are pairs $(\, B \xrightarrow{g\in\Lambda} A\,,\ \alpha \in PB)$;
- $(\, B \xrightarrow{h\in\Lambda} A\,,\ \alpha \in PB) \leq (\, D \xrightarrow{f\in\Lambda} A\,,\ \gamma \in PD)$ if there exists $w\colon B \longrightarrow D$ such that

$$\begin{array}{ccc} & & B \\ & \overset{w}{\swarrow} & \downarrow h \\ D & \xrightarrow[f]{} & A \end{array}$$

  commutes and $\alpha \leq P_w(\gamma)$.

It is easy to see that the previous data give a preorder. Let $P^e(A)$ be the partial order obtained by identifying two objects when

$$(\, B \xrightarrow{h\in\Lambda} A\,,\ \alpha \in PB) \gtreqless (\, D \xrightarrow{f\in\Lambda} A\,,\ \gamma \in PD)$$

in the usual way. With abuse of notation we denote the equivalence class of an element in the same way.

Given a morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$, let $P^e_f(\, C \xrightarrow{g\in\Lambda} B\,,\ \beta \in PC)$ be the object

$$(\, D \xrightarrow{f^*g} A\,,\ P_{g^*f}(\beta) \in PD)$$

where

$$\begin{array}{ccc} D & \xrightarrow{f^*g} & A \\ {\scriptstyle g^*f}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & B \end{array}$$

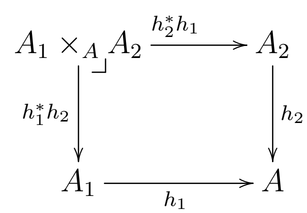
is a pullback because $g \in \Lambda$.

4.1. Proposition. *Let $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ be a primary doctrine. Then $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is a primary doctrine, in particular:*

(i) for every object A in $\mathcal{C}$, $P^e(A)$ is a inf-semilattice;

(ii) for every morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$, P^e_f is well-defined and it is an homomorphism of inf-semilattices.

Proof. (i) For every A we have the top element $(A \xrightarrow{id_A} A\ ,\ \top_A)$. Consider two elements $(A_1 \xrightarrow{h_1} A\ ,\ \alpha_1 \in PA_1)$ and $(A_2 \xrightarrow{h_2} A\ ,\ \alpha_2 \in PA_2)$. In order to define the greatest lower bound of the two objects consider a pullback



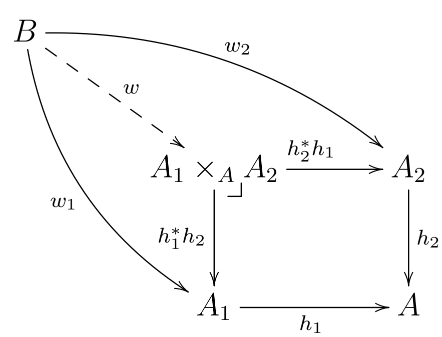
which exists because $h_1 \in \Lambda$ (and $h_2 \in \Lambda$). We claim that

$$(A_1 \times_A A_2 \xrightarrow{h_1 h_1^* h_2} A\ , P_{h_1^* h_2}(\alpha_1) \wedge P_{h_2^* h_1}(\alpha_2))$$

is such an infimum. It is easy to check that

$$(A_1 \times_A A_2 \xrightarrow{h_1 h_1^* h_2} A\ , P_{h_1^* h_2}(\alpha_1) \wedge P_{h_2^* h_1}(\alpha_2)) \leq (A_i \xrightarrow{h_i} A\ ,\ \alpha_i \in PA_i)$$

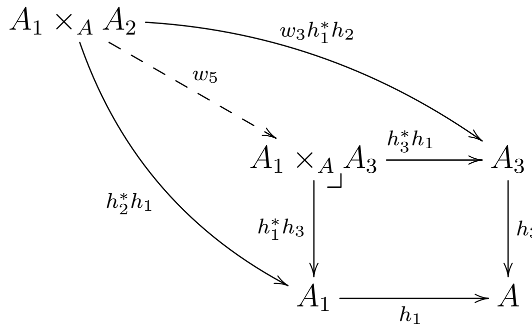
for $i = 1, 2$. Next consider $(B \xrightarrow{g} A\ ,\ \beta \in PB) \leq (A_i \xrightarrow{h_i} A\ ,\ \alpha_i \in PA_i)$ for $i = 1, 2$ and $g = h_i w_i$. Then there is a morphism $w\colon B \longrightarrow A_1 \times_A A_2$ such that



commutes and $\beta \leq P_{w_1}(\alpha_1) \wedge P_{w_2}(\alpha_2) = P_w(P_{h_1^* h_2}(\alpha_1) \wedge P_{h_2^* h_1}(\alpha_2))$. Observe that the infimum is well defined, since if, for example, we have

$$(A_2 \xrightarrow{h_2} A\ ,\ \alpha_2 \in PA_2) \gtreqless (A_3 \xrightarrow{h_3} A\ ,\ \alpha_3 \in PA_3)$$

then there exist $w_3\colon A_2 \longrightarrow A_3$ and $w_4\colon A_3 \longrightarrow A_2$ such that $h_3 w_3 = h_2$, $\alpha_2 \leq P_{w_3}(\alpha_3)$, $h_2 w_4 = h_3$ and $\alpha_3 \leq P_{w_4}(\alpha_2)$. Therefore there exists $w_5\colon A_1 \times_A A_2 \longrightarrow A_1 \times_A A_3$



such that

$$P_{h_1^* h_2}(\alpha_1) \wedge P_{h_2^* h_1}(\alpha_2) \leq P_{w_5}(P_{h_1^* h_3}(\alpha_1) \wedge P_{h_3^* h_1}(\alpha_3)).$$

Then we can conclude that

$$(A_1 \times_A A_2 \xrightarrow{h_1 h_1^* h_2} A \, , P_{h_1^* h_2}(\alpha_1) \wedge P_{h_2^* h_1}(\alpha_2)) \leq (A_1 \times_A A_3 \xrightarrow{h_1 h_1^* h_3} A \, , P_{h_1^* h_3}(\alpha_1) \wedge P_{h_3^* h_1}(\alpha_3)).$$

Using the same argument one can prove that

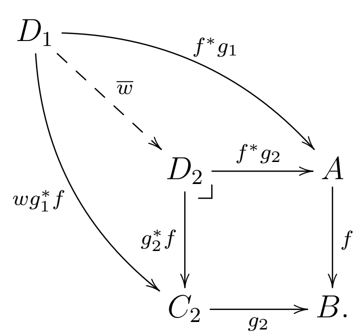
$$(A_1 \times_A A_3 \xrightarrow{h_1 h_1^* h_3} A \, , P_{h_1^* h_3}(\alpha_1) \wedge P_{h_3^* h_1}(\alpha_3)) \leq (A_1 \times_A A_2 \xrightarrow{h_1 h_1^* h_2} A \, , P_{h_1^* h_2}(\alpha_1) \wedge P_{h_2^* h_1}(\alpha_2)).$$

Therefore we can conclude that the infimum is well-defined.

