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A CATEGORICAL REDUCTION SYSTEM FOR LINEAR LOGIC

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ABSTRACT. Diagram chasing is not an easy task. The coherence holds in a generalized sense if we have a mechanical method to judge whether given two morphisms are equal to each other. A simple way to this end is to reform a concerned category into a calculus, where the instructions for the diagram chasing are given in the form of rewriting rules. We apply this idea to the categorical semantics of the linear logic. We build a calculus directly on the free category of the semantics. It enables us to perform diagram chasing as essentially one-way computations led by the rewriting rules. We verify the weak termination property of this calculus. This gives the first step towards the mechanization of diagram chasing.

1. Introduction

This work started with the naive idea that diagram chasing in category theory may be mechanized, at least in specific cases. Constructing the appropriate commutative diagrams to show equality of two morphisms is by no means straightforward. The automation of the task would be useful and thus of interest. Monoidal categories and symmetric monoidal categories admit coherence theorems ascertaining that any parallel morphisms are automatically equal to one another as far as they consist only of canonical isomorphisms [22, 19]. Autonomous categories and $\star$-autonomous categories do not have strict coherences but, through graphical presentations, checking equality of two morphisms can be automated [20, 7, 16], although the decision procedure is intractible [15].

Categorical semantics of type theories provide an equivalence between type systems and certain categories [17]. A decisively classic work is the semantics of the simply typed lambda calculus using the cartesian closed category [21]. It shows exact correspondence between $\beta\eta$-equal lambda terms and commutative diagrams. Unfortunately, the equivalence is valid only after the process of calculation is ignored. As the name suggests, the lambda calculus is a computational system. Equality between two lambda terms can be automatically checked through mechanical computation by $\beta\eta$-reduction. To ensure the equivalence between the calculus and the category, however, we have to identify all terms occurring during this calculation. Thus, the dynamic content of the calculus is lost in the categorical semantics.

Existence of computation in the side of the lambda calculus suggests that the corresponding cartesian closed category may well be given a dynamic computational mech-

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anism. The $\beta\eta$-equality in the lambda calculus corresponds to the adjunction between product and exponential, i.e., $(-) \times B \dashv B \to (-)$. For example, the β-equivalence corresponds to the following commutative triangle diagram, which arises from the adjunction:

$$\begin{tikzcd} A \times B \arrow[r, "\mathrm{abs}\cdot"] \arrow[dr, "1"'] & (B \to A \times B) \times B \arrow[d, "\mathrm{ev}"] \arrow[dl, phantom, "\circlearrowright"] \\ & A \times B \end{tikzcd}$$

where $\circlearrowright$ denotes that two legs are equal.[1] When we regard this diagram as a reduction, we modify it as

$$\begin{tikzcd} A \times B \arrow[r, "\mathrm{abs}\cdot"] \arrow[dr, "1"'] & (B \to A \times B) \times B \arrow[d, "\mathrm{ev}"] \arrow[dl, phantom, "\Swarrow"] \\ & A \times B \end{tikzcd}$$

where the 2-cell double arrow $\Rightarrow$ means one-way rewriting. The morphism $(\mathrm{abs} \times 1_B); \mathrm{ev}$ contracts to $1_{A \times B}$. Rewriting in the reverse direction is prohibited. This idea, essentially due to Seely [28], looks natural but does not seem to be pursued further. Previous works by Seely [28] and Jay [18] construct 2-categories employing ordinary lambda terms. These are the lambda calculus presented in the style of 2-categories, not the rewriting system directly built on categories.

We install a calculus on the categorical semantics of the linear logic. In the lambda calculus, when an argument of a function is accessed n times, its n copies are created to be substituted simultaneously. The duplication process is, however, encapsulated in the β-reduction rule, inseparable from other operations. The linear logic isolates duplication so that the timing and the amount of it can be controlled. The categorical semantics of the linear logic has a symmetric monoidal adjunction $(-) \otimes B \dashv B \multimap (-)$ equipped with comonad $!$, such that $!A$ comes equipped with a commutative coalgebra structure on it. The β-reduction of the linear logic becomes

$$\begin{tikzcd} A \otimes B \arrow[r, "\mathrm{abs}\cdot"] \arrow[dr, "1"'] & (B \multimap A \otimes B) \otimes B \arrow[d, "\mathrm{ev}"] \arrow[dl, phantom, "\Swarrow"] \\ & A \otimes B \end{tikzcd}$$

Alternatively, if we take a $\star$-autonomous category as its base,

$$\begin{tikzcd} \mathbf{1} \otimes A \arrow[r, "\tau\cdot"] \arrow[d, "\sim"'] & (A \parr A^*) \otimes A \arrow[d, "\partial'"] \arrow[dl, phantom, "\Swarrow"] \\ A \arrow[dr, "\sim"'] & A \parr (A^* \otimes A) \arrow[d, "\cdot\gamma"] \\ & A \parr \bot \end{tikzcd}$$

[1]To save space, we occasionally use a dot to signify the position where a suitable identity is inserted, and omit tensor and cotensor on morphisms.

Among the defining diagrams of the categorical semantics, twenty-one diagrams are regarded as reduction rules. For example,

$$\begin{tikzcd} !A \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Swarrow"] & !!A \arrow[d, "\delta"] \\ !!A \arrow[r, "!\delta"'] & !!!A \end{tikzcd}$$

replaces one of the defining commutative diagrams of comonad. The naturality of certain morphisms is also replaced with rewriting rules. For example, naturality of the diagonal (comultiplication) of the comonoid gives rise to

$$\begin{tikzcd} !A \arrow[r, "!f"] \arrow[d, "d"'] \arrow[dr, phantom, "\Swarrow"] & !B \arrow[d, "d"] \\ !A \otimes !A \arrow[r, "!f \otimes !f"'] & !B \otimes !B \end{tikzcd}$$

that realizes duplication. The choice of the directions of rewriting rules is justified by comparison to the conversion rules in the type theory. Details are given in section 5. These rules, in addition to $\beta\eta$-rules, provide twenty-three reduction rules in total. We contend that the categorical model of the lambda calculus is too coarse to incorporate a computational system in it. If we clearly separate copying from the other functions using the linear logic, we can directly implement rewriting on a category so that the obtained calculus has the desirable properties.

Our purpose is, however, not to transcribe a carbon copy of the type system in a category. We build a calculus worth existing in its own right. The linear logic allows finer control on duplication than the lambda calculus, yet the unit of substitutions is coarse. A term is duplicated in one stroke no matter how large it is. To improve the situation, graphical reduction systems have been considered [14]. Each link in a graph can be individually duplicated so that optimal efficiency is attained by ultimate usage of sharing. However, graphical systems have a drawback. Arbitrary connections of links do not form a syntactically lawful graph in general. Moreover, it is not obvious how to ensure that the graphs occurring in the process of rewriting remains meaningful. A system by Ghani [12] and one by Asperti [1] are examples of the calculi inspired by the category theory. The former is term rewriting and the latter is graph rewriting.

Our categorical rewriting system lies between term rewriting and graph rewriting. It enables fine-grained control of resources. We can dissect terms in order to duplicate them piece by piece. A morphism of the form $!f$ corresponds to a box in the linear logic. The linear logic has no function to split boxes, thus a box $!(g; f)$ must be copied as an assembled unit. In contrast our system permits to decompose it into $!g; !f$ to activate partial duplication $!f; d \Rightarrow d; (!f \otimes !f)$ by naturality of the diagonal. A similar property is presented in a system based on lambda terms by Jay [18]. For graph rewriting, in contrast, extremely fine control is enabled since duplication per link is allowed. However, there is a risk that the intermediate graphs appearing in computation may lose semantical justification. As our system deals with only those which are meaningful as morphisms, the duplicated unit f always keeps its semantical meaning. Our calculus rewrites the entities that have mathematical "meaning", whilst finer control on duplication is enabled than ordinary term rewriting.

Early works that view reductions as 2-cells are [28, 26]. Seely and Jay constructed 2-categories from lambda terms as mentioned above [28, 18]. A graphical system based on the categorical semantics is [1, 2]. The categorical abstract machine [9] is a virtual machine based on categorical combinators [10]. To the author's best knowledge, no previous works built computational systems directly on categories.

Our system satisfies the properties that computational systems require. We verify normalizability in this paper. Confluence will be discussed in a forthcoming paper.

2. Linear category

We start with the definition of categorical models of the linear logic. Among several equivalent definitions [5, 6, 24, 25, 27], we take the following [23, Prop.25]. A reason for this choice is that we can write down all defining conditions as commutative diagrams. The reader may consult these papers for comparison between various models.

2.1. Definition. An (intuitionistic or classical) *linear category* is a pair of a category $\mathbf{C}$ and a functor $! : \mathbf{C} \to \mathbf{C}$ with the following additional structures:

(i) If an intuitionisic linear category is concerned with, the underlying category is a symmetric monoidal closed category $(\mathbf{C}, \otimes, \mathbf{1}, \multimap)$. If a classical linear category is concerned with, the underlying category is a $\star$-autonomous category $(\mathbf{C}, \otimes, \parr, \mathbf{1}, \perp, (-)^*)$.

(ii) ! is equipped with the structure of a symmetric monoidal functor $(!, \tilde{\varphi}, \varphi_0)$.

(iii) ! is equipped with the structure of a comonad $(!, \delta, \varepsilon)$ where $\delta : ! \to !!$ and $\varepsilon : ! \to Id$ are monoidal natural transformations.

(iv) The objects of the shape $!A$ are equipped with the structure of a commutative comonoid $(!A, d_A, e_A)$ where collectively $d_A : !A \to !A \otimes !A$ and $e_A : !A \to \mathbf{1}$ are monoidal natural transformations in A.

Moreover, these structures are related in the following way:

(v) Each d_A and each e_A give rise to coalgebra morphisms when $!A, \mathbf{1}$, and $!A \otimes !A$ are naturally regarded as coalgebras.

(vi) Each δ_A is a comonoid morphism.

We give the list of all defining commutative diagrams, although they are absolutely standard. In the next section, we pick up some of the diagrams and turn them into rewriting rules to build up a calculus. So it will be instructive to give a full list as a preparation.

A symmetric monoidal category has a 2-place functor $\otimes$ and an object $\mathbf{1}$, and is equipped with natural isomorphisms $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, $\lambda_A : \mathbf{1} \otimes A \to A$, $\rho_A : A \otimes \mathbf{1} \to A$, and $\sigma_{A,B} : A \otimes B \to B \otimes A$. The naturality of these isomorphisms are

$$\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, "\alpha"] \arrow[d, "(f\otimes g)\otimes h"'] \arrow[dr, phantom, "\circlearrowleft"] & A\otimes(B\otimes C) \arrow[d, "f\otimes(g\otimes h)"] \\
(A'\otimes B')\otimes C' \arrow[r, "\alpha"'] & A'\otimes(B'\otimes C')
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1}\otimes A \arrow[r, "\lambda"] \arrow[d, "1\otimes f"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, "f"] \\
\mathbf{1}\otimes A' \arrow[r, "\lambda"'] & A'
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes B \arrow[r, "\sigma"] \arrow[d, "f\otimes g"'] \arrow[dr, phantom, "\circlearrowleft"] & B\otimes A \arrow[d, "g\otimes f"] \\
A'\otimes B' \arrow[r, "\sigma"'] & B'\otimes A'
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes\mathbf{1} \arrow[r, "\rho"] \arrow[d, "f\otimes 1"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, "f"] \\
A'\otimes\mathbf{1} \arrow[r, "\rho"'] & A'
\end{tikzcd}$$

where subscripts are omitted for simplicity. Moreover, these are subject to the following coherence conditions [19]:

$$\begin{tikzcd}
& ((A\otimes B)\otimes C)\otimes D \arrow[dl, "\alpha\otimes 1"'] \arrow[dr, "\alpha"] & \\
(A\otimes(B\otimes C))\otimes D \arrow[d, "\alpha"'] & \circlearrowleft & (A\otimes B)\otimes(C\otimes D) \arrow[d, "\alpha"] \\
A\otimes((B\otimes C)\otimes D) \arrow[rr, "1\otimes\alpha"'] & & A\otimes(B\otimes(C\otimes D))
\end{tikzcd}
\qquad
\begin{tikzcd}
(A\otimes\mathbf{1})\otimes B \arrow[rr, "\alpha"] \arrow[dr, "\rho\otimes 1"'] & \circlearrowleft & A\otimes(\mathbf{1}\otimes B) \arrow[dl, "1\otimes\lambda"] \\
& A\otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
& (A\otimes B)\otimes C \arrow[dl, "\sigma\otimes 1"'] \arrow[dr, "\alpha"] & \\
(B\otimes A)\otimes C \arrow[d, "\alpha"'] & \circlearrowleft & A\otimes(B\otimes C) \arrow[d, "\sigma"] \\
B\otimes(A\otimes C) \arrow[dr, "1\otimes\sigma"'] & & (B\otimes C)\otimes A \arrow[dl, "\alpha"] \\
& B\otimes(C\otimes A) &
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes B \arrow[rr, "\sigma"] \arrow[dr, "1"'] & \circlearrowleft & B\otimes A \arrow[dl, "\sigma"] \\
& A\otimes B &
\end{tikzcd}$$

A symmetric monoidal closed category [20] is further equipped with an adjoint $(-)\otimes B \dashv B \multimap (-)$. We write the unit as $\mathrm{abs}_A^B : A \to B \multimap (A\otimes B)$ and the counit as $\mathrm{ev}_A^B : (B\multimap A)\otimes B \to A$. These satisfy the naturality as

$$\begin{tikzcd}
A \arrow[r, "\mathrm{abs}_A^B"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & B\multimap(A\otimes B) \arrow[d, "1\multimap(f\otimes 1)"] \\
A' \arrow[r, "\mathrm{abs}_{A'}^B"'] & B\multimap(A'\otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
(B\multimap A)\otimes B \arrow[r, "\mathrm{ev}_A^B"] \arrow[d, "(1\multimap f)\otimes 1"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d, "f"] \\
(B\multimap A')\otimes B \arrow[r, "\mathrm{ev}_{A'}^B"'] & A'
\end{tikzcd}$$

and render the following adjoint triangles commutative:

$$\begin{tikzcd}
A\otimes B \arrow[rr, "\mathrm{abs}\otimes 1"] \arrow[dr, "1"'] & \circlearrowleft & (B\multimap A\otimes B)\otimes B \arrow[dl, "\mathrm{ev}"] \\
& A\otimes B &
\end{tikzcd}
\qquad
\begin{tikzcd}
B\multimap A \arrow[rr, "\mathrm{abs}"] \arrow[dr, "1"'] & \circlearrowleft & B\multimap(B\multimap A)\otimes B \arrow[dl, "1\multimap\mathrm{ev}"] \\
& B\multimap A &
\end{tikzcd}$$

(product has higher precedence than exponential).