(ii) We first prove that, for every morphism $f\colon A \longrightarrow B$, P^e_f is a morphism of preorders. By showing this, P^e_f will be a well-defined morphism of partial orders since we identify two elements $\overline{\alpha}$ and $\overline{\beta}$ of $P^e(B)$ if $\overline{\alpha} \gtreqless \overline{\beta}$. Consider $(C_1 \xrightarrow{g_1 \in \Lambda} B \, , \; \alpha_1 \in PC_1) \leq (C_2 \xrightarrow{g_2 \in \Lambda} B \, , \; \alpha_2 \in PC_2)$ with $g_2 w = g_1$ and $\alpha_1 \leq P_w(\alpha_2)$. We want to prove that

$$(D_1 \xrightarrow{f^* g_1} A \, , \; P_{g_1^* f}(\alpha_1) \in PD_1) \leq (D_2 \xrightarrow{f^* g_2} A \, , \; P_{g_2^* f}(\alpha_2) \in PD_1).$$

We can observe that $g_2 w g_1^* f = g_1 g_1^* f = f f^* g_1$. Then there exists a unique $\overline{w}\colon D_1 \longrightarrow D_2$ such that the following diagram commutes



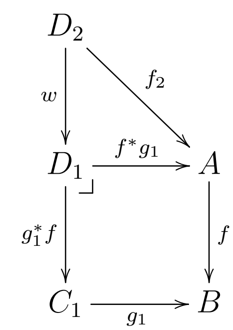
Moreover $P_{\overline{w}}(P_{g_2^* f}(\alpha_2)) = P_{g_1^* f}(P_w(\alpha_2)) \geq P_{g_1^* f}(\alpha_1)$, and it is easy to see that P^e_f preserves top elements. Finally it is straightforward to prove that $P^e_f(\overline{\alpha} \wedge \overline{\beta}) = P^e_f(\overline{\alpha}) \wedge P^e_f(\overline{\beta})$. ∎

4.2. Proposition. *Given a morphism* $f\colon A \longrightarrow B$ *of* Λ*, let*

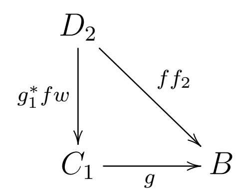
$$\exists^e_f(\,C \xrightarrow{h} A\,, \alpha \in PC) := (\,C \xrightarrow{fh} B\,, \alpha \in PC)$$

when $(\,C \xrightarrow{h} A\,, \alpha \in PC)$ *is in* $P^e(A)$*. Then* $\exists^e_f$ *is left adjoint to* P^e_f*.*

Proof. Let $\overline{\alpha} := (\,C_1 \xrightarrow{g_1} B\,,\ \alpha_1 \in PC_1)$ and $\overline{\beta} := (\,D_2 \xrightarrow{f_2} A\,,\ \beta_2 \in PD_2)$. Now we assume that $\overline{\beta} \leq P^e_f(\overline{\alpha})$. This means that

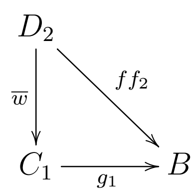


and $\beta_2 \leq P_w(P_{g_1^*f}(\alpha_1))$. Then we have

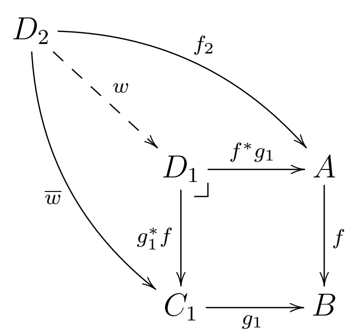


and $\beta_2 \leq P_{wg_1^*f}(\alpha_1)$. Then $\exists^e_f(\overline{\beta}) \leq \overline{\alpha}$.

Now assume $\exists^e_f(\overline{\beta}) \leq \overline{\alpha}$



with $\beta_2 \leq P_{\overline{w}}(\alpha_1)$. Then there exists $w\colon D_2 \longrightarrow D_1$ such that the following diagram commutes



and $\beta_1 \leq P_{\overline{w}}(\alpha_1) = P_w(P_{g_1^* f}(\alpha_1))$. Then we can conclude that $\overline{\beta} \leq P^e_f(\overline{\alpha})$. ∎

4.3. Theorem. *For every primary doctrine* $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$, $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *satisfies:*

(i) **Beck-Chevalley Condition***: for any pullback*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & A' \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X & \xrightarrow[g]{} & A \end{array}$$

with $g \in \Lambda$ (hence also $g' \in \Lambda$), for any $\overline{\beta} \in P^e(X)$ the following equality holds

$$\exists^e_{g'} P^e_{f'}(\overline{\beta}) = P^e_f \exists^e_g(\overline{\beta}).$$

(ii) **Frobenius Reciprocity***: for every morphism $f\colon X \longrightarrow A$ of Λ, for every element $\overline{\alpha} \in P^e(A)$ and $\overline{\beta} \in P^e(X)$, the following equality holds*

$$\exists^e_f(P^e_f(\overline{\alpha}) \wedge \overline{\beta}) = \overline{\alpha} \wedge \exists^e_f(\overline{\beta}).$$

Proof. (i) Consider the following pullback square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & A' \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X & \xrightarrow[g]{} & A \end{array}$$

where $g, g' \in \Lambda$, and let $\overline{\beta} := (\ C_1 \xrightarrow{h_1} X\ ,\ \beta_1 \in PC_1) \in P^e(X)$. Consider the following diagram

$$\begin{array}{ccccc} D_1 & \xrightarrow{f'^* h_1} & X' & \xrightarrow{g'} & A' \\ {\scriptstyle h_1^* f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f'} & \lrcorner & \downarrow{\scriptstyle f} \\ C_1 & \xrightarrow[h_1]{} & X & \xrightarrow[g]{} & A. \end{array}$$

Since the two square are pullbacks, then the big square is a pullback, and then

$$(\ D_1 \xrightarrow{g' f'^* h_1} A\ , P_{h_1^* f'}(\beta_1)) = (\ D_1 \xrightarrow{f^* g h_1} A\ , P_{g h_1^* f}(\beta_1))$$

and these are by definition

$$\exists^e_{g'} P^e_{f'}(\overline{\beta}) = P^e_f \exists^e_g(\overline{\beta}).$$

Therefore the Beck-Chevalley Condition is satisfied.

(ii) Consider a morphism $f\colon X \longrightarrow A$ of Λ, an element $\overline{\alpha} := (\, C_1 \xrightarrow{h_1} A\,,\ \alpha_1 \in PC_1)$ in $P^e(A)$, and an element $\overline{\beta} = (\, D_2 \xrightarrow{h_2} X\,,\ \beta_2 \in PD_2)$ in $P^e(X)$. Observe that the following diagram is a pullback

$$\begin{array}{ccccc} D_1 \times_X D_2 & \xrightarrow{(f^*h_1^*h_2)} & D_1 & \xrightarrow{h_1^*f} & C_1 \\ {\scriptstyle h_2^*(f^*h_1)}\downarrow & & \downarrow{\scriptstyle f^*h_1} & & \downarrow{\scriptstyle h_1} \\ D_2 & \xrightarrow[h_2]{} & X & \xrightarrow[f]{} & A \end{array}$$

and this means that

$$\exists^e_f(P^e_f(\overline{\alpha}) \wedge \overline{\beta}) = \overline{\alpha} \wedge \exists^e_f(\overline{\beta}).$$

Therefore the Frobenius Reciprocity is satisfied. ■

4.4. Remark. Observe that Proposition 4.2 and Theorem 4.3 just rely on the closure of the class Λ of morphisms under composition and pullback and on the values of functors in meet semilattices, while the finite product structure of $\mathcal{C}$ is not used.

We recall a useful lemma, which allows us to apply the previous construction on the class of projections, in order to obtain an existential doctrine in the sense of Definition 3.3.

4.5. Lemma. *Let $\mathcal{C}$ be a category with finite products. Then the class of projections is closed under pullbacks, compositions and it contains identities.*

Proof. It is direct to check that projections compose and that identities are projections. We show that this class is closed under pullbacks. Consider a projection $pr_A\colon A \times B \longrightarrow A$ and an arbitrary morphism $f\colon C \longrightarrow A$ of $\mathcal{C}$. It is direct to verify that the square

$$\begin{array}{ccc} A \times B \times C & \xrightarrow{pr_C} & C \\ {\scriptstyle \langle f\,pr_C, pr_B\rangle}\downarrow & & \downarrow{\scriptstyle f} \\ A \times B & \xrightarrow[pr_A]{} & A \end{array}$$

commutes and it is a pullback. ■

4.6. Corollary. *Let $P\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** be a primary doctrine. If Λ is the class of projections then the doctrine $P^e\colon \mathcal{C}^{op} \longrightarrow$ **InfSL** is existential.*

4.7. Remark. In the case that Λ is the class of the projections, then from a primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$, we can construct an existential doctrine $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ in the sense of Definition 3.3. Therefore the notion of existential doctrine can be generalized in the sense that an existential doctrine can be defined as a pair

$$(P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL} , \Lambda)$$

where $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is a primary doctrine and Λ is a class of morphisms of $\mathcal{C}$ closed by pullbacks, composition and identities, which satisfies the conditions of Theorem 4.3.