A linearly distributive (or weakly distribuitve) category is a symmetric monidal category that has an additional monoidal category structure $(\parr, \perp)$ and linear distribution morphisms $\partial_{A,B,C} : A \otimes (B \parr C) \to (A \otimes B) \parr C$, which are natural. We write bars over the natural isomorphism of the added monoidal structure for distinction. The naturality turns out to be

$$\begin{tikzcd}
(A \parr B) \parr C \arrow[r, "\bar{\alpha}"] \arrow[d, "(f \parr g) \parr h"'] \arrow[dr, phantom, "\circlearrowright"] & A \parr (B \parr C) \arrow[d, "f \parr (g \parr h)"] \\
(A' \parr B') \parr C' \arrow[r, "\bar{\alpha}"'] & A' \parr (B' \parr C')
\end{tikzcd}
\qquad
\begin{tikzcd}
\perp \parr A \arrow[r, "\bar{\lambda}"] \arrow[d, "1 \parr f"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[d, "f"] \\
\perp \parr A' \arrow[r, "\bar{\lambda}"'] & A'
\end{tikzcd}$$

$$\begin{tikzcd}
A \parr B \arrow[r, "\bar{\sigma}"] \arrow[d, "f \parr g"'] \arrow[dr, phantom, "\circlearrowright"] & B \parr A \arrow[d, "g \parr f"] \\
A' \parr B' \arrow[r, "\bar{\sigma}"'] & B' \parr A'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \parr \perp \arrow[r, "\bar{\rho}"] \arrow[d, "f \parr 1"'] \arrow[dr, phantom, "\circlearrowright"] & A \arrow[d, "f"] \\
A' \parr \perp \arrow[r, "\bar{\rho}"'] & A'
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (B \parr C) \arrow[r, "\partial"] \arrow[d, "f \otimes (g \parr h)"'] \arrow[dr, phantom, "\circlearrowright"] & (A \otimes B) \parr C \arrow[d, "(f \otimes g) \parr h"] \\
A' \otimes (B' \parr C') \arrow[r, "\partial"'] & (A' \otimes B') \parr C'
\end{tikzcd}$$

The coherence conditions for the added monoidal structure are

$$\begin{tikzcd}[column sep=small]
& ((A \parr B) \parr C) \parr D \arrow[dl, "\bar{\alpha} \parr 1"'] \arrow[dr, "\bar{\alpha}"] & \\
(A \parr (B \parr C)) \parr D \arrow[d, "\bar{\alpha}"'] & \circlearrowright & (A \parr B) \parr (C \parr D) \arrow[d, "\bar{\alpha}"] \\
A \parr ((B \parr C) \parr D) \arrow[rr, "1 \parr \bar{\alpha}"'] & & A \parr (B \parr (C \parr D))
\end{tikzcd}
\qquad
\begin{tikzcd}
(A \parr \mathbf{1}) \parr B \arrow[rr, "\bar{\alpha}"] \arrow[dr, "\bar{\rho} \parr 1"'] & \circlearrowright & A \parr (\mathbf{1} \parr B) \arrow[dl, "1 \parr \bar{\lambda}"] \\
& A \parr B &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& (A \parr B) \parr C \arrow[dl, "\bar{\sigma} \parr 1"'] \arrow[dr, "\bar{\alpha}"] & \\
(B \parr A) \parr C \arrow[d, "\bar{\alpha}"'] & \circlearrowright & A \parr (B \parr C) \arrow[d, "\bar{\sigma}"] \\
B \parr (A \parr C) \arrow[dr, "1 \parr \bar{\sigma}"'] & & (B \parr C) \parr A \arrow[dl, "\bar{\alpha}"] \\
& B \parr (C \parr A) &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \parr B \arrow[rr, "\bar{\sigma}"] \arrow[dr, "1"'] & \circlearrowright & B \parr A \arrow[dl, "\bar{\sigma}"] \\
& A \parr B &
\end{tikzcd}$$

The following are the coherences of linear distribution morphisms [8]. Here $\partial'_{ABC} : (A \parr B) \otimes C \to A \parr (B \otimes C)$ is induced from ∂_{ABC} by symmetry of tensor and cotensor. The label $\sim$ represents appropriate structural isomorphisms.

$$
\begin{tikzcd}
\mathbf{1}\otimes(A\parr B) \arrow[rr,"\partial"] \arrow[dr,"\sim"'] & \arrow[d, phantom, "\circlearrowleft"] & (\mathbf{1}\otimes A)\parr B \arrow[dl,"\sim"] \\
& A\parr B &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A\otimes(B\parr(C\parr D)) \arrow[dl,"\sim"'] \arrow[dr,"\partial"] & \\
A\otimes((B\parr C)\parr D) \arrow[d,"\partial"'] & \circlearrowleft & (A\otimes B)\parr(C\parr D) \arrow[d,"\sim"] \\
(A\otimes(B\parr C))\parr D \arrow[rr,"\partial\cdot"'] & & ((A\otimes B)\parr C)\parr D
\end{tikzcd}
$$

$$
\begin{tikzcd}
& (A\parr B)\otimes(C\parr D) \arrow[dl,"\partial'"'] \arrow[dr,"\partial"] & \\
A\parr(B\otimes(C\parr D)) \arrow[d,"\cdot\partial"'] & \circlearrowleft & ((A\parr B)\otimes C)\parr D \arrow[d,"\partial'\cdot"] \\
A\parr((B\otimes C)\parr D) \arrow[rr,"\sim"'] & & (A\parr(B\otimes C))\parr D
\end{tikzcd}
$$

and their duals:

$$
\begin{tikzcd}
& A\otimes B \arrow[dl,"\sim"'] \arrow[dr,"\sim"] \arrow[d, phantom, "\circlearrowleft"] & \\
A\otimes(B\parr\perp) \arrow[rr,"\partial"'] & & (A\otimes B)\parr\perp
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes(B\otimes(C\parr D)) \arrow[rr,"\cdot\partial"] \arrow[d,"\sim"'] & & A\otimes((B\otimes C)\parr D) \arrow[d,"\partial"] \\
(A\otimes B)\otimes(C\parr D) \arrow[dr,"\partial"'] & \circlearrowleft & (A\otimes(B\otimes C))\parr D \arrow[dl,"\sim"] \\
& ((A\otimes B)\otimes C)\parr D &
\end{tikzcd}
$$

$$
\begin{tikzcd}
A\otimes((B\parr C)\otimes D) \arrow[rr,"\sim"] \arrow[d,"\cdot\partial'"'] & & (A\otimes(B\parr C))\otimes D \arrow[d,"\partial\cdot"] \\
A\otimes(B\parr(C\otimes D)) \arrow[dr,"\partial"'] & \circlearrowleft & ((A\otimes B)\parr C)\otimes D \arrow[dl,"\partial'"] \\
& (A\otimes B)\parr(C\otimes D) &
\end{tikzcd}
$$

These three diagrams are obtained if we upend a transparent sheet on which the former three are written so that the back surface comes to the front, reverse the direction of the arrow, and interchange tensor and cotensor.

A $*$-autonomous category is a linearly distributive category equipped with a map $A \mapsto A^*$ on objects as well as two families of morphisms: $\tau_A : \mathbf{1} \to A \parr A^*$ and $\gamma_A : A^* \otimes A \to \perp$. We need to demand neither that (-)* be functorial, nor that τ_A and γ_A be natural [8]. We distinguish between A^{**} and A, which are naturally isomorphic. The coherence conditions for these are

$$
\begin{tikzcd}
\mathbf{1}\otimes A \arrow[rr,"\tau\otimes 1"] \arrow[d,"\sim"'] & & (A\parr A^*)\otimes A \arrow[d,"\partial'"] \\
A \arrow[dr,"\sim"'] & \circlearrowleft & A\parr(A^*\otimes A) \arrow[dl,"1\parr\gamma"] \\
& A\parr\perp &
\end{tikzcd}
\qquad
\begin{tikzcd}
A^*\otimes\mathbf{1} \arrow[rr,"1\otimes\tau"] \arrow[d,"\sim"'] & & A^*\otimes(A\parr A^*) \arrow[d,"\partial"] \\
A^* \arrow[dr,"\sim"'] & \circlearrowleft & (A^*\otimes A)\parr A^* \arrow[dl,"\gamma\parr 1"] \\
& \perp\parr A^* &
\end{tikzcd}
$$

The condition (ii) of Def. 2.1 requires that $! : \mathbf{C} \to \mathbf{C}$ is a symmetric monoidal functor. This means that the functor ! is equipped with a natural transformation $\tilde{\varphi}_{A,B} : !A \otimes !B \to !(A \otimes B)$ and a morphism $\varphi_0 : \mathbf{1} \to !\mathbf{1}$. The tilde is added so that these are dinstinguished if the subscripts are omitted. The naught signifies it to be nullary. The naturality of $\tilde{\varphi}$ is

$$\begin{tikzcd}
!A \otimes !B \arrow[r, "\tilde{\varphi}"] \arrow[d, "!f\otimes !g"'] \arrow[dr, phantom, "\circlearrowleft"] & !(A \otimes B) \arrow[d, "!(f\otimes g)"] \\
!A' \otimes !B' \arrow[r, "\tilde{\varphi}"'] & !(A' \otimes B')
\end{tikzcd}$$

and the coherence conditions are

$$\begin{tikzcd}
 & (!A \otimes !B) \otimes !C \arrow[dl, "\alpha"'] \arrow[dr, "\tilde{\varphi}\otimes 1"] & \\
!A \otimes (!B \otimes !C) \arrow[d, "1\otimes\tilde{\varphi}"'] & \circlearrowleft & !(A \otimes B) \otimes !C \arrow[d, "\tilde{\varphi}"] \\
!A \otimes !(B \otimes C) \arrow[dr, "\tilde{\varphi}"'] & & !((A \otimes B) \otimes C) \arrow[dl, "!\alpha"] \\
 & !(A \otimes (B \otimes C)) &
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \otimes !B \arrow[r, "\tilde{\varphi}"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circlearrowleft"] & !(A \otimes B) \arrow[d, "!\sigma"] \\
!B \otimes !A \arrow[r, "\tilde{\varphi}"'] & !(B \otimes A)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{1} \otimes !A \arrow[r, "\varphi_0\otimes 1"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\circlearrowleft"] & !\mathbf{1} \otimes !A \arrow[d, "\tilde{\varphi}"] \\
!A \arrow[r, "\sim"'] & !(\mathbf{1} \otimes A)
\end{tikzcd}$$

where $\sim$ denotes an appropriate structural isomorphism.

The condition (iii) of the linear category is the requirement that the functor ! is endowed with a comonad structure. Namely, two natural transformations $\delta_A : !A \to !!A$ and $\varepsilon_A : !A \to A$ are associated. The naturality is

$$\begin{tikzcd}
!A \arrow[r, "!f"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\circlearrowleft"] & !B \arrow[d, "\delta"] \\
!!A \arrow[r, "!!f"'] & !!B
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[r, "!f"] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "\circlearrowleft"] & !B \arrow[d, "\varepsilon"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

and the coherence conditions are

$$\begin{tikzcd}
!A \arrow[r, "\delta"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\circlearrowleft"] & !!A \arrow[d, "\delta"] \\
!!A \arrow[r, "!\delta"'] & !!!A
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[rr, "\delta"] \arrow[dr, "1"'] & \circlearrowleft & !!A \arrow[dl, "\varepsilon"] \\
 & !A &
\end{tikzcd}$$

$$\begin{tikzcd}
!A \arrow[rr, "\delta"] \arrow[dr, "1"'] & \circlearrowleft & !!A \arrow[dl, "!\varepsilon"] \\
 & !A &
\end{tikzcd}$$

Moreover, δ and ε are required to be monoidal natural transformations. In general, for monoidal functors F and G, a natural transformation $\nu : F \to G$ is monoidal if it commutes with $(\tilde{\varphi}, \varphi_0)$ for F and G in an obvious sense. Whenever ! is a monoidal functor, so is !!. An identity functor is always a monoidal functor. Hence, it makes sense to require $\delta : ! \to !!$ and $\varepsilon : ! \to Id$ to be monoidal. This amounts to the following four diagrams:

$$
\begin{tikzcd}
 & !A \otimes !B \arrow[dl, "\delta\otimes\delta"'] \arrow[dr, "\tilde{\varphi}"] \arrow[dd, phantom, "\circlearrowright"] & \\
!!A \otimes !!B \arrow[d, "\tilde{\varphi}"'] & & !(A \otimes B) \arrow[d, "\delta"] \\
!(!A \otimes !B) \arrow[rr, "!\tilde{\varphi}"'] & & !!(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \otimes !B \arrow[rr, "\tilde{\varphi}"] \arrow[dr, "\varepsilon\otimes\varepsilon"'] \arrow[r, phantom, shift right=3, "\circlearrowright"] & & !(A \otimes B) \arrow[dl, "\varepsilon"] \\
 & A \otimes B &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{1} \arrow[r, "\varphi_0"] \arrow[d, "\varphi_0"'] \arrow[dr, phantom, "\circlearrowright"] & !\mathbf{1} \arrow[d, "\delta"] \\
!\mathbf{1} \arrow[r, "!\varphi_0"'] & !!\mathbf{1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{1} \arrow[rr, "\varphi_0"] \arrow[dr, "1"'] \arrow[r, phantom, shift right=3, "\circlearrowright"] & & !\mathbf{1} \arrow[dl, "\varepsilon"] \\
 & \mathbf{1} &
\end{tikzcd}
$$

The condition (iv) requires that the objects of the form $!A$ have the structure of commutative comonoids. Namely, there are family of morphisms $d_A : !A \to !A \otimes !A$ and $e_A : !A \to \mathbf{1}$, rendering the following diagrams commutative:

$$
\begin{tikzcd}
 & !A \arrow[dl, "d"'] \arrow[dr, "d"] \arrow[dd, phantom, "\circlearrowright"] & \\
!A \otimes !A \arrow[d, "d\otimes 1"'] & & !A \otimes !A \arrow[d, "1\otimes d"] \\
(!A \otimes !A) \otimes !A \arrow[rr, "\alpha"'] & & !A \otimes (!A \otimes !A)
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[rr, "d"] \arrow[dr, "d"'] \arrow[r, phantom, shift right=3, "\circlearrowright"] & & !A \otimes !A \arrow[dl, "\sigma"] \\
 & !A \otimes !A &
\end{tikzcd}
$$

$$
\begin{tikzcd}
!A \arrow[rr, "d"] \arrow[dr, "\sim"'] \arrow[r, phantom, shift right=3, "\circlearrowright"] & & !A \otimes !A \arrow[dl, "e\otimes 1"] \\
 & \mathbf{1} \otimes !A &
\end{tikzcd}
$$

Moreover d_A and e_A must be monoidal natural transformations. Consequently, $!f$ is always a comonoid morphism. The naturality is given as

$$
\begin{tikzcd}
!A \arrow[r, "!f"] \arrow[d, "d"'] \arrow[dr, phantom, "\circlearrowright"] & !B \arrow[d, "d"] \\
!A \otimes !A \arrow[r, "!f\otimes !f"'] & !B \otimes !B
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \arrow[rr, "!f"] \arrow[dr, "e"'] \arrow[r, phantom, shift right=3, "\circlearrowright"] & & !B \arrow[dl, "e"] \\
 & \mathbf{1} &
\end{tikzcd}
$$

and requiring them to be monoidal amounts to

$$
\begin{tikzcd}
 & !A \otimes !B \arrow[dl, "d\otimes d"'] \arrow[dr, "\tilde{\varphi}"] \arrow[dd, phantom, "\circlearrowright"] & \\
(!A \otimes !A) \otimes (!B \otimes !B) \arrow[d, "\sim"'] & & !(A \otimes B) \arrow[d, "d"] \\
(!A \otimes !B) \otimes (!A \otimes !B) \arrow[rr, "\tilde{\varphi}\otimes\tilde{\varphi}"'] & & !(A \otimes B) \otimes !(A \otimes B)
\end{tikzcd}
\qquad
\begin{tikzcd}
!A \otimes !B \arrow[r, "\tilde{\varphi}"] \arrow[d, "e\otimes e"'] \arrow[dr, phantom, "\circlearrowright"] & !(A \otimes B) \arrow[d, "e"] \\
\mathbf{1} \otimes \mathbf{1} \arrow[r, "\sim"'] & \mathbf{1}
\end{tikzcd}
$$

$$\begin{tikzcd} \mathbf{1} \arrow[r, "\varphi_0"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\circlearrowright"] & !\mathbf{1} \arrow[d, "d"] \\ \mathbf{1}\otimes\mathbf{1} \arrow[r, "\varphi_0\otimes\varphi_0"'] & !\mathbf{1}\otimes !\mathbf{1} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{1} \arrow[rr, "\varphi_0"] \arrow[dr, "1"'] & \circlearrowright & !\mathbf{1} \arrow[dl, "e"] \\ & \mathbf{1} & \end{tikzcd}$$

Note that $A \mapsto\, !A \otimes !A$ and the constant $\mathbf{1}$ are monoidal functors.

The condition (v) is concerned with coalgebra morphisms. The notion of coalgebras is associated with comonads. If $X \xrightarrow{f} !X$ and $Y \xrightarrow{g} !Y$ are coalgebras, a coalgebra morphism between these is simply a morphism $X \xrightarrow{k} Y$ satisfying $f; !k = k; g$. The morphism $!A \xrightarrow{\delta_A} !!A$ gives rise to a coalgebra, often called a free coalgebra. Similarly $!A \otimes !A$ and $\mathbf{1}$ have natural coalgebra structures on them. The linear category requires that $d_A : !A \to !A \otimes !A$ and $e_A : !A \to \mathbf{1}$ are coalgebra morphisms. Namely, the following diagrams are commutative:

$$\begin{tikzcd} & !A \arrow[dl, "d"'] \arrow[dr, "\delta"] & \\ !A\otimes !A \arrow[d, "\delta\otimes\delta"'] & \circlearrowright & !!A \arrow[d, "!d"] \\ !!A\otimes !!A \arrow[rr, "\tilde{\varphi}"'] & & !(!A\otimes !A) \end{tikzcd} \qquad \begin{tikzcd} !A \arrow[r, "\delta"] \arrow[d, "e"'] \arrow[dr, phantom, "\circlearrowright"] & !!A \arrow[d, "!e"] \\ \mathbf{1} \arrow[r, "\varphi_0"'] & !\mathbf{1} \end{tikzcd}$$

Finally, the condition (vi) says that $\delta_A : !A \to !!A$ is a comonoid morphism. Namely, it is commutative to the comonoid structure:

$$\begin{tikzcd} !A \arrow[r, "\delta"] \arrow[d, "d"'] \arrow[dr, phantom, "\circlearrowright"] & !!A \arrow[d, "d"] \\ !A\otimes !A \arrow[r, "\delta\otimes\delta"'] & !!A\otimes !!A \end{tikzcd} \qquad \begin{tikzcd} !A \arrow[rr, "\delta"] \arrow[dr, "e"'] & \circlearrowright & !!A \arrow[dl, "e"] \\ & \mathbf{1} & \end{tikzcd}$$

As a consequence, every coalgebra morphism between free coalgebras turn out to be a comonoid morphism.

3. A categorical reduction system

Suppose that a set of atomic objects is given. A *free* (intuitionistic or classical) linear category is naturally defined. It is obtained by freely generating by the constructors and taking quotients with regard to the due conditions in Def. 2.1. Since all conditions are given as commutative diagrams, taking quotients makes sense.

We regard the free linear category as a syntactic structure. Our goal is to develop a dynamic calculus installed directly on the category. In analogy to type systems, objects of the category correspond to types, while morphisms correspond to terms. As type systems are designed as rewriting calculi of terms, our categorical system is realized as a rewriting calculus of morphisms.

If we consider the free intuitionistic linear category, the base of which is a symmetric monoidal closed category, the set of objects are generated by

$$A \quad ::= \quad X \;\mid\; \mathbf{1} \;\mid\; A \otimes A \;\mid\; A \multimap A \;\mid\; !A$$

where X ranges over a given set of atomic objects. Atomic morphisms are identities, with *structural isomorphisms*:

$$\begin{array}{ll}
A \xrightarrow{1_A} A & \\
(A \otimes B) \otimes C \xrightarrow{\alpha_{ABC}} A \otimes (B \otimes C) & A \otimes (B \otimes C) \xrightarrow{\alpha^{-1}_{ABC}} (A \otimes B) \otimes C \\
\mathbf{1} \otimes A \xrightarrow{\lambda_A} A & A \xrightarrow{\lambda^{-1}_A} \mathbf{1} \otimes A \\
A \otimes \mathbf{1} \xrightarrow{\rho_A} A & A \xrightarrow{\rho^{-1}_A} A \otimes \mathbf{1} \\
A \otimes B \xrightarrow{\sigma_{AB}} B \otimes A & B \otimes A \xrightarrow{\sigma^{-1}_{AB}} A \otimes B,
\end{array}$$

the units and the counits of the adjunction:

$$A \xrightarrow{\mathrm{abs}^B_A} B \multimap A \otimes B \qquad\qquad (B \multimap A) \otimes B \xrightarrow{\mathrm{ev}^B_A} A$$

together with the morphisms given in Def. 2.1:

$$\begin{array}{ll}
!A \otimes !B \xrightarrow{\tilde{\varphi}_{AB}} !(A \otimes B) & !A \xrightarrow{\delta_A} !!A \\
\mathbf{1} \xrightarrow{\varphi_0} !\mathbf{1} & !A \xrightarrow{\varepsilon_A} A \\
 & !A \xrightarrow{d_A} !A \otimes !A \\
 & !A \xrightarrow{e_A} \mathbf{1}
\end{array}$$

For future reference, the last six atomic morphisms are called *algebraic morphisms*. The set of morphisms is generated from the atomic morphisms by (diagramatic order) composition $f; g$ and the functorial operations $f \otimes g$, $!f$, and $1_B \multimap f$. Subscripts will often be omitted. Our system will be designed so that subscripts have no significance. It is analogous to ordinary type systems where rules depend only on the shape of terms, not on types.

We introduce a congruence relation over morphisms. Two morphisms that are equivalent under this congruence are understood to be able to be rewritable from one another. First, we have the axioms of categories and the elementary property of functors:

$$\begin{array}{ll}
f; 1 = f = 1; f & F1 = 1 \\
(f; g); h = f; (g; h) & F(f; g) = Ff; Fg
\end{array}$$

where F is one of either $(-) \otimes (-)$, $B \multimap (-)$, or $!(-)$. For the tensor product, we appropriately reform the equality as it is a 2-place functor. Second, each structural isomorphism and its inverse are actual inverses:

$$\alpha; \alpha^{-1} = 1 \qquad\qquad \alpha^{-1}; \alpha = 1 \qquad\qquad \cdots$$

Next, we consider the case where the base category is $\star$-autonomous. Among several equivalent definitions known for $\star$-autonomous categories [4, 16], we adopt the one using the linearly distributive category [8]. The set of objects is generated by

$$A \quad ::= \quad X \;\mid\; \mathbf{1} \;\mid\; \bot \;\mid\; A \otimes A \;\mid\; A \parr A \;\mid\; A^* \;\mid\; !A$$

In addition to the structural isomorphisms for $\otimes$, we include the isomorphisms giving the symmetric monoidal structure on $\parr$:

$$
\begin{array}{ll}
(A \parr B) \parr C \xrightarrow{\bar{\alpha}_{ABC}} A \parr (B \parr C) & A \parr (B \parr C) \xrightarrow{\bar{\alpha}^{-1}_{ABC}} (A \parr B) \parr C \\
\bot \parr A \xrightarrow{\bar{\lambda}_A} A & A \xrightarrow{\bar{\lambda}^{-1}_A} \bot \parr A \\
A \parr \bot \xrightarrow{\bar{\rho}_A} A & A \xrightarrow{\bar{\rho}^{-1}_A} A \parr \bot \\
A \parr B \xrightarrow{\bar{\sigma}_{AB}} B \parr A & B \parr A \xrightarrow{\bar{\sigma}^{-1}_{AB}} A \parr B
\end{array}
$$

Moreover the linear distribution morphisms

$$A \otimes (B \parr C) \xrightarrow{\partial_{ABC}} (A \otimes B) \parr C$$

are added. The morphisms abs and ev are removed. In place, we add duality morphisms:

$$\mathbf{1} \xrightarrow{\tau_A} A \parr A^* \qquad\qquad A^* \otimes A \xrightarrow{\gamma_A} \bot$$

The six algebraic morphisms are the same. The set of morphisms is generated from the atomic morphisms above by composition $f; g$ and the functorial operations $f \otimes g$, $f \parr g$, and $!f$. We note that $(-)^*$ is not regarded as a contravariant functor [8]. We distinguish A^{**} from A.

The core of our calculus lies in the orientation of diagrams, which we shortly provide. Among the commutative diagrams listed in the previous section, twenty-three diagrams are selected and reformed into rewriting rules. The rest remain to be equivalences. The orientation of rewriting is denoted by a double arrow. We can rewrite only in the designated direction, whilst between equivalent morphisms we allow rewriting in either direction. In other words, we give a rewriting system modulo congruence. The selected diagrams comprise of one diagram in (ii) of Def. 2.1, all of (iii), all save two diagrams of (iv), all of (v), and all of (vi). Moreover, the adjoint triangles of monoidal closedness and the defining diagrams for $*$-autonomy are turned to rewriting rules. Our tactic is to select diagrams as much as the resulting calculus keeps the desirable properties. It is hopeless to reform all diagrams to rewriting rules. For example, consider the following diagram, which is one of (iv):

$$
\begin{tikzcd}
 & !A \arrow[dl, "d"'] \arrow[dr, "d"] & \\
!A \otimes !A \arrow[d, "d \otimes 1"'] & \circlearrowleft & !A \otimes !A \arrow[d, "1 \otimes d"] \\
(!A \otimes !A) \otimes !A \arrow[rr, "\alpha"'] & & !A \otimes (!A \otimes !A)
\end{tikzcd}
$$

This diagram has symmetry. Enforcing a rule so that only one-way rewriting is permitted would yield a useless calculus.

The following are the list of the twenty-three rewriting rules. The first twenty-one diagrams are common for both the symmetric monoidal closed base and the $\star$-autonomous base. The last two depend on the selected base. In the diagrams, the label $\sim$ denotes appropriate structural isomorphisms and f is an arbitrary morphism.

$$\begin{tikzcd} !A \arrow[r,"\delta"] \arrow[d,"\delta"'] \arrow[dr, phantom, "\Swarrow"] & !!A \arrow[d,"\delta"] \\ !!A \arrow[r,"!\delta"'] & !!!A \end{tikzcd} \qquad \cdots \ (1)$$

$$\begin{tikzcd} !A \arrow[rr,"\delta"] \arrow[dr,"1"'] \arrow[r, phantom, "\Swarrow"] & & !!A \arrow[dl,"\varepsilon"] \\ & !A & \end{tikzcd} \qquad \cdots \ (2)$$

$$\begin{tikzcd} !A \arrow[rr,"\delta"] \arrow[dr,"1"'] \arrow[r, phantom, "\Swarrow"] & & !!A \arrow[dl,"!\varepsilon"] \\ & !A & \end{tikzcd} \qquad \cdots \ (3)$$

$$\begin{tikzcd} !A \arrow[r,"\delta"] \arrow[d,"d"'] \arrow[dr, phantom, "\Swarrow"] & !!A \arrow[d,"d"] \\ !A\otimes !A \arrow[r,"\delta\otimes\delta"'] & !!A\otimes !!A \end{tikzcd} \qquad \cdots \ (4)$$

$$\begin{tikzcd} & !A \arrow[dl,"d"'] \arrow[dr,"\delta"] \arrow[dd, phantom, "\Swarrow"] & \\ !A\otimes !A \arrow[d,"\delta\otimes\delta"'] & & !!A \arrow[d,"!d"] \\ !!A\otimes !!A \arrow[rr,"\tilde{\varphi}"'] & & !(!A\otimes !A) \end{tikzcd} \qquad \cdots \ (5)$$

$$\begin{tikzcd} !A \arrow[rr,"\delta"] \arrow[dr,"e"'] \arrow[r, phantom, "\Swarrow"] & & !!A \arrow[dl,"e"] \\ & \mathbf{1} & \end{tikzcd} \qquad \cdots \ (6)$$

$$\begin{tikzcd} !A \arrow[r,"\delta"] \arrow[d,"e"'] \arrow[dr, phantom, "\Swarrow"] & !!A \arrow[d,"!e"] \\ \mathbf{1} \arrow[r,"\varphi_0"'] & !\mathbf{1} \end{tikzcd} \qquad \cdots \ (7)$$

$$\begin{tikzcd} !A \arrow[rr,"d"] \arrow[dr,"\sim"'] \arrow[r, phantom, "\Swarrow"] & & !A\otimes !A \arrow[dl,"e\otimes 1"] \\ & \mathbf{1}\otimes !A & \end{tikzcd} \qquad \cdots \ (8)$$

$$\begin{tikzcd} & !A\otimes !B \arrow[dl,"\delta\otimes\delta"'] \arrow[dr,"\tilde{\varphi}"] \arrow[dd, phantom, "\Swarrow"] & \\ !!A\otimes !!B \arrow[d,"\tilde{\varphi}"'] & & !(A\otimes B) \arrow[d,"\delta"] \\ !(!A\otimes !B) \arrow[rr,"!\tilde{\varphi}"'] & & !!(A\otimes B) \end{tikzcd} \qquad \cdots \ (9)$$

$$\begin{tikzcd} !A\otimes !B \arrow[rr,"\tilde{\varphi}"] \arrow[dr,"\varepsilon\otimes\varepsilon"'] \arrow[r, phantom, "\Swarrow"] & & !(A\otimes B) \arrow[dl,"\varepsilon"] \\ & A\otimes B & \end{tikzcd} \qquad \cdots \ (10)$$

$$\begin{tikzcd} & !A\otimes !B \arrow[dl,"d\otimes d"'] \arrow[dr,"\tilde{\varphi}"] \arrow[dd, phantom, "\Swarrow"] & \\ (!A\otimes !A)\otimes(!B\otimes !B) \arrow[d,"\sim"'] & & !(A\otimes B) \arrow[d,"d"] \\ (!A\otimes !B)\otimes(!A\otimes !B) \arrow[rr,"\tilde{\varphi}\otimes\tilde{\varphi}"'] & & !(A\otimes B)\otimes !(A\otimes B) \end{tikzcd} \qquad \cdots \ (11)$$

$$\begin{tikzcd} !A\otimes !B \arrow[r,"\tilde{\varphi}"] \arrow[d,"e\otimes e"'] \arrow[dr, phantom, "\Swarrow"] & !(A\otimes B) \arrow[d,"e"] \\ \mathbf{1}\otimes\mathbf{1} \arrow[r,"\sim"'] & \mathbf{1} \end{tikzcd} \qquad \cdots \ (12)$$

$$\begin{tikzcd} \mathbf{1} \arrow[r,"\varphi_0"] \arrow[d,"\varphi_0"'] \arrow[dr, phantom, "\Swarrow"] & !\mathbf{1} \arrow[d,"\delta"] \\ !\mathbf{1} \arrow[r,"!\varphi_0"'] & !!\mathbf{1} \end{tikzcd} \qquad \cdots \ (13)$$

$$\begin{tikzcd} \mathbf{1} \arrow[rr,"\varphi_0"] \arrow[dr,"1"'] \arrow[r, phantom, "\Swarrow"] & & !\mathbf{1} \arrow[dl,"\varepsilon"] \\ & \mathbf{1} & \end{tikzcd} \qquad \cdots \ (14)$$