4.8. Remark. Let $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{Pos}_{\top}$ be a functor where $\mathbf{Pos}_{\top}$ is the category of posets with top element. We can apply the existential completion since we have not used the hypothesis that PA has infimum in the proofs; we have proved that if it has a infimum it is preserved by the completion. In this case we must avoid to require Frobenius reciprocity.

Since a poset of the category $\mathbf{Pos}_{\top}$ has a top element, one has an injection from the doctrine $P\colon \mathcal{C} \longrightarrow \mathbf{Pos}_{\top}$ into $P^e\colon \mathcal{C} \longrightarrow \mathbf{Pos}_{\top}$. From a logical point of view, one can think of extending a theory without existential quantification to one with that quantifier, requiring that the theorems of the previous theory are preserved.

We refer to [Hofstra, 2010] for a general presentation of constructions which freely add quantification to a fibration, and their applications in logic.

In the rest of the section we assume that the morphisms of Λ are all the projections, since by Lemma 4.5 this class is closed under pullbacks, compositions and it contains identities.

We define a 2-functor $\mathrm{E}\colon \mathbf{PD} \longrightarrow \mathbf{ED}$ from the 2-category of primary doctrines to the 2-category of existential doctrines, see Definition 3.5, which sends a primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ to the existential doctrine $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$. For all the standard notions about 2-category theory we refer to [Borceux, 1994; Leinster, 2003].

4.9. Proposition. *Consider the category* $\mathbf{PD}(P,R)$. *We define*

$$\mathrm{E}_{P,R}\colon \mathbf{PD}(P,R) \longrightarrow \mathbf{ED}(P^e,R^e)$$

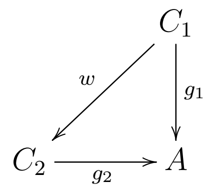
as follow:

- *for every 1-cell* (F,b), $\mathrm{E}_{P,R}(F,b) := (F,b^e)$, *where* $b^e_A\colon P^eA \longrightarrow R^eFA$ *sends an object* $(C \xrightarrow{g} A\ ,\ \alpha)$ *in the object* $(FC \xrightarrow{Fg} FA\ ,\ b_C(\alpha))$;
- *for every 2-cell* $\theta\colon (F,b) \Longrightarrow (G,c)$, $\mathrm{E}_{P,R}\theta$ *is essentially the same.*

With the previous assignment E *is a 2-functor.*

PROOF. We prove that $(F, b^e)\colon P^e \longrightarrow R^e$ is a 1-cell of $\mathbf{ED}(P^e, R^e)$. We first prove that for every $A \in \mathcal{C}$, b^e_A preserves the order.

If $(\ C_1 \xrightarrow{g_1} A\ ,\ \alpha_1) \leq (\ C_2 \xrightarrow{g_2} A\ ,\ \alpha_2)$, we have a morphism $w\colon C_1 \longrightarrow C_2$ such that the following diagram commutes



and $\alpha_1 \leq P_w(\alpha_2)$. Since b is a natural transformation, we have that $b_{C_1}P_w = R_{Fw}b_{C_2}$. Then we can conclude that $(\ FC_1 \xrightarrow{Fg_1} FA\ ,\ b_{C_1}(\alpha_1)) \leq (\ FC_2 \xrightarrow{Fg_2} FA\ ,\ b_{C_2}(\alpha_2))$ because $Fg_2Fw = Fg_1$ and $b_{C_1}(\alpha_1) \leq b_{C_1}P_w(\alpha_2) = R_{Fw}(b_{C_2}\alpha_2)$. Moreover, since F preserves products, we can conclude that b^e_A preserves inf.

One can prove that $b^e\colon P^e \longrightarrow R^eF^{op}$ is a natural transformation using the facts that F preserves products, which is needed to preserve projections. Moreover we can easily see that b^e preserves the left adjoints along projections. Then (F, b^e) is a 1-cell of $\mathbf{ED}$.

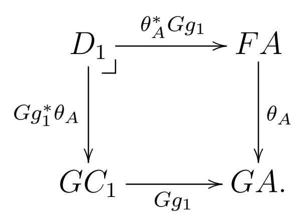
Now consider a 2-cell $\theta\colon (F, b) \Longrightarrow (G, c)$, and let $\overline{\alpha} = (\ C_1 \xrightarrow{g_1} A\ ,\ \alpha_1)$ be an object of $P^e(A)$. Then

$$b^e_A(\overline{\alpha}) = (\ FC_1 \xrightarrow{Fg_1} FA\ ,\ b_{C_1}(\alpha_1))$$

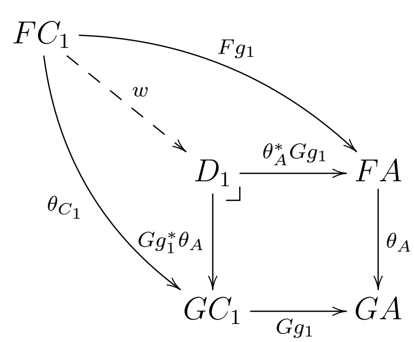
and

$$R^e_{\theta_A}c^e_A(\overline{\alpha}) = (\ D_1 \xrightarrow{\theta^*_A Gg_1} FA\ ,\ R_{Gg_1^*\theta_A}c_{C_1}(\alpha_1))$$

where



Now observe that since $\theta\colon F \longrightarrow G$ is a natural transformation, there exists a unique $w\colon FC_1 \longrightarrow D_1$ such that the diagram



commutes and then $b_{C_1}(\alpha_1) \leq R_{\theta_{C_1}} c_{C_1}(\alpha_1) = R_w R_{G g_1^* \theta_A} c_{C_1}(\alpha_1)$. Therefore we can conclude that $b^e_A(\overline{\alpha}) \leq R^e_{\theta_A} c^e_A(\overline{\alpha})$, and then $\theta\colon F \longrightarrow G$ can is a 2-cell $\theta\colon (F, b^e) \Longrightarrow (G, c^e)$, and $\mathrm{E}_{P,R}(\theta\gamma) = \mathrm{E}_{P,R}(\theta)\mathrm{E}_{P,R}(\gamma)$.

Finally one can prove that the following diagram commutes observing that for every $(F, b) \in \mathbf{PD}(P, R)$ and $(G, c) \in \mathbf{PD}(R, D)$, $(GF, c^e \star b^e) = (GF, (c \star b)^e)$

$$\begin{array}{ccc} \mathbf{PD}(P,R) \times \mathbf{PD}(R,D) & \xrightarrow{c_{PRD}} & \mathbf{PD}(P,D) \\ {\scriptstyle \mathrm{E}_{PR} \times \mathrm{E}_{RD}} \downarrow & & \downarrow {\scriptstyle \mathrm{E}_{PD}} \\ \mathbf{ED}(P^e,R^e) \times \mathbf{ED}(R^e,D^e) & \xrightarrow[c_{P^eR^eD^e}]{} & \mathbf{ED}(P^e,D^e) \end{array}$$

where c_{PRD} and $c_{P^eR^eD^e}$ denote the composition functors of the 2-categories **PD** and **ED**, and the same for the unit diagram. Therefore we can conclude that E is a 2-functor. ■

Now we prove the 2-functor $\mathrm{E}\colon \mathbf{PD} \longrightarrow \mathbf{ED}$ given by the assignment $\mathrm{E}(P) = P^e$ and by the functors $\mathrm{E}_{P,R}$ defined in Proposition 4.9, is left adjoint to the functor $\mathrm{U}\colon \mathbf{ED} \longrightarrow \mathbf{PD}$ which *forgets* the existential structure, i.e. it sends P to $\mathrm{U}(P) = P$.

4.10. Proposition. *Let $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ be a primary doctrine. Then*

$$(id_{\mathcal{C}}, \iota_P)\colon P \longrightarrow P^e$$

where $\iota_{PA}\colon PA \longrightarrow P^eA$ sends α into $(\, A \xrightarrow{id_A} A\,,\ \alpha)$ is a 1-cell of primary doctrines. Moreover the assignment

$$\eta\colon id_{\mathbf{PD}} \longrightarrow \mathrm{UE}$$

where $\eta_P := (id_{\mathcal{C}}, \iota_P)$, is a 2-natural transformation.