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "\varphi_0"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\Swarrow"] & !\mathbf{1} \arrow[d, "d"] \\
\mathbf{1}\otimes\mathbf{1} \arrow[r, "\varphi_0\otimes\varphi_0"'] & !\mathbf{1}\otimes !\mathbf{1}
\end{tikzcd} \qquad \cdots (15)$$

$$\begin{tikzcd}
\mathbf{1} \arrow[rr, "\varphi_0"] \arrow[dr, "1"'] & \arrow[d, phantom, "\Swarrow"] & !\mathbf{1} \arrow[dl, "e"] \\
& \mathbf{1} &
\end{tikzcd} \qquad \cdots (16)$$

$$\begin{tikzcd}
\mathbf{1}\otimes !A \arrow[r, "\varphi_0\otimes 1"] \arrow[d, "\sim"'] \arrow[dr, phantom, "\Swarrow"] & !\mathbf{1}\otimes !A \arrow[d, "\tilde{\varphi}"] \\
!A \arrow[r, "\sim"'] & !(\mathbf{1}\otimes A)
\end{tikzcd} \qquad \cdots (17)$$

$$\begin{tikzcd}
!A \arrow[r, "!f"] \arrow[d, "\delta"'] \arrow[dr, phantom, "\Swarrow"] & !B \arrow[d, "\delta"] \\
!!A \arrow[r, "!!f"'] & !!B
\end{tikzcd} \qquad \cdots (18)$$

$$\begin{tikzcd}
!A \arrow[r, "!f"] \arrow[d, "\varepsilon"'] \arrow[dr, phantom, "\Swarrow"] & !B \arrow[d, "\varepsilon"] \\
A \arrow[r, "f"'] & B
\end{tikzcd} \qquad \cdots (19)$$

$$\begin{tikzcd}
!A \arrow[r, "!f"] \arrow[d, "d"'] \arrow[dr, phantom, "\Swarrow"] & !B \arrow[d, "d"] \\
!A\otimes !A \arrow[r, "!f\otimes !f"'] & !B\otimes !B
\end{tikzcd} \qquad \cdots (20)$$

$$\begin{tikzcd}
!A \arrow[rr, "!f"] \arrow[dr, "e"'] & \arrow[d, phantom, "\Swarrow"] & !B \arrow[dl, "e"] \\
& \mathbf{1} &
\end{tikzcd} \qquad \cdots (21)$$

The remaining two rules are interchanged depending on which base category is adopted. If we choose the symmetric monoidal closed category [20],

$$\begin{tikzcd}
A\otimes B \arrow[rr, "\mathrm{abs}\otimes 1"] \arrow[dr, "1"'] & \arrow[d, phantom, "\Swarrow"] & (B\multimap A\otimes B)\otimes B \arrow[dl, "\mathrm{ev}"] \\
& A\otimes B &
\end{tikzcd} \qquad \cdots (22)$$

$$\begin{tikzcd}
B\multimap A \arrow[rr, "\mathrm{abs}"] \arrow[dr, "1"'] & \arrow[d, phantom, "\Swarrow"] & B\multimap (B\multimap A)\otimes B \arrow[dl, "1\multimap\mathrm{ev}"] \\
& B\multimap A &
\end{tikzcd} \qquad \cdots (23)$$

If we select the $\star$-autonomous category we replace the above by the following two, where $\partial'_{ABC} : (A \parr B)\otimes C \to A \parr (B\otimes C)$ is induced from ∂_{ABC} by the symmetry of tensor and cotensor,

$$\begin{tikzcd}
\mathbf{1}\otimes A \arrow[rr, "\tau\otimes 1"] \arrow[d, "\sim"'] & & (A\parr A^*)\otimes A \arrow[d, "\partial'"] \\
A \arrow[dr, "\sim"'] & \Swarrow & A\parr (A^*\otimes A) \arrow[dl, "1\parr\gamma"] \\
& A\parr\bot &
\end{tikzcd} \qquad \cdots (22)$$

$$\begin{tikzcd}
A^*\otimes\mathbf{1} \arrow[rr, "1\otimes\tau"] \arrow[d, "\sim"'] & & A^*\otimes (A\parr A^*) \arrow[d, "\partial"] \\
A^* \arrow[dr, "\sim"'] & \Swarrow & (A^*\otimes A)\parr A^* \arrow[dl, "\gamma\parr 1"] \\
& \bot\parr A^* &
\end{tikzcd} \qquad \cdots (23)$$

If we define $X \multimap Y$ with $Y \parr X^*$, abs_A^B with $A \xrightarrow{\sim} A\otimes\mathbf{1} \xrightarrow{\cdot\tau_B} A\otimes(B\parr B^*) \xrightarrow{\partial} (A\otimes B)\parr B^*$ and ev_A^B with $(A\parr B^*)\otimes B \xrightarrow{\partial'} A\parr(B^*\otimes B) \xrightarrow{\cdot\gamma_B} A\parr\bot \xrightarrow{\sim} A$, then rule (22) and (23) for the symmetric monoidal closed base are a consequence of the corresponding rules for the $\star$-autonomous base.

In place of referring to the rules by numbers, we call them by the shape of their redexes.

For example, rule (1) is called $(\delta;\delta)$-type, rule (5) $(\delta;!d)$-type, and rule (17) $(\varphi_0;\tilde{\varphi})$-type. We call (1) through (7) collectively δ-type as the redexes start with δ. Likewise we call (9) through (12) $\tilde{\varphi}$-type, and (13) through (17) φ_0-type. If we collectively deal with (1) through (17) starting with one of $\delta, d, \tilde{\varphi}, \varphi_0$, we call a reduction in the group an *algebraic reduction*. We call (18) through (21) *naturality reductions*[2]. Rule (22) and (23) are called β and η reductions respectively.

4. Example: local confluence

We give several examples of computation in our calculus. We consider a few cases of local confluence. Global confluence will be discussed in a forthcoming paper.

First, let us consider $\varphi_0;\delta;\varepsilon$. If we contract $\delta;\varepsilon$ by rule (1), we obtain

$$\begin{tikzcd}
\mathbf{1} \arrow[r, "\varphi_0"] \arrow[d, "\varphi_0"'] & !\mathbf{1} \arrow[d, "\delta"] \arrow[dl, "1"] \arrow[l, phantom, "\circlearrowright", shift right=6ex] \\
!\mathbf{1} & !!\mathbf{1} \arrow[l, "\varepsilon"] \arrow[ul, phantom, "\Leftarrow", shift right=3ex]
\end{tikzcd}$$

while, if we contract $\varphi_0;\delta$ by rule (13), then we have a sequence of contractions as follows:

$$\begin{tikzcd}
\mathbf{1} \arrow[rr, "\varphi_0"] \arrow[d, "1"'] \arrow[dr, "\varphi_0"] & & !\mathbf{1} \arrow[dd, "\delta"] \arrow[dl, phantom, "\Leftarrow"] \\
\mathbf{1} \arrow[d, "\varphi_0"'] \arrow[ur, phantom, "\Leftarrow"] & !\mathbf{1} \arrow[l, "\varepsilon"] \arrow[dr, "!\varphi_0"] & \\
!\mathbf{1} \arrow[ur, phantom, "\Leftarrow"] & & !!\mathbf{1} \arrow[ll, "\varepsilon"]
\end{tikzcd}$$

where (14) and (19) are used in addition. The leftmost vertical arrows in the two diagrams are both equal to φ_0.

Second, let us consider $\delta;!d;\delta$. If we contract $!d;\delta$ first by a naturality reduction,

[2] The naturality of $\tilde{\varphi}$ is taken to be equivalence. We comment that if we turn the naturality into a rewriting rule in either orientation, the local confluence discussed in the next section fails.

$$
\begin{tikzcd}
 & & !A \arrow[rr, "\delta"] \arrow[dll, "d"'] \arrow[ddr, "\delta"] & & !!A \arrow[drr, "!d"] \arrow[dd, "\delta"] \arrow[l, phantom, "\Lleftarrow" description] & & \\
!A \otimes !A \arrow[dd, "\delta\otimes\delta"'] \arrow[drr, phantom, "\Lleftarrow"] & & & & & & !(!A \otimes !A) \arrow[dd, "\delta"] \arrow[dll, phantom, "\Lleftarrow"] \\
 & & & !!A \arrow[r, "!\delta"] \arrow[ddl, "!d"] & !!!A \arrow[drr, "!!d"] & & \\
!!A \otimes !!A \arrow[drr, "\tilde{\varphi}"'] & & & & \arrow[d, phantom, "\Lleftarrow"] & & !!(!A \otimes !A) \\
 & & !(!A \otimes !A) \arrow[rr, "!(\delta\otimes\delta)"'] & & !(!!A \otimes !!A) \arrow[urr, "!\tilde{\varphi}"'] & &
\end{tikzcd}
$$

If we contract $\delta; !d$ first,

$$
\begin{tikzcd}
 & & !A \arrow[rr, "\delta"] \arrow[dll, "d"'] & \arrow[d, phantom, "\Lleftarrow"] & !!A \arrow[drr, "!d"] & & \\
!A \otimes !A \arrow[drrr, "\delta\otimes\delta"] \arrow[dd, "\delta\otimes\delta"'] & & & & & & !(!A \otimes !A) \arrow[dd, "\delta"] \\
 & \arrow[d, phantom, "\Lleftarrow"] & & !!A \otimes !!A \arrow[urrr, "\tilde{\varphi}"'] \arrow[d, "\delta\otimes\delta"] & & \arrow[d, phantom, "\Lleftarrow"] & \\
!!A \otimes !!A \arrow[rrr, "!\delta\otimes!\delta"'] \arrow[drr, "\tilde{\varphi}"'] & & \arrow[d, phantom, "\circlearrowright"] & !!!A \otimes !!!A \arrow[dr, "\tilde{\varphi}"] & & & !!(!A \otimes !A) \\
 & & !(!A \otimes !A) \arrow[rr, "!(\delta\otimes\delta)"'] & & !(!!A \otimes !!A) \arrow[urr, "!\tilde{\varphi}"'] & &
\end{tikzcd}
$$

The next example of critical pairs starts with $\tilde{\varphi}; \delta; !d$. If we contract $\delta; !d$ first,

$$
\begin{tikzcd}
 & & {!A \otimes !B} \arrow[dl, "d\otimes d"'] \arrow[dr, "\tilde{\varphi}"] & & \\
 & {!A \otimes !A \otimes !B \otimes !B} \arrow[ddl, "\delta\otimes\delta\otimes\delta\otimes\delta"'] \arrow[dr, "\sim"] \arrow[rr, phantom, "\Swarrow"] & & {!(A \otimes B)} \arrow[ddl, "d"'] \arrow[ddr, "\delta"] & \\
 & & {!A \otimes !B \otimes !A \otimes !B} \arrow[d, "\tilde{\varphi}\otimes\tilde{\varphi}"] \arrow[dddll, "\delta\otimes\delta\otimes\delta\otimes\delta"] & & \\
{!!A \otimes !!A \otimes !!B \otimes !!B} \arrow[dd, "\sim"'] & & {!(A \otimes B) \otimes !(A \otimes B)} \arrow[d, "\delta\otimes\delta"] \arrow[rr, phantom, "\Swarrow"] & & {!!(A \otimes B)} \arrow[dd, "!d"] \\
 & & {!!(A \otimes B) \otimes !!(A \otimes B)} \arrow[drr, "\tilde{\varphi}"] & & \\
{!!A \otimes !!B \otimes !!A \otimes !!B} \arrow[dr, "\tilde{\varphi}\otimes\tilde{\varphi}"'] \arrow[rr, phantom, "\Swarrow"] & & & & {!(!(A \otimes B) \otimes !(A \otimes B))} \\
 & {!(!A \otimes !B) \otimes !(!A \otimes !B)} \arrow[uur, "!\tilde{\varphi}\otimes!\tilde{\varphi}"] \arrow[rr, "\tilde{\varphi}"'] & & {!(!A \otimes !B \otimes !A \otimes !B)} \arrow[ur, "!(\tilde{\varphi}\otimes\tilde{\varphi})"'] \arrow[u, phantom, "\circlearrowright"] &
\end{tikzcd}
$$

If we contract $\tilde{\varphi};\delta$ first

$$
\begin{tikzcd}
 & & {!A \otimes !B} \arrow[dl, "d\otimes d"'] \arrow[dr, "\tilde{\varphi}"] \arrow[dd, "\delta\otimes\delta"] & & \\
 & {!A \otimes !A \otimes !B \otimes !B} \arrow[dl, "\delta\otimes\delta\otimes\delta\otimes\delta"'] \arrow[r, phantom, "\Swarrow"] & & {!(A \otimes B)} \arrow[dr, "\delta"] \arrow[l, phantom, "\Swarrow"] & \\
{!!A \otimes !!A \otimes !!B \otimes !!B} \arrow[dd, "\tilde{\varphi}\otimes\tilde{\varphi}"'] & & {!!A \otimes !!B} \arrow[ddll, "!d\otimes!d"] \arrow[dr, "\tilde{\varphi}"] & & {!!(A \otimes B)} \arrow[dd, "!d"] \\
 & & & {!(!A \otimes !B)} \arrow[ur, "!\tilde{\varphi}"] \arrow[ddll, "!(d\otimes d)"] & \\
{!(!A \otimes !A) \otimes !(!B \otimes !B)} \arrow[dr, "\tilde{\varphi}"'] \arrow[r, phantom, "\circlearrowright"] & & & & {!(!(A \otimes B) \otimes !(A \otimes B))} \arrow[l, phantom, "\Swarrow"] \\
 & {!(!A \otimes !A \otimes !B \otimes !B)} \arrow[rr, "\sim"'] & & {!(!A \otimes !B \otimes !A \otimes !B)} \arrow[ur, "!(\tilde{\varphi}\otimes\tilde{\varphi})"'] &
\end{tikzcd}
$$

The obtained sequences of morphisms are not exactly equal. By the coherence theorem of symmetric monoidal functors, however, they are equivalent.

5. Comparison to a type theory

We briefly discuss the relation of our calculus to a type system of intuitionistic linear logic. It is intended to justify the design of the calculus. The following comparison shows that our categorical calculus is a refinement of a term calculus. Furthermore, it explains why twenty-three diagrams are oriented in that way.

We use the dual intuitionistic linear logic due to Barber [3], modified slightly. The modality $!A$ is used to decorate types while $\sharp\, M$ is used instead to decorate terms[3]. In the type environment, we use $\sharp\, x : A$ and $x : A$ to distinguish the intuitionistic part and the linear part, instead of partitioning by a semicolon as in the original. The modality $\sharp$ means that x is in the intuitionistic part. So an environment Γ is a finite sequence of $\sharp\, x_i : A_i$ or $x_i : A_i$ where the order has no significance. In the original system the intuitionistic part is strictly separated from the linear part. Instead, we use lifting to change the modality of a variable:

$$\frac{\Gamma,\ x : A \vdash M : B}{\Gamma,\ \sharp x : A \vdash M : B}$$

Accordingly, we limit the axiom to the shape of $x : A \vdash x : A$. Weakening and contraction are given explicitly:

$$\frac{\Gamma \vdash M : B}{\Gamma,\ \sharp x : A \vdash M : B} \qquad\qquad \frac{\Gamma,\ \sharp x' : A,\ \sharp x'' : A \vdash M : B}{\Gamma,\ \sharp x : A \vdash M[x/x'x''] : B}$$

In the contraction rule $M[x/x'x'']$ denotes the operation to substitute x simultaneously for x' and x''.

For terms we use postfix notation $M\{\sharp\, x \mapsto N\}$ in place of the prefix let-operator. It replaces $\mathsf{let}\ !x\ \mathsf{be}\ N\ \mathsf{in}\ M$ in Barber's system. The β-rule for $\sharp$ is given as $M\{\sharp\, x \mapsto \sharp\, N\} \Rightarrow M[N/x]$, and the η-rule as $\sharp\, x\{\sharp\, x \mapsto M\} \Rightarrow M$.

The type system is interpreted in the free intuitionistic linear category in a standard way. A type judgement $\Gamma \vdash M : B$ corresponds to a morphism $f : \Gamma \to B$ where Γ denotes the sequence of $!A_i$ or A_i connected by $\otimes$. If the type environment contains $\sharp\, x_i : A_i$ we use $!A_i$, and if it contains $x_i : A_i$ we use A_i. Here we associate a morphism with a derivation tree, rather than with a term, It is necessary for a fine analysis of the relation between the type system and our calculus.

First, let us justify the β-reduction for $\sharp$:

$$\frac{\begin{array}{c}\vdots\pi_1\\ \Gamma,\ \sharp x : A\ \vdash\ M : B\end{array} \qquad \dfrac{\begin{array}{c}\vdots\pi_2\\ \sharp\Delta\ \vdash\ K : A\end{array}}{\sharp\Delta\ \vdash\ \sharp K : !A}}{\Gamma,\ \sharp\Delta\ \vdash\ M\{\sharp x \mapsto \sharp K\} : B} \qquad \Rightarrow \qquad \begin{array}{c}\vdots\pi_1\sharp^x\pi_2\\ \Gamma,\ \sharp\Delta\ \vdash\ M[K/x] : B\end{array}$$

where $\sharp\,\Delta$ denotes that all type assignments in the environment have the shape of $\sharp\, x_i : A_i$. The derivation $\pi_1\sharp^x\pi_2$ is obtained by connecting π_2 at the place of the axiom involving x in π_1.

We split cases according to the last rule involving the variable x in π_1. If the last inference is lifting:

[3]Simply for immediate viewability. Barber uses ! for both.

$$\dfrac{\begin{array}{c}\vdots\rho_1\\ \Gamma,\, x : A \;\vdash\; M : B\end{array}}{\Gamma,\, \sharp x : A \;\vdash\; M : B}$$

then let ρ_1 be interpreted by $\Gamma \otimes A \xrightarrow{f} B$ and π_2 by $!\Delta \xrightarrow{h} A$. For example, if Δ consists of two types C_1 and C_2, the derivation before rewriting is interpreted as the counterclockwise sequence of arrows from $\Gamma \otimes !C_1 \otimes !C_2$ to B in the following diagram, and the one after reduction is the other extreme going clockwise. The contraction of derivations is realized by computation in our calculus as

$$\begin{tikzcd}
\Gamma \otimes !C_1 \otimes !C_2 \arrow[d, "\cdot\delta\delta"'] \arrow[dr, "1"] & & \\
\Gamma \otimes !!C_1 \otimes !!C_2 \arrow[r, "\cdot\varepsilon\varepsilon"'] \arrow[d, "\cdot\tilde{\varphi}"'] \arrow[ur, phantom, "\Nearrow"] & \Gamma \otimes !C_1 \otimes !C_2 \arrow[d, "\cdot h"] & \\
\Gamma \otimes !(!C_1 \otimes !C_2) \arrow[ur, "\cdot\varepsilon"'] \arrow[d, "\cdot !h"'] \arrow[ur, phantom, shift left=3, "\Nearrow"] \arrow[r, phantom, "\Nearrow"] & \Gamma \otimes A \arrow[r, "f"'] & B \\
\Gamma \otimes !A \arrow[ur, "\cdot\varepsilon"'] & &
\end{tikzcd}$$

The rules used here are (2), (10), and (19). If Δ is empty, rule (14) is used since φ_0 is employed in place of $\tilde{\varphi}$. If the last inference of π_1 is contraction:

$$\dfrac{\begin{array}{c}\vdots\rho_1\\ \Gamma,\, \sharp x' : A,\, \sharp x'' : A \;\vdash\; M : B\end{array}}{\Gamma,\, \sharp x : A \;\vdash\; M[x/x'x''] : B}$$

a similar analysis shows that the rules (4), (11), (15), and (20) are utilized. If the last inference of π_1 is weakening

$$\dfrac{\begin{array}{c}\vdots\rho_1\\ \Gamma \;\vdash\; M : B\end{array}}{\Gamma,\, \sharp x : A \;\vdash\; M : B}$$

the rules (6), (12), (16), and (21) are used. If the last inference of π_1 is $\sharp$-introduction,

$$\dfrac{\begin{array}{c}\vdots\rho_1\\ \sharp\Gamma,\, \sharp x : A \;\vdash\; M : B\end{array}}{\sharp\Gamma,\, \sharp x : A \;\vdash\; \sharp M : !B}$$

the rules (1), (9), (13), and (18) are used. When Δ is empty, rule (17) is also used.

Second, we consider the η-reduction for $\sharp$ modality:

$$\dfrac{\dfrac{\dfrac{\overline{x : A \vdash x : A}}{\sharp x : A \vdash x : A}}{\sharp x : A \vdash \sharp x : !A} \qquad \begin{array}{c} \vdots^{\pi} \\ \Delta \vdash K : !A \end{array}}{\Delta \vdash \sharp x\{\sharp x \mapsto K\} : !A} \quad \Rightarrow \quad \begin{array}{c} \vdots^{\pi} \\ \Delta \vdash K : !A \end{array}$$

The left-hand side is interpreted by $\Delta \xrightarrow{h} !A \xrightarrow{\delta} !!A \xrightarrow{!\varepsilon} !A$, and rule (3) is used.

If the type assignment introduced by weakening is deleted by contraction, both are superfluous. Hence we can have the following simplifying rule:

$$\dfrac{\dfrac{\begin{array}{c} \vdots \\ \Gamma, \sharp x' : A \vdash M : B \end{array}}{\Gamma, \sharp x' : A, \sharp x'' : A \vdash M : B}}{\Gamma, \sharp x : A \vdash M[x/x'] : B} \quad \Rightarrow \quad \begin{array}{c} \vdots \\ \Gamma, \sharp x : A \vdash M[x/x'] : B \end{array}$$

where rule (8) is used.

The rest are the interchanging rule of the $\sharp$ modality with weakening and contraction:

$$\dfrac{\dfrac{\begin{array}{c} \vdots \\ \sharp\Gamma, \sharp x' : A, \sharp x'' : A \vdash M : B \end{array}}{\sharp\Gamma, \sharp x : A \vdash M[x/x'x''] : B}}{\sharp\Gamma, \sharp x : A \vdash \sharp(M[x/x'x'']) : !B} \quad \Rightarrow \quad \dfrac{\dfrac{\begin{array}{c} \vdots \\ \sharp\Gamma, \sharp x' : A, \sharp x'' : A \vdash M : B \end{array}}{\sharp\Gamma, \sharp x' : A, \sharp x'' : A \vdash \sharp M : !B}}{\sharp\Gamma, \sharp x : A \vdash (\sharp M)[x/x'x''] : !B}$$

$$\dfrac{\dfrac{\begin{array}{c} \vdots \\ \sharp\Gamma \vdash M : B \end{array}}{\sharp\Gamma, \sharp x : A \vdash M : B}}{\sharp\Gamma, \sharp x : A \vdash \sharp M : !B} \quad \Rightarrow \quad \dfrac{\dfrac{\begin{array}{c} \vdots \\ \sharp\Gamma \vdash M : B \end{array}}{\sharp\Gamma \vdash \sharp M : !B}}{\sharp\Gamma, \sharp x : A \vdash \sharp M : !B}$$

The former uses rule (5) and the latter uses rules (7) and (17).

Finally the β-reduction for abstraction $(\lambda x.\, M)K \Rightarrow M[K/x]$ corresponds to rule (22) and the η-reduction $\lambda x.\, Mx \Rightarrow M$ to rule (23).

Every rule, save rule (17), is used exactly once, as observed from the analysis above. Each rule has an intrinsic role. The rewriting orientation of the diagrams is determined so that the contractions of derivation trees are simulated by our calculus. In addition, a single rewriting step of terms is realized by several steps of categorical rewriting. Therefore we conclude that the categorical calculus is a refinement of the term calculus. Furthermore, as mentioned in the introduction, we permit the rewriting $!(f;g) \rightsquigarrow !f;!g$. So our system incarnates a mechanism to decompose a term and substitute only a subterm obtained by decomposition. Our calculus is also a refinement in this sense.

6. Normalizability

We show the weak termination of the categorical reduction system. Hereafter we consider the system with base a $\star$-autonomous category. A reason for the choice is that naturality of abs_A and ev_A in the symmetric monoidal closed category is awkward and cumbersome to handle. Moreover, the latter is simulated by the former.

6.1. DEFINITION. A *normal form* is a morphism that is equivalent to the shape that has no redexes.

A redex may, however, be created from none as a consequence of congruence. For example, the obvious normal form $!A \xrightarrow{\varepsilon_A} A$ is, by composing $1_{!A} = !1_A$, equivalent to $!A \xrightarrow{!1_A} !A \xrightarrow{\varepsilon_A} A$ that has a naturality redex.

6.2. DEFINITION. A *reversible* reduction is one of the naturality rules (18) through (21) where f is an identity, a structural isomorphism or its inverse, or their compositions.

We can cancel reversible reductions. For example, suppose that $!A \xrightarrow{!f} !B \xrightarrow{d_B} !B \otimes !B$ is contracted to $!A \xrightarrow{d_A} !A \otimes !A \xrightarrow{!f \otimes !f} !B \otimes !B$. Then, by attaching $1_{!A} = !f; !f^{-1}$ in front and transferring $!f^{-1}$ by naturality, we restore a morphism that is equivalent to the original. Reversible reductions are regarded to be inessential.

6.3. LEMMA. *Only reversible reductions occur in a reduction sequence from a normal form.*

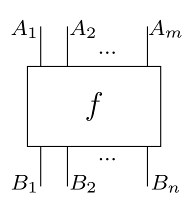
PROOF. The morphisms equivalent to identities are written as the composition of structural isomorphisms. ■

Once a morphism reaches a normal form, we can have only inessential reductions afterwards. In the following, we ignore reversible redexes. We assume they are removed by contraction implicitly.

6.4. DEFINITION. A morphism (weakly) *terminates* if there is a finite reduction sequence ending with a normal form[4].

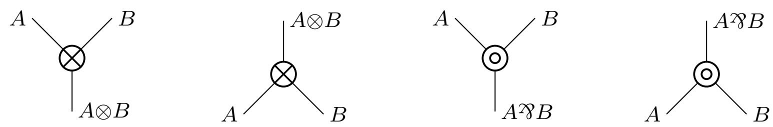
We are not motivated by constructing a graph reduction system, yet it is helpful to use graphs to avoid a nuisance incurred by structural isomorphisms and their coherences. In this paper, we only modestly use graphs, that are introduced informally to enhance intuitive understanding. To discuss confluence in a forthcoming paper, we will rely on full graphical visualization. We will not intend to construct a graph reduction system, though.

As in [7], we represent a morphism $f : A_1 \otimes A_2 \otimes \cdots \otimes A_m \to B_1 \parr B_2 \parr \cdots \parr B_n$ in the $\star$-autonomous category as a figure

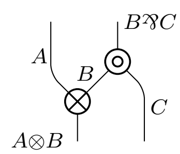


[4]In this paper, strong termination scarcely occurs. Hence we omit "weakly" for simplicity.

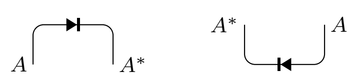
Each of tensor and cotensor has two gates



We use a double circle in place of $\parr$ as the latter symbol is not symmetric under a vertical flip. The linear distribution morphism $\partial : A \otimes (B \parr C) \to (A \otimes B) \parr C$, for example, corresponds to

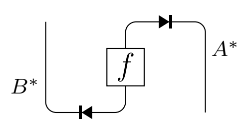


Duality morphisms τ_A and γ_A are represented by bends:



We add a diode-like symbol to signify which side has the duality star, so that it is restored if the labels attached to wires are omitted. The duality $\beta\eta$-reduction corresponds to the operation straightening double bends:

For $f : A \to B$, its dual $f^* : B^* \to A^*$ is depicted as



The verification of the weak termination is based on the standard reducibility method. The following proof strategy is inspired by [13].

6.5. Definition. A *positive funnel* on object A is a set $\mathcal{S}$ of morphisms $X \xrightarrow{f} A$ for varied X satisfying the following two conditions:

(i) The identity $A \xrightarrow{1} A$ is a member of $\mathcal{S}$.

(ii) Each $f \in \mathcal{S}$ terminates.

A *negative funnel* on A is a set $\mathcal{S}$ of morphisms $A \xrightarrow{f} X$ for varied X satisfying the same conditions (i) and (ii).

When we simply say a funnel, it means either a positive funnel or a negative funnel.

6.6. Definition. Given an object A, let $\mathcal{S}$ be a set of morphisms of the form $X \xrightarrow{f} A$ (resp. $A \xrightarrow{f} X$). The *complement* $\mathcal{S}^{\perp}$ is the set of all morphisms $A \xrightarrow{g} Y$ (resp. $Y \xrightarrow{g} A$) subject to the condition that $f; g$ (resp. $g; f$) terminates for all $f \in \mathcal{S}$.

6.7. Lemma. *If $\mathcal{S}$ is a positive (negative) funnel on A, the complement $\mathcal{S}^{\perp}$ is a negative (resp. positive) funnel.*

Proof. Condition (i) of $\mathcal{S}^{\perp}$ follows from (ii) of $\mathcal{S}$. Condition (ii) of $\mathcal{S}^{\perp}$ follows from (i) of $\mathcal{S}$. ∎

6.8. Lemma. *Let $\mathcal{R}$ and $\mathcal{S}$ be a set of morphisms of the form $X \xrightarrow{f} A$ or of the form $A \xrightarrow{f} X$.*

(i) *If $\mathcal{R} \subseteq \mathcal{S}$, then $\mathcal{S}^{\perp} \subseteq \mathcal{R}^{\perp}$.*
(ii) $\mathcal{S} \subseteq \mathcal{S}^{\perp\perp}$.
(iii) $\mathcal{S}^{\perp} = \mathcal{S}^{\perp\perp\perp}$.