Proof. It is easy to prove that $\iota_{PA}\colon PA \longrightarrow P^eA$ preserves all the structures. For every morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$, it one can see that the following diagram commutes

$$\begin{array}{ccc} PB & \xrightarrow{P_f} & PA \\ {\scriptstyle \iota_{PB}} \downarrow & & \downarrow {\scriptstyle \iota_{PA}} \\ P^eB & \xrightarrow[P^e_f]{} & P^eA. \end{array}$$

Then we can conclude that $(id_{\mathcal{C}}, \iota_P)\colon P \longrightarrow P^e$ is a 1-cell of **PD** and it is a direct verification the proof the η is a 2-natural transformation. ■

4.11. Proposition. *Let* $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *be an existential doctrine. Then*

$$(id_{\mathcal{C}}, \zeta_P)\colon P^e \longrightarrow P$$

where $\zeta_{P A}\colon P^e A \longrightarrow PA$ *sends* $(C \xrightarrow{f} A\ ,\ \alpha)$ *in* $\exists_f(\alpha)$ *is a 1-cell of existential doctrines. Moreover the assignment*

$$\varepsilon\colon \mathrm{EU} \longrightarrow id_{\mathbf{ED}}$$

where $\varepsilon_P = (id_{\mathcal{C}}, \zeta_P)$, *is a 2-natural transformation.*

Proof. Suppose $(C_1 \xrightarrow{g_1} A\ ,\ \alpha_1) \leq (C_2 \xrightarrow{g_2} A\ ,\ \alpha_2)$, with $w\colon C_1 \longrightarrow C_2$, $g_2 w = g_1$ and $\alpha_1 \leq P_w(\alpha_2)$. Then by Beck-Chevalley we have the equality

$$\exists_{g_1^* g_2} P_{g_2^* g_1}(\alpha_2) = P_{g_1}\exists_{g_2}(\alpha_2)$$

and

$$\alpha_1 \leq P_w(\alpha_2) \leq P_w P_{g_2}\exists_{g_2}(\alpha_2) = P_{g_1}\exists_{g_2}(\alpha_2).$$

Then

$$\exists_{g_1}(\alpha_1) \leq \exists_{g_2}(\alpha_2)$$

since $\exists_{g_1} \dashv P_{g_1}$, and $\top_A = \zeta_A(A \xrightarrow{id_A} A\ ,\ \top_A)$. Now we prove the naturality of ζ_P. Let $f\colon A \longrightarrow B$ be a morphism of $\mathcal{C}$. Then the following diagram commutes

$$\begin{array}{ccc} P^e B & \xrightarrow{P^e_f} & P^e A \\ \downarrow{\scriptstyle \zeta_B} & & \downarrow{\scriptstyle \zeta_A} \\ PB & \xrightarrow[P_f]{} & PA \end{array}$$

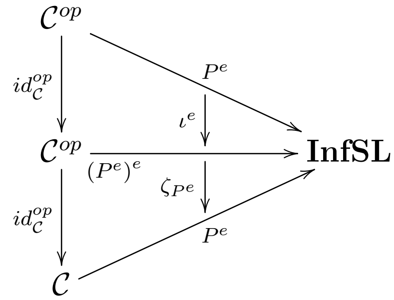
because it corresponds to the Beck-Chevalley condition. Moreover it is easy to see that ζ_P preserves left-adjoints. Then we an conclude that for every elementary existential doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$, ζ_P is a 1-cell of **ED**.

The proof of the naturality of ε is a routine verification. One must use the fact that we are working in **ED**, and then for every 1-cell (F, b), b preserves left-adjoints along the projections. ∎

4.12. Proposition. *For every primary doctrine* $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *we have*

$$\varepsilon_{P^e} \circ {\eta_P}^e = id_P\,.$$

PROOF. Consider the following diagram



and let $(C \xrightarrow{g} A\ ,\ \alpha \in PA)$ be an element of P^eA. Then

$$\iota_{P^e_A}(C \xrightarrow{g} A\ ,\ \alpha \in PC) = (A \xrightarrow{id_A} A\ ,\ (C \xrightarrow{g} A\ ,\ \alpha \in PC) \in P^eA)$$

and

$$\zeta_{P^e A}(A \xrightarrow{id_A} A\ ,\ (C \xrightarrow{g} A\ ,\ \alpha \in PC) \in P^eA) = \exists^e_{id_A}(C \xrightarrow{g} A\ ,\ \alpha \in PC).$$

By definition of $\exists^e$ we have

$$\exists^e_{id_A}(C \xrightarrow{g} A\ ,\ \alpha \in PC) = (C \xrightarrow{g} A\ ,\ \alpha \in PC).$$

Then we can conclude that for every $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$, we have $\varepsilon_{P^e} \circ \eta_P{}^e = id_{P^e}$. ∎

4.13. COROLLARY. $\varepsilon \mathrm{E} \circ \mathrm{E}\eta = id_{\mathrm{E}}$.

4.14. THEOREM. *The 2-functor* E *is 2-adjoint to the 2-functor* U.

PROOF. It is direct to verify that for every existential doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ we have

$$\varepsilon_P \circ \eta_P = id_P$$

and then $\mathrm{U}\varepsilon \circ \eta\mathrm{U} = id_{\mathrm{U}}$. Therefore, by Corollary 4.13, we can conclude that the 2-functor E is 2-adjoint to the forgetful functor U, where η is the unit of this 2-adjunction, and ε is the counit. ∎

5. The 2-monad T_e

In this section we construct a 2-monad $\mathrm{T}_e\colon \mathbf{PD} \longrightarrow \mathbf{PD}$, and we prove that every existential doctrine can be seen as an algebra for this 2-monad. Finally we show that the 2-monad T_e is lax-idempotent.

We define:

- $\mathrm{T}_e\colon \mathbf{PD} \longrightarrow \mathbf{PD}$ the 2-functor $\mathrm{T} = \mathrm{U} \circ \mathrm{E}$;
- $\eta\colon id_{\mathbf{PD}} \longrightarrow \mathrm{T}_e$ is the 2-natural transformation defined in Proposition 4.10;
- $\mu\colon \mathrm{T}_e^2 \longrightarrow \mathrm{T}_e$ is the 2-natural transformation $\mu = \mathrm{U}\varepsilon\mathrm{E}$.

5.1. PROPOSITION. T_e *is a 2-monad.*

PROOF. One can easily check that the following diagrams commute

$$\begin{array}{ccc} \mathrm{T}_e^3 & \xrightarrow{\mu \mathrm{T}_e} & \mathrm{T}_e^2 \\ {\scriptstyle \mathrm{T}_e\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ \mathrm{T}_e^2 & \xrightarrow[\mu]{} & \mathrm{T}_e \end{array}$$

$$\begin{array}{ccccc} id_{\mathbf{PD}} \circ \mathrm{T}_e & \xrightarrow{\eta \mathrm{T}_e} & \mathrm{T}_e^2 & \xleftarrow{\mathrm{T}_e \eta} & \mathrm{T}_e \circ id_{\mathbf{PD}} \\ & {\scriptstyle id}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle id} & \\ & & \mathrm{T}_e. & & \end{array}$$

■

5.2. PROPOSITION. *Let* $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *be an existential doctrine. Then* $(P, (id_{\mathcal{C}}, \zeta_P))$ *is a* T_e*-algebra, where* $\varepsilon_P = (id_{\mathcal{C}}, \zeta_P)\colon P^e \longrightarrow P$ *is the 1-cell of existential doctrines defined in Proposition 4.11, i.e.* $\zeta_{PA}\colon P^e A \longrightarrow PA$ *sends* $(\, C \xrightarrow{f} A\,,\ \alpha)$ *to* $\exists_f(\alpha)$*.*

PROOF. It is a direct verification. ■

5.3. PROPOSITION. *Let* $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *be an primary doctrine, and let* $(P, (F, a))$ *be a* T_e*-algebra. Then* $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *is existential,* $F = id_{\mathcal{C}}$ *and* $a = \zeta_P$*.*

PROOF. By the identity axiom for T_e-algebras, we know that F must be the identity functor, and $a_A \iota_A = id_{PA}$.

$$\begin{array}{ccc} P & \xrightarrow{\eta_P} & P^e \\ & {\scriptstyle id_P}\searrow & \downarrow{\scriptstyle (F,a)} \\ & & P. \end{array}$$

For every morphism $f\colon A \longrightarrow B$ of $\mathcal{C}$, where f is a projection, we claim that

$$\exists_f(\alpha) := a_B \exists^e_f \iota_A(\alpha)$$

is left adjoint to P_f. Let $\alpha \in PA$ and $\beta \in PB$, and suppose that $\alpha \leq P_f(\beta)$. Then we have that

$$(\, A \xrightarrow{f} B\,,\ \alpha) \leq (\, B \xrightarrow{id_B} B\,,\ \beta)$$

in $P^e B$ and $(\, A \xrightarrow{f} B\,,\ \alpha) = \exists^e_f(\, A \xrightarrow{id_A} A\,,\ \alpha)$. Therefore, by definition of ι, we have

$$\exists^e_f \iota_A(\alpha) \leq \iota_B(\beta).$$

Hence

$$a_B \exists^e_f \iota_A(\alpha) \leq a_B \iota_B(\beta) = \beta.$$

Now suppose that $\exists_f(\alpha) \leq \beta$. Then

$$a_B(A \xrightarrow{f} B , \alpha) \leq \beta$$

so

$$P_f a_B(A \xrightarrow{f} B , \alpha) \leq P_f(\beta).$$

By the naturality of a, we have

$$P_f a_B(A \xrightarrow{f} B , \alpha) = a_A P^e_f(A \xrightarrow{f} B , \alpha).$$

Now observe that $\iota_A(\alpha) = (A \xrightarrow{id_A} A , \alpha) \leq P^e_f(A \xrightarrow{f} B , \alpha)$. Therefore we have that

$$\alpha \leq P_f(\beta)$$

follows from the unit law and the naturality of a.

Now we prove that Beck-Chevalley holds. Consider the following pullback

$$\begin{array}{ccc} X' & \xrightarrow{g'} & A' \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ X & \xrightarrow[g]{} & A \end{array}$$

and $\alpha \in PX$. Then we have

$$\exists_{g'} P_{f'}(\alpha) = a_{A'} \exists^e_{g'} \iota_{X'}(P_{f'}\alpha) = a_{A'}(X' \xrightarrow{g'} A' , P_{f'}(\alpha)).$$

Observe that

$$(X' \xrightarrow{g'} A' , P_{f'}(\alpha)) = P^e_f(X \xrightarrow{g} A , \alpha)$$

and since a is a natural transformation, we have

$$a_{A'} P^e_f(X \xrightarrow{g} A , \alpha) = P_f a_A(X \xrightarrow{g} A , \alpha).$$

Finally we can conclude that Beck-Chevalley holds because

$$P_f \exists_g(\alpha) = P_f a_A \exists^e_g \iota_X(\alpha) = P_f a_A(X \xrightarrow{g} A , \alpha).$$

Hence

$$\exists_{g'} P_{f'}(\alpha) = P_f \exists_g(\alpha).$$

Now consider a projection $f\colon A \longrightarrow B$, and two elements $\beta \in PB$ and $\alpha \in PA$. We want to prove that the Frobenius reciprocity holds.

$$\exists_f(P_f(\beta)\wedge\alpha) = a_B\exists^e_f\iota_A(P_f(\beta)\wedge\alpha) = a_B(\ A \xrightarrow{f} B\ ,\ P_f(\beta)\wedge\alpha)$$

and

$$\beta\wedge\exists_f(\alpha) = a_B\iota_B(\beta)\wedge a_B(\ A \xrightarrow{f} B\ ,\ \alpha)$$

and

$$a_B\iota_B(\beta)\wedge a_B(\ A \xrightarrow{f} B\ ,\ \alpha) = a_B((\ B \xrightarrow{id_B} B\ ,\ \beta)\wedge(\ A \xrightarrow{f} B\ ,\ \alpha)).$$

We can observe that

$$a_B((\ B \xrightarrow{id_B} B\ ,\ \beta)\wedge(\ A \xrightarrow{f} B\ ,\ \alpha)) = a_B(\ A \xrightarrow{f} B\ ,\ P_f(\beta)\wedge\alpha)$$

and conclude that

$$\exists_f(P_f(\beta)\wedge\alpha) = \beta\wedge\exists_f(\alpha).$$

Therefore the primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is existential. Finally we can observe that

$$a_A(\ C \xrightarrow{g} A\ ,\alpha) = a_A\exists^e_g(\ C \xrightarrow{id_C} C\ ,\ \alpha) = a_A\exists^e_g\iota_C(\alpha) = \exists_g(\alpha).$$

Observe that all the previous calculations just depend on the naturality of a and its unit law. ∎

5.4. Proposition. *Let $(P,(id_{\mathcal{C}},\zeta_P))$ and $(R,(id_{\mathcal{D}},\zeta_R))$ be two* T_e*-algebras. Then every morphism $(F,b)\colon(P,(id_{\mathcal{C}},\zeta_P)) \longrightarrow (R,(id_{\mathcal{D}},\zeta_R))$ of* T_e*-algebras is a 1-cell of* **ED**. *Moreover every 1-cell of* **ED** *induces a morphism of* T_e*-algebras.*

Proof. Let $(F,b)\colon(P,(id_{\mathcal{C}},\zeta_P)) \longrightarrow (R,(id_{\mathcal{D}},\zeta_R))$ be a 1-cell of T_e-algebras. By definition of morphism of T_e-algebras, the following diagram commutes

$$\begin{array}{ccc} P^e & \xrightarrow{(F,b^e)} & R^e \\ {\scriptstyle (id_{\mathcal{C}},\zeta_P)}\big\downarrow & & \big\downarrow{\scriptstyle (id_{\mathcal{D}},\zeta_R)} \\ P & \xrightarrow[(F,b)]{} & R. \end{array}$$

Then for every object $(\ C \xrightarrow{g} A\ ,\ \alpha\in PC)$ of P^eA we have

$$\exists^R_{Fg}b_C(\alpha) = b_A\exists^P_g(\alpha)$$

and this means that for every projection $g\colon C \longrightarrow A$ the following diagram commutes

$$\begin{array}{ccc} PC & \xrightarrow{\exists^P_g} & PA \\ {\scriptstyle b_C}\big\downarrow & & \big\downarrow{\scriptstyle b_A} \\ RFC & \xrightarrow[\exists^R_{Fg}]{} & RFA. \end{array}$$

Similarly one can prove that every 1-cell of **ED** induces a morphism of T_e-algebras. ∎

5.5. Corollary. *We have the following isomorphism*

$$\mathrm{T}_e\text{-}\mathbf{Alg} \cong \mathbf{ED}$$

Proof. It follows from Proposition 5.4 and Proposition 5.3. ∎

Now we are going to prove that the 2-monad $\mathrm{T}_e\colon \mathbf{PD} \longrightarrow \mathbf{PD}$ is lax-idempotent. This means that the 2-monad T_e has both uniqueness of algebra structure and uniqueness of morphism structure, and then we can say that the *existential* structure for a doctrine is a *property* of the doctrine.