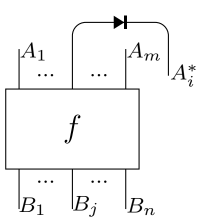
Proof. Standard. ∎

6.9. Definition. We define sets $\mathcal{R} \otimes \mathcal{S}$, $\mathcal{R} \parr \mathcal{S}$, $!\mathcal{S}$, and $\mathcal{S}^*$ for positive (or negative) funnels $\mathcal{R}$ and $\mathcal{S}$.

$$\begin{array}{lcl} \mathcal{R} \otimes \mathcal{S} & = & \{f \otimes g \mid f \in \mathcal{R},\ g \in \mathcal{S}\} \\ \mathcal{R} \parr \mathcal{S} & = & \{f \parr g \mid f \in \mathcal{R},\ g \in \mathcal{S}\} \\ !\mathcal{S} & = & \{!f \mid f \in \mathcal{S}\} \\ \mathcal{S}^* & = & \{f^* \mid f \in \mathcal{S}\}. \end{array}$$

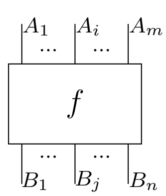
The first three sets $\mathcal{R} \otimes \mathcal{S}$, $\mathcal{R} \parr \mathcal{S}$, and $!\mathcal{S}$ are clearly positive (or negative) funnels. However, the last set $\mathcal{S}^*$ is not a funnel. In fact, identity 1_{A^*} is not a member of $\mathcal{S}^*$, since it does not equal $(1_A)^*$, as (-)* is not a contravariant functor.

6.10. Lemma. *Let $A_1 \otimes A_2 \otimes \cdots \otimes A_m \xrightarrow{f} B_1 \parr B_2 \parr \cdots \parr B_n$ be a morphism. Then*

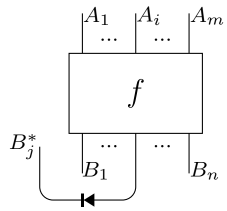


terminates

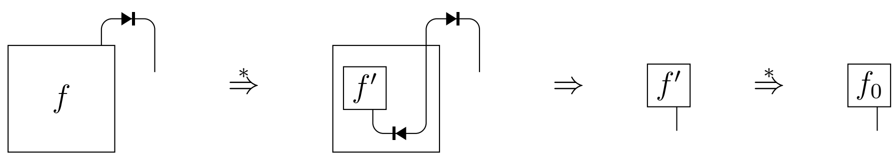
$\Longleftrightarrow$



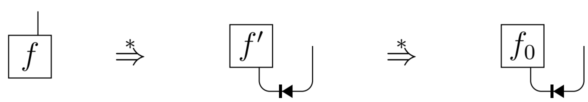
terminates

$\Longleftrightarrow$  *terminates.*

Proof. As they are symmetric, we will verify only the first equivalence. Suppose that f with a bend terminates. If it leads to a normal form where the bend is intact, there is an obvious terminating reduction sequence from f. Otherwise, the bend vanishes by η-reduction as in

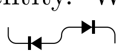


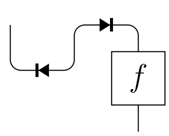
where f_0 is a normal form. Then we have



The last is a normal form unless the bend is connected to another bend, forming a β-redex. If so the contraction of the redex leads to a normal form. The converse is straightforward. ∎

6.11. Lemma. *If $\mathcal{S}$ is a negative (or positive) funnel, then $(\mathcal{S}^*)^\perp$ is a negative (or positive) funnel.*

Proof. Suppose $\mathcal{S}$ is a negative funnel. Since every morphism in $\mathcal{S}^*$ terminates by Lem. 6.10, $(\mathcal{S}^*)^\perp$ contains an identity. We verify that each morphism $f \in (\mathcal{S}^*)^\perp$ terminates. As $\mathcal{S}$ contains an identity,  is a member of $\mathcal{S}^*$. Hence



terminates. By Lem. 6.10, we can eliminate the leftmost bend. With a similar argument, we can eliminate the other bend as well. ∎

6.12. Definition. For each object A, we define a negative funnel $R^-(A)$ and a positive funnel $R^+(A)$ by induction on the construction of A.

$$
\begin{array}{lcll}
R^-(A) & = & \{1_A\}^\perp & \text{if } A \text{ is atomic} \\
R^-(\mathbf{1}) & = & \{1_{\mathbf{1}}\}^\perp & \\
R^+(\perp) & = & \{1_\perp\}^\perp & \\
R^-(A \otimes B) & = & (R^+(A) \otimes R^+(B))^\perp & \\
R^+(A \invamp B) & = & (R^-(A) \invamp R^-(B))^\perp & \\
R^-(A^*) & = & ((R^-(A))^*)^\perp & \\
R^-(!A) & = & (!R^+(A))^\perp &
\end{array}
$$

Only one of $R^-(A)$ and $R^+(A)$ is listed above. The other is defined as its complement $(–)^\perp$. For example, $R^+(!A)$ is $(R^-(!A))^\perp$. Evidently, $R^+(A)$ and $R^-(A)$ are the complements of each other.

6.13. Lemma. *Let $A \xrightarrow{f} B$ and $C \xrightarrow{g} D$ be morphisms. The following implications hold:*

$$
\begin{array}{lcl}
f \in R^-(A) & \Longrightarrow & !f \in R^-(!A) \\
f \in R^+(B) & \Longrightarrow & !f \in R^+(!B)
\end{array}
$$

$$
\begin{array}{lcl}
f \in R^-(A),\ g \in R^-(C) & \Longrightarrow & f \otimes g \in R^-(A \otimes C) \\
f \in R^+(B),\ g \in R^+(D) & \Longrightarrow & f \otimes g \in R^+(B \otimes D)
\end{array}
$$

$$
\begin{array}{lcl}
f \in R^-(A),\ g \in R^-(C) & \Longrightarrow & f \invamp g \in R^-(A \invamp C) \\
f \in R^+(B),\ g \in R^+(D) & \Longrightarrow & f \invamp g \in R^+(B \invamp D)
\end{array}
$$

$$
\begin{array}{lcl}
f \in R^-(A) & \Longrightarrow & f^* \in R^+(A^*) \\
f \in R^+(B) & \Longrightarrow & f^* \in R^-(B^*)
\end{array}
$$

Proof. We verify the case for !. We take an arbitrary $!g$ from $!R^+(A)$. Composition $g; f$ terminates by hypothesis. So $!g; !f = !(g; f)$ terminates as well. Thus the first assertion follows. The second assertion is a consequence of the inflation property of $(–)^{\perp\perp}$. The rest are similar. For $(–)^*$, we note that $g^*; f^*$ contracts to $(f; g)^*$ by β-reduction. ■

6.14. Definition. A *reducible* morphism is a morphism $A \xrightarrow{f} B$ satisfying that $g; f; h$ terminates for every pair of $g \in R^+(A)$ and $h \in R^-(B)$.

6.15. Lemma. *For a morphism $A \xrightarrow{f} B$, the following are equivalent:*

(i) *f is reducible.*
(ii) *If $g \in R^+(A)$ then $g; f \in R^+(B)$.*
(iii) *If $h \in R^-(B)$ then $f; h \in R^-(A)$.*

Proof. Straightforward as each of $R^+(A)$ and $R^-(A)$ is the complement of the other. ■

6.16. Proposition. *Each reducible morphism terminates.*

Proof. All funnels contain identity morphisms. ■

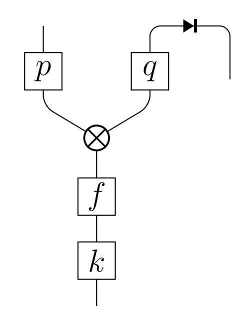
To verify the termination property, therefore it suffices to show that all morphisms in the free classical linear category are reducible. We start with the easy cases.

6.17. Lemma. *The following hold:*
(i) *A morphism* $!A \xrightarrow{f} B$ *is reducible iff* $!g; f$ *lies in* $R^+(B)$ *for every* g *in* $R^+(A)$.
(ii) *A morphism* $A \otimes B \xrightarrow{f} C$ *is reducible iff* $(g \otimes h); f$ *lies in* $R^+(C)$ *for every pair of* g *in* $R^+(A)$ *and* h *in* $R^+(B)$.
(iii) *A morphism* $A \xrightarrow{f} B \parr C$ *is reducible iff* $f; (g \parr h)$ *lies in* $R^-(A)$ *for every pair of* g *in* $R^-(B)$ *and* h *in* $R^-(C)$.
(iv) *A morphism* $\mathbf{1} \xrightarrow{f} A$ *is reducible iff* f *lies in* $R^+(A)$.
(v) *A morphism* $B^* \xrightarrow{f} C$ *is reducible iff* $g^*; f$ *lies in* $R^+(C)$ *for every* g *in* $R^-(B)$.

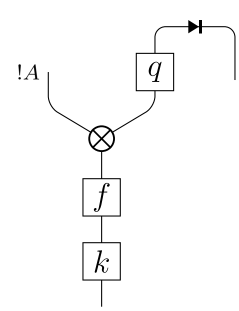
Proof. We prove (i) as the argument is similar. Supposed that f is reducible, $f; h \in R^-(!A)$ for every $h \in R^-(B)$ by Lem. 6.15. By definition of $R^-(!A)$, then, $!g; f; h$ terminates for every $g \in R^+(A)$. As h is arbitrary, $!g; f$ belongs to $(R^-(B))^\perp = R^+(B)$. The converse is also true. ∎

6.18. Lemma. *A morphism* $!A \otimes !B \xrightarrow{f} C$ *is reducible iff* $(!g \otimes !h); f$ *lies in* $R^+(C)$ *for every pair of* $g \in R^+(A)$ *and* $h \in R^+(B)$.

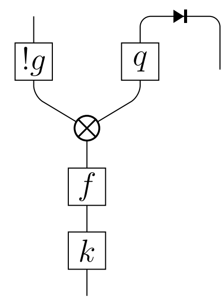
Proof. By Lem. 6.17 f is reducible iff, for every p in $R^+(!A)$, every q in $R^+(!B)$, and every k in $R^-(C)$, the morphism $(p \otimes q); f; k$ terminates. By the bending technique of Lem. 6.10, it means termination of



Since p is arbitrary, it amounts to that



lies in $R^{-}(!A)$. By Lem. 6.17, it says that



terminates for every $g \in R^{+}(A)$. Straightening the bend over q, we have succeeded in replacing p with $!g$. Applying the same process to the right wire as well, we obtain the lemma. ∎

6.19. Proposition. *The following hold:*

(i) *Identities and structural isomorphisms are reducible.*
(ii) *The composition of reducible morphisms is reducible.*

Proof. (i) is obvious. (ii) is an immediate consequence of Lem. 6.15. ∎

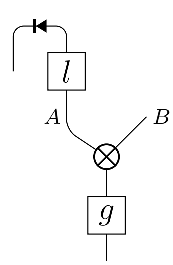
6.20. Proposition. *The following hold:*

(i) *If f is reducible, $!f$ is reducible.*
(ii) *If f and g are reducible, $f \otimes g$ is reducible.*
(iii) *If f and g are reducible, $f \parr g$ is reducible.*
(iv) *If f is reducible, f^* is reducible.*

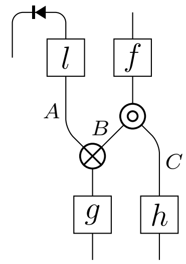
Proof. (i) through (iii) are consequences of Lem. 6.17. We prove (iv). Suppose that $f : A \to B$ is reducible. Again by Lem. 6.17, it suffices to show that $g^*; f^*; h$ terminates for every pair of $g \in R^{-}(B)$ and $h \in R^{-}(A^*)$. We note that $g^*; f^*$ contracts to $(f; g)^*$ by a duality β-reduction. Here $f; g$ lies in $R^{-}(A)$, thus $(f; g)^*$ lies in $R^{+}(A^*)$. So $(f; g)^*; h$ terminates. ∎

6.21. Proposition. *∂ is reducible.*

Proof. Suppose $\partial : A \otimes (B \parr C) \to (A \otimes B) \parr C$. We must show that $(l \otimes f); \partial; (g \parr h)$ terminates for every $l \in R^{+}(A)$, every $f \in R^{+}(B \parr C)$, every $g \in R^{-}(A \otimes B)$, and every $h \in R^{-}(C)$. For every $k \in R^{+}(B)$, $(l \otimes k); g$ terminates. Thus, by the bending technique of Lem. 6.10,



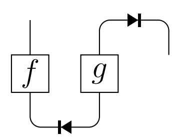
lies in $R^-(B)$. Hence



terminates. Straightening the bend over l, we obtain the proposition. ∎

6.22. Proposition. *τ_A and γ_A are reducible.*

Proof. Verification is the same for both cases. We give the proof for $\gamma_A : A^* \otimes A \to \bot$. We must show that $(f \otimes g);\gamma_A$ terminates for every pair of $f \in R^+(A^*)$ and $g \in R^+(A)$. Composition $f;g^*$ terminates since $g^* \in R^-(A^*)$ by Lem 6.13. Graphically this composition means



Straightening the bend over g, we obtain the proposition. ∎

It remains to verify that the six algebraic morphisms are reducible. There are technical difficulties that are unable to be covered by the reducibility method. To that end, we introduce several notions.

Let X range over a chosen non-empty family of objects. Later we use the case where the family is a singleton or consists of two elements. We regard X as if they are atomic objects. We consider two classes of objects generated by the following generative grammar.

$$\begin{array}{lllllllll} A & ::= & & & X & \mid & A \otimes A & \mid & !A \\ B & ::= & \mathbf{1} & \mid & X & \mid & B \otimes B & \mid & !B \end{array}$$

6.23. Definition.

(i) A *composite algebraic morphism* t is a member of the class generated from 1_B and $\varphi_0, \tilde{\varphi}_{B,B'}, \delta_B, \varepsilon_B, d_B, e_B$ as well as $\alpha_{B,B',B''}, \sigma_{B,B'}, \lambda_B, \rho_B$ and their inverses, closed under operations $t \otimes t'$, $!t$ and composition $t;t'$.

(ii) A *strict composite algebraic morphism* s is a member of the class generated from 1_A and $\tilde{\varphi}_{A,A'}, \delta_A, \varepsilon_A, d_A, e_A$ as well as $\alpha_{A,A',A''}, \sigma_{A,A'}$ and their inverses, closed under operations $s \otimes s'$, $!s$ and composition $s;s'$.

Namely, composite algebraic morphisms can use everything unrelated to ⅋ or $(-)^*$ as long as X is regarded as an atomic object. Strict composite algebraic morphisms preclude φ_0 and the isomorphisms involving $\mathbf{1}$. We comment that the target of $e_A : !A \to \mathbf{1}$ is not a member of class A. Save this exception, composite algebraic morphisms are between members of class B and strict composite algebraic morphisms are between members of A.

6.24. EXAMPLE. $!X \xrightarrow{\delta_X} !!X \xrightarrow{d_{!X}} !!X \otimes !!X \xrightarrow{\cdot !e_X} !!X \otimes !\mathbf{1}$ is a strict composite algebraic morphism. It contracts to $!X \xrightarrow{d_X} !X \otimes !X \xrightarrow{\delta_X \delta_X} !!X \otimes !!X \xrightarrow{\cdot !e_X} !!X \otimes !\mathbf{1}$, which is strict composite algebraic. It further contracts to $!X \xrightarrow{d_X} !X \otimes !X \xrightarrow{\delta_X e_X} !!X \otimes \mathbf{1} \xrightarrow{\cdot \varphi_0} !!X \otimes !\mathbf{1}$, which is composite algebraic but not strictly composite algebraic since it contains φ_0. Finally, it contracts to $!X \xrightarrow{\delta_X} !!X \xrightarrow{\sim} !!X \otimes \mathbf{1} \xrightarrow{\cdot \varphi_0} !!X \otimes !\mathbf{1}$, which is composite algebraic. We will return to this sequence of contractions in Example 6.28.