5.6. Theorem. *Let* $(P,(id_{\mathcal{C}},\zeta_P))$ *and* $(R,(id_{\mathcal{D}},\zeta_R))$ *be* T_e*-algebras, and let* $(F,b)\colon P \longrightarrow R$ *be a 1-cell of* **PD**. *Then* $((F,b),id_F)$ *is a lax-morphism of algebras, and the 2-cell of primary doctrines* $id_F\colon (id_{\mathcal{D}},\zeta_R)(F,b^e) \Longrightarrow (F,b)(id_{\mathcal{C}},\zeta_P)$ *is the unique 2-cell making* $((F,b),id_F)$ *a lax-morphism. Therefore the 2-monad* $\mathrm{T}_e\colon \mathbf{PD}\longrightarrow\mathbf{PD}$ *is lax-idempotent.*

Proof. Consider the following diagram where, following the notation of Proposition 4.11, $\varepsilon_P=(id_{\mathcal{C}},\zeta_P)$ and $\varepsilon_R=(id_{\mathcal{D}},\zeta_R)$

$$\begin{array}{ccc} P^e & \xrightarrow{(F,b^e)} & R^e \\ {\scriptstyle \varepsilon_P}\downarrow & \Downarrow id_F & \downarrow{\scriptstyle \varepsilon_R} \\ P & \xrightarrow[(F,b)]{} & R. \end{array}$$

We must prove that for every object A of $\mathcal{C}$ and every $(\,C \xrightarrow{f} A\,,\ \alpha)$ in P^eA

$$\exists^R_{Ff} b_C(\alpha) \leq b_A \exists^P_f(\alpha)$$

but the previous property holds if and only if

$$b_C(\alpha) \leq R_{Ff} b_A \exists^P_f(\alpha) = b_C P_f \exists^P_f(\alpha)$$

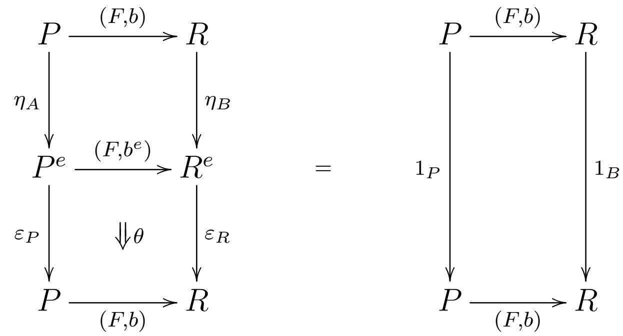
and this holds since $\alpha \leq P_f\exists^P_f(\alpha)$.

Finally it is easy to see that $id_F\colon \varepsilon_R(F,b^e) \Longrightarrow (F,b)\varepsilon_P$ satisfies the coherence conditions for lax-T_e-morphisms.

Now suppose there exists another 2-cell $\theta\colon \varepsilon_R(F,b^e) \Longrightarrow (F,b)\varepsilon_P$ such that $((F,b),\theta)$ is a lax-morphism

$$\begin{array}{ccc} P^e & \xrightarrow{(F,b^e)} & R^e \\ {\scriptstyle \varepsilon_P}\downarrow & \Downarrow \theta & \downarrow{\scriptstyle \varepsilon_R} \\ P & \xrightarrow[(F,b)]{} & R. \end{array}$$

Then it must satisfy the following condition



and this means that $\theta = id_F$. ∎

5.7. Remark. Observe that the family $\lambda_P \colon id_{P^{ee}} \Longrightarrow \eta_{P^e}\mu_P$ defined as $\lambda_P := id_{\mathcal{C}}$ is a 2-cell in **ED**.

It is clearly a natural transformation. We must check that for every $\alpha \in (P^e)^e A$

$$\alpha \leq \iota_{P^e A}\zeta_{P^e A}(\alpha).$$

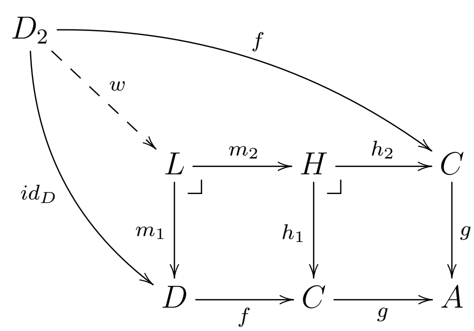
Let $\alpha := (\ C \xrightarrow{g} A\ ,\ (\ D \xrightarrow{f} C\ ,\ \beta \in PD))$. Then we have

$$\iota_{P^e A}\zeta_{P^e A}(\alpha) = \iota_{P^e A}(\ D \xrightarrow{gf} A\ ,\ \beta \in PD) = (\ A \xrightarrow{id_A} A\ ,(\ D \xrightarrow{gf} A\ ,\ \beta \in PD)).$$

Now we want to prove that

$$(\ D \xrightarrow{f} C\ ,\ \beta \in PD) \leq P^e_g(\ D \xrightarrow{gf} A\ ,\ \beta \in PD).$$

To see this inequality we can observe that the following diagram commutes



since every square is a pullback, hence $P_w(P_{m_1}(\beta)) = \beta$.

Moreover one can check that 2-cell $\lambda \colon id_{T_e^2} \longrightarrow \eta T_e \mu$ is a modification. See [Borceux, 1994] for the formal definition of modifications.

Finally, observe that the 2-cell μ is left adjoint to $\eta \mathrm{T}_e$, where the unit of the adjunction is λ and the counit is the identity. This result follows from the fact that for every $P \colon \mathcal{C}^{op} \longrightarrow$ **InfSL**, the first component of the 1-cells μ_P, $\eta \mathrm{T}_e$ are the identity functor, and since λ_P is the identity natural transformation, when we look at the conditions of adjoint 1-cell in the 2-category **Cat**, we can observe that all the components are identities.

5.8. Remark. By Proposition 5.3 and Proposition 5.3 we have that a doctrine is existential if and only if it has a structure of T_e-algebra, but we can show that this also holds in the general setting of pseudo-algebras: if $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is a primary doctrine, and if $(P,(F,a))$ is a pseudo-T_e-algebra, then the doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is existential (the converse holds since strict algebras are a particular case of pseudo-algebras).

We refer to [Lack, 2010; Tanaka, 2004] for all the details about the formal definition of pseudo-algebras, and their properties.

If $(P,(F,a))$ is a pseudo-algebra, then there exists an invertible 2-cell

$$\begin{array}{ccc} P & \xrightarrow{\eta_A} & P^e \\ & \searrow^{id_P} \; \not\swarrow a_\eta & \big\downarrow (F,a) \\ & & P \end{array}$$

and by definition, it is a natural transformation $a_\eta\colon F \longrightarrow id_{\mathcal{C}}$, and for every $A \in \mathcal{C}$ and $\alpha \in PA$ we have $a_A \iota_A(\alpha) = P_{a_{\eta_A}}(\alpha)$.

Now consider a morphism $f\colon A \longrightarrow B$ in $\mathcal{C}$ and $\alpha \in PA$. We define

$$\exists_f(\alpha) := P_{{a_{\eta_A}}^{-1}} a_B \exists^e_f \iota_A(\alpha).$$

Using the same argument of Proposition 5.3 we can conclude that the primary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is existential.

5.9. Example. Consider the fragment $\mathcal{L}$ of first order intuitionistic logic with logical symbols $\top$ and $\wedge$, and let $\mathcal{L}_\exists$ be the fragment whose logical symbols are $\top$, $\wedge$ and $\exists$. Then we have that, following the notation used in Example 3.6, the syntactic doctrine

$$LT_{\mathcal{L}_\exists}\colon \mathcal{C}^{op}_{\mathcal{L}_\exists} \longrightarrow \mathbf{InfSL}$$

is isomorphic to the existential completion

$${LT_{\mathcal{L}}}^e\colon \mathcal{C}^{op}_{\mathcal{L}} \longrightarrow \mathbf{InfSL}$$

of the primary doctrine $LT_{\mathcal{L}}\colon \mathcal{C}^{op}_{\mathcal{L}} \longrightarrow \mathbf{InfSL}$.

Observe that we have this isomorphism because the operation of extending a language with the existential quantification is a free operation on the logic, so by the known equivalence between doctrines and logic given by the internal language, see for example [Pitts, 1995], and since by Theorem 4.14 the existential completion is a free completion, these two doctrines must be isomorphic.

More specific categorical definitions of internal language are in [Maietti, 2005; Maietti et al., 2005].

6. Exact completion for elementary doctrines

It is proved in [Maietti and Rosolini, 2013c] that there is a biadjunction **EED** $\rightarrow$ **Xct** between the 2-category of elementary existential doctrines and the 2-category of exact categories given by the composition of the following 2-functors: the first is the left biadjoint to the inclusion of **CEED** into **EED**, see [Maietti and Rosolini, 2013c, Theorem 3.1]. The second is the biequivalence between **CEED** and the 2-category **LFS** of categories with finite limits and a proper stable factorization system, see [Hughes and Jacobs, 2003]. The third is provided in [Kelly, 1992], where it is proved that the inclusion of the 2-category **Reg** of regular categories (with exact functors) into **LFS** has a left biadjoint. The last functor is the biadjoint to the forgetful functor from the 2-category **Xct** into **Reg**, see [Carboni and Vitale, 1998].

In this section we combine these results with the existential completion for elementary doctrines, by proving the following result.

6.1. PROPOSITION. *The elementary structure is preserved by the existential completion, in the sense that if* $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *is an elementary doctrine, then* $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *is an elementary existential doctrine.*

Let $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ be an elementary doctrine, and consider its existential completion $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$. Given two objects A and C of $\mathcal{C}$ we define

$$\exists^e_{\Delta_A \times id_C}\colon P^e(A\times C) \longrightarrow P^e(A\times A\times C)$$

on $\overline{\alpha} = (\ A\times C\times D \xrightarrow{pr} A\times C\ ,\ \alpha\in P(A\times C\times D))$ as

$$\exists^e_{\Delta_A\times id_C}(\overline{\alpha}) := (\ A\times A\times C\times D \xrightarrow{pr} A\times A\times C\ ,\ \exists_{\Delta_A\times id_{C\times D}}(\alpha)\in P(A\times A\times C\times D)).$$

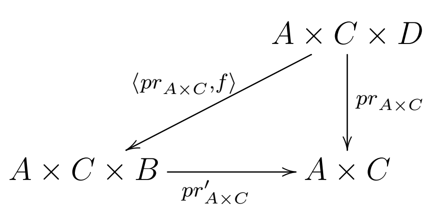
6.2. REMARK. We can prove that $\exists^e_{\Delta_A\times id_C}$ is a well defined functor for every A and C: consider two elements of $P^e(A\times C)$

$$\overline{\alpha} = (\ A\times C\times D \xrightarrow{pr} A\times C\ ,\ \alpha\in P(A\times C\times D))$$

and

$$\overline{\beta} = (\ A\times C\times B \xrightarrow{pr'} A\times C\ ,\ \beta\in P(A\times C\times B))$$

and suppose that $\overline{\alpha} \leq \overline{\beta}$. By definition there exists a morphism $f\colon A\times C\times D \longrightarrow B$ such that the following diagram commutes



and $\alpha \leq P_{\langle pr_{A\times C}, f\rangle}(\beta)$. Since the doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ is elementary we have

$$\beta \leq P_{\Delta_A \times id_{C\times B}} \exists_{\Delta_A \times id_{C\times B}}(\beta)$$

and then

$$\alpha \leq P_{\langle pr_{A\times C}, f\rangle}(P_{\Delta_A \times id_{C\times B}} \exists_{\Delta_A \times id_{C\times B}}(\beta)).$$

Now observe that $(\Delta_A \times id_{C\times B})(\langle pr_{A\times C}, f\rangle) = (\langle pr_{A\times A\times C}, f\, pr_{A\times C\times D}\rangle)(\Delta_A \times id_{C\times D})$, and this implies

$$\alpha \leq P_{\Delta_A \times id_{C\times D}}(P_{\langle pr_{A\times A\times C}, f\, pr_{A\times C\times D}\rangle} \exists_{\Delta_A \times id_{C\times B}}(\beta)).$$

Therefore we conclude

$$\exists_{\Delta_A \times id_{C\times D}}(\alpha) \leq P_{\langle pr_{A\times A\times C}, f\, pr_{A\times C\times D}\rangle} \exists_{\Delta_A \times id_{C\times B}}(\beta).$$

It is easy to observe that the last inequality implies

$$\exists^e_{\Delta_A \times id_C}(\overline{\alpha}) \leq \exists^e_{\Delta_A \times id_C}(\overline{\beta}).$$

6.3. Proposition. *With the notation used before the functor*

$$\exists^e_{\Delta_A \times id_C}\colon P^e(A\times C) \longrightarrow P^e(A\times A\times C)$$

is left adjoint to the functor

$$P^e_{\Delta_A \times id_C}\colon P^e(A\times A\times C) \longrightarrow P^e(A\times C).$$

Proof. Consider an element $\overline{\alpha} \in P^e(A\times C)$,

$$\overline{\alpha} := (\ A\times C\times B \xrightarrow{pr} A\times C\ ,\ \alpha \in P(A\times C\times B))$$

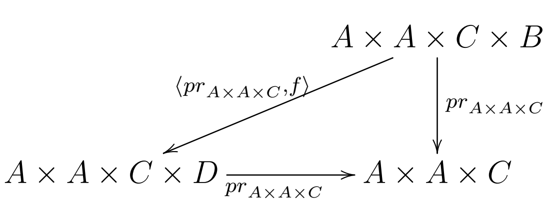
and an element $\overline{\beta} \in P^e(A\times A\times C)$,

$$\overline{\beta} := (\ A\times A\times C\times D \xrightarrow{pr'} A\times A\times C\ ,\ \beta \in P(A\times A\times C\times D))$$

and suppose that

$$\exists^e_{\Delta_A \times id_C}(\overline{\alpha}) \leq \overline{\beta}$$

which means that there exists $f\colon A\times A\times C\times B \longrightarrow D$



such that $\exists_{\Delta_A\times id_{C\times B}}(\alpha)\leq P_{\langle pr_{A\times A\times C},f\rangle}(\beta)$. Therefore we have

$$\alpha\leq P_{\Delta_A\times id_{C\times B}}P_{\langle pr_{A\times A\times C},f\rangle}(\beta)$$

and since

$$(\langle pr_{A\times A\times C},f\rangle)(\Delta_A\times id_{C\times B})=(\Delta_A\times id_{C\times D})\,pr_{A\times C\times D}(\langle pr_{A\times A\times C},f\rangle)(\Delta_A\times id_{C\times B})$$

we can conclude that

$$\alpha\leq P_{pr_{A\times C\times D}(\langle pr_{A\times A\times C},f\rangle)(\Delta_A\times id_{C\times B})}(P_{\Delta_A\times id_{C\times D}}(\beta)).$$

Then

$$\overline{\alpha}\leq P^e_{\Delta_A\times id_C}(\overline{\beta})$$

because

$$P^e_{\Delta_A\times id_C}(\overline{\beta})=(\,A\times C\times D\xrightarrow{pr_{A\times C}}A\times C\,,\;P_{\Delta_A\times id_{C\times D}}(\beta)).$$

In the same way we can prove that $\overline{\alpha}\leq P^e_{\Delta_A\times id_C}(\overline{\beta})$ implies $\exists^e_{\Delta_A\times id_C}(\overline{\alpha})\leq\overline{\beta}$. ∎

6.4. Proposition. *Let δ^e_A be $\exists^e_{\Delta_A}(\overline{\top}_A)$. For every element $\overline{\alpha}$ of the fibre $P^e(A\times C)$ we have*

$$\exists^e_{\Delta_A\times id_C}(\overline{\alpha})=P^e_{\langle pr_2,pr_3\rangle}(\overline{\alpha})\wedge P^e_{\langle pr_1,pr_2\rangle}(\delta^e_A)$$

where pr_i, $i=1,2,3$, are the projections from $A\times A\times C$. In particular we have

$$\exists^e_{\Delta_A}(\overline{\beta})=P^e_{pr_2}(\overline{\beta})\wedge\delta^e_A$$

for every element $\overline{\beta}$ of the fibre $P^e(A\times A)$.