A strict composite algebraic morphism s that has no naturality redexes at the beginning may create naturality through reduction. For example, $!X \xrightarrow{\delta} !!X \xrightarrow{\delta} !!!X \xrightarrow{e} \mathbf{1}$ contracts to $!X \xrightarrow{\delta} !!X \xrightarrow{!\delta} !!!X \xrightarrow{e} \mathbf{1}$, the latter containing naturality redex $!\delta; e$ while the former has none. This happens because rule (1) produces $!\delta$ wrapped by $!(–)$. Likewise rule (9) produces $!\tilde{\varphi}$. We do not have to consider rule (13) as $\varphi_0; \delta$ is not allowed in strict composite algebraic morphisms.

6.25. LEMMA. *Let u denote one of δ, ε, d and e. Suppose that a strict composite algebraic morphism s has no naturality redexes other than those of the form $!f; u_A$ where f consists of δ and $\tilde{\varphi}$ only. Any morphism obtained by contraction of s satisfies the same property for naturality redexes.*

PROOF. Simple case analysis. We cannot create naturality redexes when f contains something other than δ and $\tilde{\varphi}$ by contraction unless we have such redexes from the outset. ■

A *restricted* naturality redex is $!f; u_A$ where f consists solely of δ and $\tilde{\varphi}$. The above lemma asserts that if the naturality redexes of a strict composite algebraic morphism are restricted then the property is preserved under contraction.

We verify that strict composite algebraic morphisms (strongly) terminate if their naturality redexes are restricted. Although X may run over a family of two or more objects, the following argument is irrelevant to the number of distinct X. So we describe the case when X is unique. If there are two or more, each X should read one of them appropriately. We write $A = A[X, X, \dots, X]$ displaying each occurrence of X. We further write $A = A[X^{x_1}, X^{x_2}, \dots, X^{x_n}]$. At this stage, the x_i are merely the labels to distinguish occurrences. As we explain shortly, however, we assign natural numbers greater than or equal to 2.

Suppose that

$$s : A[X^{y_1}, X^{y_2}, \dots, X^{y_n}] \to B[X^{x_1}, X^{x_2}, \dots, X^{x_m}].$$

Each y_i is computed by applying a function $|s|$ determined by the shape of s to some of $x_1, x_2, \dots, x_m$. We show that if s is contracted by applying a certain type of reduction rules, then $y_1 + y_2 + \cdots + y_n$ strictly decreases. To define $|s|$ we need some auxiliary data given below.

6.26. DEFINITION. Let x denote an occurrence of X in A. We define $\theta_A(x)$ recursively as follows: (i) If $A = X$ then we set $\theta_X(x) = x$. (ii) For the exponential, we set $\theta_{!A}(x) =$

$2\theta_A(x)$. (iii) For the tensor, we set $\theta_{A\otimes A'}(x) = b + \theta_A(x)$ and symmetrically $\theta_{A'\otimes A}(x) = b + \theta_A(x)$ where b denotes the number of occurrences of X in A'.

This recursive definition is applied to each occurrence of X. For example if $A = !(X^{x_1} \otimes !!X^{x_2})$ then $\theta_A(x_1) = 2(1 + x_1)$ and $\theta_A(x_2) = 2(1 + 4x_2)$. Observe that θ_A is not a single function, the shape of which changes per occurrence. We remark that a structural isomorphism does not affect θ_A since it does not change the number of occurrences of X. For example $\theta_{(A\otimes B)\otimes C}(x) = \theta_{A\otimes(B\otimes C)}(x)$.

Let $A[X^x]$ denote a specific occurrence of X in A. Suppose that $s : A \to B$ is a strict composite algebraic morphism. With each occurrence $A[X^y]$ of X in A, we can naturally associate a finite number of occurrences $B[X^{x_1}, X^{x_2}, \ldots, X^{x_n}]$ of X. The number n depends on the shape of s. If $s = d_A$ then $n = 2$ and we set $d_A : !A[X^y] \to !A[X^{x_1}] \otimes !A[X^{x_2}]$, where $A[X^{x_i}]$ signifies the occurrence of X at the same position as $A[X^y]$. If $s = e_A$ then $n = 0$. For $\tilde{\varphi}_{A,A'}, \delta_A, \varepsilon_A, d_A, e_A, \alpha_{A,A',A''}$, and $\sigma_{A,A'}$, we have $n = 1$ and the association is straightforward. This association is naturally extended to the composition $s;t$: if X^z associates with $X^{y_1}, X^{y_2}, \ldots, X^{y_n}$ in s, and if each X^{y_i} associates with $X^{x_{i1}}, X^{x_{i2}}, \ldots, X^{x_{im_i}}$ in t, then z associates with all of $X^{x_{11}}$ through $X^{x_{nm_n}}$ in $s;t$. The tensor $s \otimes t$ and the exponentiation $!s$ do not alter the association.

6.27. Definition. We define the arithmetic expression $y = |s|(x_1, x_2, \ldots, x_n)$ for each strict composite algebraic morphism $s : A[X^y] \to B[X^{x_1}, X^{x_2}, \ldots, X^{x_n}]$ and each occurrence $A[X^y]$ of X in A.

(i) For $\delta_A : !A[X^y] \to !!A[X^x]$ we associate $y = |\delta_A|(x)$ where $|\delta_A|(x) = 2^{\theta_A(x)}$.
(ii) For $d_A : !A[X^y] \to !A[X^x] \otimes !A[X^{x'}]$ we set $y = |d_A|(x, x')$ where $|d_A|(x, x') = \theta_A(x) + \theta_A(x')$.
(iii) For $\varepsilon_A : !A[X^y] \to A[X^x]$ we set $y = |\varepsilon_A|(x)$ where $|\varepsilon_A|(x) = \theta_A(x)$.
(iv) For $e_A : !A[X^y] \to \mathbf{1}$ we set $y = |e_A|()$ where $|e_A|() = \theta_A(2)$.
(v) For the other components $1_A, \tilde{\varphi}_{A,A'}, \alpha_{A,A',A''}$, and $\sigma_{A,A'}$, we set $y = x$.
(vi) For the composition, $|s;t|$ is defined by the composition of the reverse order, $|s| \circ |t|$. Namely, if $z = |s|(y_1, y_2, \ldots, y_n)$ and $y_i = |t|(x_{i1}, x_{i2}, \ldots, x_{im_i})$, then $z = |s;t|(x_{11}, \ldots, x_{nm_n})$ is obtained by substitutions for y_i. For the tensor, $|s \otimes t|$ is either $|s|$ or $|t|$, depending on which side of tensor X^y lies in. Finally, we set $|!s| = |s|$.

The definition is applied to each occurrence separately. For instance, when $A = !(X \otimes !!X)$ and $d_A : !(X^{y_1} \otimes !!X^{y_2}) \to !(X^{x_1} \otimes !!X^{x_2}) \otimes !(X^{x_1'} \otimes !!X^{x_2'})$ then $y_1 = |d_A|(x_1, x_1') = 2(1 + x_1) + 2(1 + x_1')$ and $y_2 = |d_A|(x_2, x_2') = 2(1 + 4x_2) + 2(1 + 4x_2')$. As usual, we can interpret the arithmetic expression $|s|(x_1, x_2, \ldots, x_n)$ as a function. For the reason explained in Lem. 6.30, we assume x_i are natural numbers greater than or equal to 2. As is clear from the definition, the functions are increasing.

6.28. Example. We consider the morphisms in Example 6.24. If we label the first morphism as in

$$!X^y \xrightarrow{\delta_X} !!X \xrightarrow{d_{!X}} !!X \otimes !!X \xrightarrow{\cdot !e_X} !!X^x \otimes !\mathbf{1},$$

we have $y = 2^{\theta_{!X}(x)+\theta_{!X}(2)} = 2^{2x+4}$. It contracts to

$$!X^{y'} \xrightarrow{d_X} !X \otimes !X \xrightarrow{\delta_X \delta_X} !!X \otimes !!X \xrightarrow{\cdot !e_X} !!X^x \otimes !\mathbf{1},$$

for which we have $y' = 2^x + 2^2$. It further contracts to

$$!X^{y''} \xrightarrow{d_X} !X \otimes !X \xrightarrow{\delta_X e_X} !!X \otimes \mathbf{1} \xrightarrow{\cdot \varphi_0} !!X^x \otimes !\mathbf{1},$$

for which $y'' = 2^x + 2$. Finally it contracts to

$$!X^{y'''} \xrightarrow{\delta_X} !!X \xrightarrow{\sim} !!X \otimes \mathbf{1} \xrightarrow{\cdot \varphi_0} !!X^x \otimes !\mathbf{1},$$

for which $y''' = 2^x$. We observe $2^{2x+4} > 2^x + 2^2 > 2^x + 2 > 2^x$. We will verify that this is universally true.

6.29. Remark. Tranquilli assigns natural numbers to show the termination of net rewriting [29]. Exact correspondence to our assignment is not immediate. We stop here by commenting that the assignment to diagonal d has similarity.

6.30. Lemma. *Algebraic reduction in a strict composite algebraic morphism decreases the natural numbers involved in the redex.*

Proof. As φ_0-type contraction never occurs, it suffices to consider rule (1) through (12). Since $\mathbf{1}$ is not involved, in the definition of $\theta_{A\otimes B}(x) = b + \theta_A(x)$, the number b is greater than or equal to 1. We verify several subtle cases, leaving the others to the reader.

Rule (1). Suppose $!!!A[X^x]$ where X^x denotes an arbitrary occurrence in A. Then

$$\begin{aligned} |\delta_A; !\delta_A|(x) &= 2^{\theta_A(2^{\theta_A(x)})} \\ |\delta_A; \delta_{!A}|(x) &= 2^{\theta_A(2^{2\theta_A(x)})} \end{aligned}$$

as $\theta_{!A}(x) = 2\theta_A(x)$. Since $\theta_A(x) < 2\theta_A(x)$ we have $|\delta_A; !\delta_A|(x) < |\delta_A; \delta_{!A}|(x)$.

Rule (3). Suppose that $!(A[X^x] \otimes A[X^{x'}])$ displays two occurrences at the corresponding same positions in A. Then

$$\begin{aligned} |d_A; (\delta_A \otimes \delta_A); \tilde{\varphi}_{!A,!A}|(x, x') &= \theta_A(2^{\theta_A(x)}) + \theta_A(2^{\theta_A(x')}) \\ |\delta_A; !d_A|(x, x') &= 2^{\theta_A(\theta_A(x)+\theta_A(x'))}. \end{aligned}$$

So, putting $u = \theta_A(x)$ and $v = \theta_A(x')$, we must show that $\theta_A(2^u) + \theta_A(2^v) < 2^{\theta_A(u+v)}$. This inequality is verified by induction. If θ_A is an identity function, the inequality amounts to $2^u + 2^v < 2^{u+v}$, which is correct as we have $u, v \geq 2$ since we assumed that $x_i \geq 2$. If $\theta_A = 2\theta_B$ the inequality amounts to $2\theta_B(2^u) + 2\theta_B(2^v) < 2^{2\theta_B(u+v)}$. By the induction hypothesis (LHS) $< 2 \cdot 2^{\theta_B(u+v)}$. By $1 < \theta_B(u+v)$ this is smaller than (RHS). If $\theta_A = b + \theta_B$ the inequality amounts to $2b + \theta_B(2^u) + \theta_B(2^v) < 2^{b+\theta_B(u+v)}$. By the induction hypothesis (LHS) $< 2b + 2^{\theta_B(u+v)}$. This is less than or equal to $2^b + 2^{\theta_B(u+v)} \leq$ (RHS). For the last inequality we use $1 \leq b$.

$\tilde{\varphi}$-type rules. These are manipulated uniformly. For example, let us consider rule (9). Suppose $!!(A[X^x] \otimes B)$. Then

$$\begin{aligned} |(\delta_A \otimes \delta_B); \tilde{\varphi}_{!A,!B}; !\tilde{\varphi}_{A,B}|(x) &= |\delta_A|(x) = 2^{\theta_A(x)} \\ |\tilde{\varphi}_{A,B}; \delta_{A\otimes B}|(x) &= |\delta_{A\otimes B}|(x) = 2^{b+\theta_A(x)} \end{aligned}$$

where b is the number of occurrences of X in B. Since $1 \leq b$ the former is smaller than the latter. The case when the specified X occurs in B is similar. ■

Next, we verify that restricted naturality reductions decrease the assigned natural numbers. We give several lemmata towards it.

6.31. LEMMA. *The inequality* $1 + \theta_A(x) \leq \theta_A(1 + x)$ *holds.*

PROOF. By induction on the construction of A. ■

6.32. LEMMA. *The inequality* $2\theta_A(x) \leq \theta_A(2b^- + 2x)$ *holds where* b^- *is one less than the number of occurrences of* X *in* A.

PROOF. If we integrate serial applications of tensor, we have

$$\theta_A(x) = 2^{k_0}b_0 + 2^{k_0+k_1}b_1 + \cdots + 2^{k_0+k_1+\cdots+k_{q-1}}b_{q-1} + 2^{k_0+k_1+\cdots+k_q}x.$$

This is the case, for example, if $A = !^{k_0}(A_0 \otimes !^{k_1}(A_1 \otimes \cdots !^{k_{q-1}}(A_{q-1} \otimes !^{k_q}X)\cdots))$ and each A_i contains b_i occurrences of X. We have $b_i \geq 1$. The two numbers k_0 and k_q in both ends are non-negative while the other k_i are strictly positive. If $q = 0$, i.e., when A is $!^{k_0}X$, we have $\theta_A(x) = 2^{k_0}x$. The equality holds in this case as $b^- = 0$. If $q > 0$ we note $\max\{b_0, b_1, \ldots, b_{q-1}\} \leq b^-$. We observe that $2^{k_0} + 2^{k_0+k_1} + \cdots + 2^{k_0+k_1+\cdots+k_{q-1}} < 2^{k_0+k_1+\cdots+k_{q-1}+1}$ holds, which is clear if regarded as numbers in base 2. Therefore we have $2^{k_0}b_0 + 2^{k_0+k_1}b_1 + \cdots + 2^{k_0+k_1+\cdots+k_{q-1}}b_{q-1} < 2^{k_0+k_1+\cdots+k_{q-1}+1}b^- \leq 2^{k_0+k_1+\cdots+k_{q-1}+k_q}(2b^-)$. Thus $2\theta_A(x) < 2^{k_0+k_1+\cdots+k_q}(2b^-) + \theta_A(2x) = \theta_A(2b^- + 2x)$. ■

6.33. LEMMA. *Let* b^- *be one less than the number of occurrences of* X *in* A. *Then* $b^- < \theta_A(x)$ *holds.*

PROOF. The definition of $\theta_A(x)$ sums up all occurrences of X through recursive calls. ■

6.34. LEMMA. *Restricted naturality reduction in a strict composite algebraic morphism decreases the natural numbers involved in the redex.*

PROOF. Consider the redex $!f; u_A$. It suffices to prove the case where f consists of a single δ or of a single $\tilde{\varphi}$. Since the latter is simpler, we prove it first. Namely, suppose that $f = F(\tilde{\varphi}_{A\otimes A'}) : F(!A \otimes !A') \to F(!(A \otimes A'))$ for a functor F. For example, let us suppose $u = \delta$. We have

$$\begin{aligned} |\delta_{F(!A\otimes !A')}; !!F(\tilde{\varphi}_{A,A'})|(x) &= 2^{\theta_{F(!A\otimes !A')}(x)} \\ |!F(\tilde{\varphi}_{A,A'}); \delta_{F(!(A\otimes A'))}|(x) &= 2^{\theta_{F(!(A\otimes A'))}(x)}. \end{aligned}$$

If X^x occurs in A and if b is the nubmer of occurrences of X in A', we have $\theta_{!A\otimes !A'}(x) = b+2\theta_A(x) < 2(b+\theta_A(x)) = \theta_{!(A\otimes A')}(x)$ as $1 \leq b$. Therefore the former is smaller than the latter. We note that this relies only on comparison between $\theta_{!A\otimes !A'}$ and $\theta_{!(A\otimes A')}$. Hence the same argument applies to all the naturality rules.