Proof. Let $\overline{\alpha}=(\,A\times C\times D\xrightarrow{pr_{A\times C}}A\times C\,,\;\alpha\in P(A\times C\times D))$ be an element of the fibre $P^e(A\times C)$. Observe that $P^e_{\langle pr_2,pr_3\rangle}(\overline{\alpha})$ is the element

$$P^e_{\langle pr_2,pr_3\rangle}(\overline{\alpha})=(\,A\times A\times C\times D\xrightarrow{pr_{A\times A\times C}}A\times A\times C\,,\;P_{\langle pr'_2,pr'_3,pr'_4\rangle}(\alpha))$$

where $\langle pr'_2,pr'_3,pr'_4\rangle\colon A\times A\times C\times D\longrightarrow A\times C\times D$. Moreover we have that

$$P^e_{\langle pr_1,pr_2\rangle}(\delta^e_A)=(\,A\times A\times C\xrightarrow{id}A\times A\times C\,,\;P_{\langle pr_1,pr_2\rangle}(\delta_A)).$$

Therefore $P^e_{\langle pr_2,pr_3\rangle}(\overline{\alpha})\wedge P^e_{\langle pr_1,pr_2\rangle}(\delta^e_A)$ is the element

$$(\,A\times A\times C\times D\xrightarrow{pr_{A\times A\times C}}A\times A\times C\,,\;P_{\langle pr'_2,pr'_3,pr'_4\rangle}(\alpha)\wedge P_{\langle pr'_1,pr'_2\rangle}(\delta_A)).$$

Note that $P_{\langle pr'_2,pr'_3,pr'_4\rangle}(\alpha)\wedge P_{\langle pr'_1,pr'_2\rangle}(\delta_A)=\exists_{\Delta_A\times id_{C\times D}}(\alpha)$ because the doctrine P is elementary, so we can conclude that

$$\exists^e_{\Delta_A\times id_C}(\overline{\alpha})=P^e_{\langle pr_2,pr_3\rangle}(\overline{\alpha})\wedge P^e_{\langle pr_1,pr_2\rangle}(\delta^e_A).$$

∎

6.5. Corollary. *For every elementary doctrine* $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *, the existential completion* $P^e\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ *is elementary and existential.*

6.6. Example. Using the same argument of Example 5.9, one can prove that the syntactic doctrine

$$LT_{\mathcal{L}_{=,\exists}}\colon \mathcal{C}^{op}_{\mathcal{L}_{=,\exists}} \longrightarrow \mathbf{InfSL}$$

is the existential completion of the syntactic doctrine

$$LT_{\mathcal{L}_{=}}\colon \mathcal{C}^{op}_{\mathcal{L}_{=}} \longrightarrow \mathbf{InfSL}$$

where $\mathcal{L}_{=,\exists}$ is the Regular fragment of first order intuitionistic logic, and $\mathcal{L}_{=}$ is the Horn fragment.

We combine the existential completion for elementary doctrines with the completions stated at the begin of this section, obtaining a general version of the exact completion described in [Maietti et al., 2017; Maietti and Rosolini, 2013c]. We can summarise this operation with the following diagram

$$\mathbf{ElD} \longrightarrow \mathbf{EED} \longrightarrow \mathbf{CEED} \longrightarrow \mathbf{LFS} \longrightarrow \mathbf{Reg} \longrightarrow \mathbf{Xct}.$$

It is proved in *loc.cit.* that given an elementary existential doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$ the completion $\mathbf{EED} \to \mathbf{Xct}$ produces an exact category denoted by $\mathcal{T}_P$ and this category is defined following the same idea used to define a topos from a tripos. See [Hyland et al., 1980; Pitts, 2002].

We conclude giving a complete description of the exact category $\mathcal{T}_{P^e}$ obtained from an elementary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$.

Given an elementary doctrine $P\colon \mathcal{C}^{op} \longrightarrow \mathbf{InfSL}$, consider the category $\mathcal{T}_{P^e}$, called ***exact completion of the elementary doctrine*** P, whose

objects are pair (A, ρ) such that ρ is in $P(A \times A \times C)$ for some C and satisfies:

1. there exists a morphism $f\colon A \times A \times C \longrightarrow C$ such that

$$\rho \leq P_{\langle pr_2, pr_1, f\rangle}(\rho)$$

   in $P(A \times A \times C)$ where $pr_1, pr_2\colon A \times A \times C \longrightarrow A$;

2. there exists a morphism $g\colon A \times A \times A \times C \longrightarrow C$ such that

$$P_{\langle pr_1, pr_2, pr_4\rangle}(\rho) \wedge P_{\langle pr_2, pr_3, pr_4\rangle}(\rho) \leq P_{\langle pr_1, pr_3, g\rangle}(\rho)$$

   where $pr_1, pr_2, pr_3\colon A \times A \times A \times C \longrightarrow A$;

a morphism $\phi\colon (A, \rho) \longrightarrow (B, \sigma)$, where $\rho \in P(A \times A \times C)$ and $\sigma \in P(B \times B \times D)$, is an object ϕ of $P(A \times B \times E)$ for some E such that

1. there exists a morphism $\langle f_1, f_2\rangle\colon A\times B\times E \longrightarrow C\times D$ such that
$$\phi \leq P_{\langle pr_1, pr_1, f_1\rangle}(\rho)\wedge P_{\langle pr_2, pr_2, f_2\rangle}(\sigma)$$
where the pr_i's are the projections from $A\times B\times E$;
2. there exists a morphism $h\colon A\times A\times B\times C\times E\longrightarrow E$ such that
$$P_{\langle pr_1, pr_2, pr_4\rangle}(\rho)\wedge P_{\langle pr_2, pr_3, pr_5\rangle}(\phi)\leq P_{\langle pr_1, pr_3, h\rangle}(\phi)$$
where the pr_i's are the projections from $A\times A\times B\times C\times E$;
3. there exists a morphism $k\colon A\times B\times B\times D\times E\longrightarrow E$ such that
$$P_{\langle pr_2, pr_3, pr_4\rangle}(\sigma)\wedge P_{\langle pr_1, pr_2, pr_5\rangle}(\phi)\leq P_{\langle pr_1, pr_3, k\rangle}(\phi)$$
where the pr_i's are the projections from $A\times B\times B\times D\times E$;
4. there exists a morphism $l\colon A\times B\times B\times E\longrightarrow D$ such that
$$P_{\langle pr_1, pr_2, pr_4\rangle}(\phi)\wedge P_{\langle pr_1, pr_3, pr_4\rangle}(\phi)\leq P_{\langle pr_2, pr_3, l\rangle}(\sigma)$$
where the pr_i's are the projections from $A\times B\times B\times E$;
5. there exists a morphism $\langle g_1, g_2\rangle\colon A\times C\longrightarrow B\times E$ such that
$$P_{\langle pr_1, pr_1, pr_2\rangle}(\rho)\leq P_{\langle pr_1, g_1, g_2\rangle}(\phi)$$
where the pr_i's are the projections from $A\times C$.

The composition of two morphisms is defined following the same structure of the tripos to topos.

Observe that, in particular in point 5 of the previous construction, the existential quantifiers disappear, because the usual last condition of the tripos-to-topos construction, see [Maietti and Rosolini, 2013c; Pitts, 2002], which is the requirement $P_{\langle pr_1, pr_1\rangle}(\rho)\leq \exists_{pr_2}(\phi)$, in the case P is of the form P^e, is equivalent to the condition 5 of our previous construction because of the definition of the order in the fibre $P^e(A)$.

Finally we conclude with the following theorem which generalized the exact completion for an elementary existential doctrine to an arbitrary elementary doctrine.

6.7. Theorem. *The 2-functor* $\mathbf{Xct}\to\mathbf{ElD}$ *that takes an exact category to the elementary doctrine of its subobjects has a left biadjoint which associates the exact category* $\mathcal{T}_{P^e}$ *to an elementary doctrine* $P\colon\mathcal{C}^{op}\longrightarrow\mathbf{InfSL}$.

6.8. Example. Combining Example 6.6 and [Maietti et al., 2017, Theorem 4.7], we have that an instance of the previous construction is provided by the exact completion of existential m-variational doctrines $\mathbf{Ex}_{(LT_{\mathcal{L}_{=,\exists}})_{cx}}$ defined in [Maietti et al., 2017], which is isomorphic to the exact category $\mathcal{T}_{(LT_{\mathcal{L}_=})^e}$.

Non-syntactic examples of existential completions and exact categories built from them are left to future work.

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