Next we deal with the case when $f = F(\delta_A) : F(!A) \to F(!!A)$.

Rule (18).

$$\begin{aligned} |\delta_{F(!A)}; !!F(\delta_A)|(x) &= 2^{\theta_F(2\theta_A(2^{\theta_A(x)}))} \\ |!F(\delta_A); \delta_{F(!!A)}|(x) &= 2^{\theta_A(2^{\theta_F(4\theta_A(x))})}. \end{aligned}$$

So, putting $u = \theta_A(x)$, we must show $\theta_F(2\theta_A(2^u)) < \theta_A(2^{\theta_F(4u)})$. This is verified by induction on F. We start with the case $\theta_F = 2\theta_G$. By the induction hypothesis and Lem. 6.32, (LHS) $= 2\theta_G(2\theta_A(2^u)) < \theta_A(2b^- + 2\cdot 2^{\theta_G(4u)})$ where b^- is one less than the number of occurrences of X in A. On the other hand (RHS) $= \theta_A(2^{2\theta_G(4u)})$. So, if we put $t = \theta_G(4u)$, it suffices to show that $2b^- + 2\cdot 2^t \leq 2^{2t}$. As $u \leq t$ we can assume that $0 \leq b^- < t$ by Lem. 6.33. If $b^- = 0$ and $t = 1$ the inequality is directly checked. Assume $t \geq 2$. Then $2b^- + 2\cdot 2^t < 2t + 2\cdot 2^t < 2\cdot 2^t + 2\cdot 2^t = 2^{2+t} \leq 2^{2t}$ holds. The next case is $\theta_F = c + \theta_G$. By the induction hypothesis and Lem. 6.31, (LHS) $= c + \theta_G(2\theta_A(2^u)) < \theta_A(c + 2^{\theta_G(4u)})$. Applying $1 + 2^t < 2^{1+t}$ repeatedly, we conclude that it is smaller than $\theta_A(2^{c+\theta_G(4u)}) =$ (RHS). The base case is that θ_F is an identity function. By Lem. 6.32, (LHS) $= 2\theta_A(2^u) \leq \theta_A(2b^- + 2\cdot 2^u)$ where b^- is one less than the number of X in A. On the other hand (RHS) $= \theta_A(2^{4u})$. So it suffices to show that $2b^- + 2\cdot 2^u < 2^{4u}$, for which a sharper result has been verified in the first case.

Rule (19).

$$\begin{aligned} |\varepsilon_{F(!A)}; F(\delta_A)|(x) &= \theta_F(2\theta_A(2^{\theta_A(x)})) \\ |!F(\delta_A); \varepsilon_{F(!!A)}|(x) &= 2^{\theta_A(\theta_F(4\theta_A(x)))}. \end{aligned}$$

So we must prove that $\theta_F(2\theta_A(2^u)) < 2^{\theta_A(\theta_F(4u))}$ with $u = \theta_A(x)$. It is verified by induction on F. If θ_F is an identity, we show that $2\theta_A(2^u) < 2^{\theta_A(4u)}$ by induction on A. If θ_A is an identity, then obviously $2\cdot 2^u < 2^{4u}$. If $\theta_A = b+\theta_B$, the inner induction hypothesis implies (LHS) $< 2b+2^{\theta_B(4u)}$. So, for $t = \theta_B(4u)$, we show $2b+2^t \leq 2^{b+t}$. We can assume $1 \leq b < t$ by Lem. 6.33. Then the inequality is justified as $2b+2^t < 2^b+2^t < 2^{b+t}$. If $\theta_A = 2\theta_B$ then by the inner induction hypothesis (LHS) $< 2\cdot 2^{\theta_B(4u)} \leq 2^{2\theta_B(4u)} =$ (RHS). This finishes the base case. If $\theta_F = c + \theta_G$ then by the induction hypothesis (LSH) $< c + 2^{\theta_A(\theta_G(4u))}$, which is, by $1 + 2^t < 2^{1+t}$ and Lem. 6.31, smaller than $2^{\theta_A(c+\theta_G(4u))} =$ (RHS). If $\theta = 2\theta_G$ then by the induction hypothesis (LSH) $< 2\cdot 2^{\theta_A(\theta_G(4u))}$, which is, by Lem. 6.31, smaller than $2^{\theta_A(1+\theta_G(4u))} \leq 2^{\theta_A(2\theta_G(4u))} =$ (RHS).

Rule (20).

$$\begin{aligned} |d_{F(!A)}; !F(\delta_A) \otimes !F(\delta_A)|(x, x') &= \theta_F(2\theta_A(2^{\theta_A(x)})) + \theta_F(2\theta_A(2^{\theta_A(x')})) \\ |!F(\delta_A); d_{F(!!A)}|(x, x') &= 2^{\theta_A(\theta_F(4\theta_A(x)) + \theta_F(4\theta_A(x')))}. \end{aligned}$$

We must show that $\theta_F(2\theta_A(2^u)) + \theta_F(2\theta_A(2^v)) < 2^{\theta_A(\theta_F(4u)+\theta_F(4v))}$, for $u = \theta_A(x)$ and $v = \theta_A(x')$. It is verified as in the previous case. In this time, meanwhile, $4b + 2^t \le 2^{b+t}$ appears as the inequality that must be shown. This is valid for $1 \le b < t$.

Rule (21).

$$\begin{array}{lcl} |e_{F(!A)}|() & = & \theta_F(2\theta_A(2)) \\ |!F(\delta_A); e_{F(!!A)}|() & = & 2^{\theta_A(\theta_F(4\theta_A(2)))}. \end{array}$$

Trivially the former is smaller than the latter. ■

6.35. Lemma. *Consider a strict composite algebraic morphism $s : A[X^{y_1}, X^{y_2}, \dots, X^{y_n}] \to B[X^{x_1}, X^{x_2}, \dots, X^{x_m}]$. Suppose that s contracts to $t : A[X^{y'_1}, X^{y'_2}, \dots, X^{y'_n}] \to B[X^{x_1}, X^{x_2}, \dots, X^{x_m}]$ by a reduction sequence where naturality reductions are restricted. Then $y_i \le y'_i$ for all i and strictly $y_i < y'_i$ for one or more i, provided $x_j \ge 2$ for all j.*

Proof. Since the composition of morphisms is realized by the composition of functions and all involving functions are strictly increasing, the local arguments proved in Lem. 6.30 and 6.34 imply the lemma. ■

We are interested in morphisms of the shape $f = s; t; h$ where s is a strict composite algebraic morphism, t is a composite algebraic morphism, and h is normal. Two punctuations are called *frontiers*. For reference, s is called *demesne* and t *fief*. The frontiers are not absolute since definitions (i) and (ii) in Def. 6.23 are not exclusive.

We consider the following condition: $f = s; t; h$ contains no naturality redexes except restricted ones and $t; h$ contains no redexes other than φ_0-type. We also assume that s is a strict composite algebraic morphism, t is a composite algebraic morphism, and h is normal. We denote this condition by $\circledast$.

6.36. Lemma. *Suppose that $f = s; t; h$ fulfills the condition $\circledast$ above and it contracts to f'. Then there is a decomposition $f' = s'; t'; h'$ satisfying $\circledast$. Moreover, for an arbitrary decomposition $f' = s'; t'; h'$ subject to the condition $\circledast$, there is a decomposition $f = \tilde{s}; \tilde{t}; \tilde{h}$ satisfying $\circledast$ such that s' is a contractum of $\tilde{s}$ or a part of it.*

Proof. The first assertion means that contraction creates no redexes in the fief except of φ_0-type. The second means that contraction creates no fresh part that can be added to the demesne beyond the area attached thereto at the outset. The following are crucial cases. Suppose that $\mathbf{1} \xrightarrow{\varphi_0} !\mathbf{1} \xrightarrow{\delta} !!\mathbf{1}$ contacts to $\mathbf{1} \xrightarrow{\varphi_0} !\mathbf{1} \xrightarrow{!\varphi_0} !!\mathbf{1}$ in the fief. When they are followed by $!!\mathbf{1} \xrightarrow{\delta} !!!\mathbf{1}$ for instance, contraction creates a naturality redex $!\varphi_0; \delta$ in the fief that violates the condition $\circledast$. This situation is precluded, since the fief then contained a prohibited δ-type redex $\delta; \delta$ beforehand. Next suppose that $\mathbf{1} \otimes !A \xrightarrow{\varphi_0 \cdot} !\mathbf{1} \otimes !A \xrightarrow{\tilde{\varphi}} !(\mathbf{1} \otimes A)$ contracts to $\mathbf{1} \otimes !A \xrightarrow{\sim} !A \xrightarrow{\sim} !(\mathbf{1} \otimes A)$ in the fief. Two sides encircling the part may form a new redex, or the right side of the part may be newly attached to the demesne s, as the intervening φ_0 and $\tilde{\varphi}$ vanish. For example, if the right side is $!(\mathbf{1} \otimes A) \xrightarrow{\delta} !!(\mathbf{1} \otimes A)$ such a problem may happen. However, the fief then had a redex $\tilde{\varphi}; \delta$, which was prohibited by the condition $\circledast$. Next, consider the naturality of $\tilde{\varphi}$. As typical in the equivalence of

$!!C \otimes !A \xrightarrow{\tilde{\varphi}} !(!C \otimes A) \xrightarrow{!(e\cdot)} !(\mathbf{1} \otimes A)$ and $!!C \otimes !A \xrightarrow{!e\cdot} !\mathbf{1} \otimes !A \xrightarrow{\tilde{\varphi}} !(\mathbf{1} \otimes A)$, the former $\tilde{\varphi}$ can be a part of the demesne while the latter must be in the fief. If they are followed by, for example, δ, we have a $\tilde{\varphi}$-type redex in the fief, violating the hypothesis. However, this is precluded since non-restricted naturality redex $!(e\cdot); \delta$ is forbidden by the condition $\circledast$. We remark that no redex contains e in its left half except such forbidden naturality redexes. Hence all redex crossing the frontier can be engulfed in the demesne by extending s. For example, if $\delta; \varepsilon$ crosses the frontier, δ is in the demesne while ε is in the fief, we may enlarge the demesne so that ε is a part of it. So if we take sufficiently large $\tilde{s}$ then s' is its contractum or a part of it. ∎

6.37. Lemma. *If $f = s; t; h$ satisfies the condition $\circledast$ introduced before Lem. 6.36, f satisfies the (strong) termination property.*

Proof. By Lem. 6.36, we can enlarge the demesne s at the outset so that subsequent reductions in the demesne are all done in the descendants of s. We assign natural numbers to each X in the demesne as in Def. 6.27. By Lem. 6.35, the sum of associated natural numbers y_i strictly decreases by contractions in the demesne. Occasionally φ_0-type reductions occur in the fief, but they do not alter the associated natural numbers. So the sum is eventually constant. Thereafter only φ_0-type reductions can occur. The sequence of φ_0-type reductions must be finite since they reduce the number of algebraic morphisms other than φ_0. For example $\varphi_0; \delta$ contracts to $\varphi_0; !\varphi_0$, where δ disappears. ∎

6.38. Proposition. *Algebraic morphisms $\delta, \varepsilon, d, e$, and $\tilde{\varphi}, \varphi_0$ are reducible.*

Proof. We describe the case of $\delta_A : !A \to !!A$. By Lem. 6.17 it suffices to prove that, for any $f : X \to A$ in $R^+(A)$ and any $g : !!A \to Y$ in $R^-(!!A)$, $!f; \delta_A; g$ terminates. By a naturality reduction, it contracts to $\delta_X; !!f; g$. By Lem. 6.13, $!!f$ lies in $R^+(!!A)$. Hence $!!f; g$ contracts to a normal form h. Now $\delta_X; h$ satisfies the hypothesis of Prop. 6.37 if we set the demesne to be δ_X and the fief to be empty (an identity). So it terminates. The same argument applies to $\tilde{\varphi}_{X,Y}$, which is a strict composite algebraic morphism in X, Y. We use Lem. 6.18. For φ_0, we take the demesne to be empty and the fief to be φ_0. ∎

All atomic morphisms are reducible by Prop.6.19, 6.21, 6.22, and 6.38. Moreover, Prop. 6.20 shows that all constructions preserve the property of being reducible. With Prop. 6.16, we can conclude the weak termination property:

6.39. Theorem. *Every morphism terminates.*

6.40. Remark. Each morphism terminates under the following specific strategy. First, any β-redexes are contracted. If no β-redexes remain, naturality redexes are contracted. Finally, if there are neither β-redexes nor naturality redexes, the rightmost redexes are contracted. The verification of the theorem remains applicable if we interpret "terminate" as termination under this strategy. Prop. 6.20 (iv), Prop. 6.13, and Lem. 6.38 depend on this particular strategy. The rightmost redexes are not unique in general, as both f and g may have redexes in $f \otimes g$ for example.

We conjecture that strong normalizability is fulfilled. We define strong termination to hold if all infinite reduction sequence repeats only reversible reductions after some finite number of reduction steps.

6.41. Remark. We obtain a cartesian closed category if we enforce ! to be an identity functor. The tensor turns out to be the cartesian product $\times$ and the unit object is a terminal object. Accordingly, we obtain a reduction system for a free cartesian closed category. It contains reduction $e_{A\times B} \Rightarrow e_A \times e_B$ and $e_{\mathbf{1}} \Rightarrow 1$ among others. Note that reductions depend on subscripts, i.e., the shape of objects. This system has a looping reduction sequence $e_{A\times\mathbf{1}} \Rightarrow e_A \times e_{\mathbf{1}} \Rightarrow e_A \times 1 \cong e_{A\times\mathbf{1}}$. In our system, rule (17) blocks this to happen. Moreover, all reduction rules make sense if we omit subscripts.

7. Conclusion

We define a rewriting system on the categorical semantics of the linear logic. Namely, the free (intuitionistic or classical) linear category can be regarded as a calculus. In this paper, we verify that the calculus on the free classical linear category satisfies the weak termination property. In a forthcoming paper, we will verify that it is almost confluent (we say "almost" since we cannot properly deal with the tensor/cotensor units, which are difficult to handle.) These two results together imply that each morphism has a unique normal form as far as no units are involved.

A reward brought about by introducing a calculus is the mechanization of diagram chasing. Given two morphisms, we first convert them into normal forms. We can replace the judgment of equality between morphisms by comparison between normal forms. If the tensor/cotensor units are not involved, we can automatically check whether they are equal. In this sense, our result will give a kind of (partial) coherence result. This paper provides the first step towards this purpose.

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