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LOCALLY ANISOTROPIC TOPOSES II

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Abstract. Every Grothendieck topos has internal to it a canonical group object, called its isotropy group [Funk et al., 2012]. We continue our investigation of this group, focusing again on locally anisotropic toposes [Funk and Hofstra, 2018]. Such a topos is one admitting an étale cover by an anisotropic topos. We present a structural analysis of this class of toposes by showing that a topos is locally anisotropic if and only if it is equivalent to the topos of actions of a connected groupoid internal to an anisotropic topos. In particular we may conclude that a locally anisotropic topos, whence an étendue, has isotropy rank at most one, meaning that its isotropy quotient has trivial isotropy [Funk et al., 2018].

1. Introduction

We continue the development of an aspect of topos theory we call isotropy theory for toposes. In particular, we establish a representation theorem for a certain class of toposes called *locally anisotropic*. In a sense the result we present in this paper has a flavor similar to other theorems, by now well known, that represent a topos by means of a group(oid)-theoretic part plus another part, either spatial or logical in nature. Two particular such representation theorems come to mind: Freyd's theorem [Freyd, 1987], and the Joyal-Tierney theorem [Joyal and Tierney, 1984]. The theorem we present here also represents a topos in terms of a group-theoretic part, namely the isotropy group of the topos; the other part in this case is given by the notion of an anisotropic topos, meaning one that has trivial isotropy.

Let us make the above somewhat more precise. Every Grothendieck topos $\mathscr{E}$ has internal to it a canonical group object called its *isotropy group* [Funk et al., 2012]. (We shall review this and related notions in § 2.1 below.) This group acts canonically on every object of $\mathscr{E}$. The *isotropy quotient*

$$\psi : \mathscr{E} \longrightarrow \mathscr{E}_\theta \tag{1}$$

of a topos $\mathscr{E}$ is the result of annihilating the isotropy of $\mathscr{E}$: technically, $\mathscr{E}_\theta$ is the full subcategory of $\mathscr{E}$ on those objects for which the canonical action by the isotropy group is trivial. For example, the isotropy quotient of a topos of group actions is the topos of sets. The main question we address in this paper is:

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1.1. Question. What is the nature of the quotient map ψ (1), and how is $\mathscr{E}$ recovered from its isotropy quotient $\mathscr{E}_\theta$?

Intuitively speaking, since $\mathscr{E}_\theta$ arises from $\mathscr{E}$ by canceling the action by a group, we hope to recover $\mathscr{E}$ from $\mathscr{E}_\theta$ as a topos of group(oid) actions. In general, as far as we know the question of when this is possible remains open, but here we are able to answer the question completely for the class of locally anisotropic toposes. We say that a topos $\mathscr{E}$ is *locally anisotropic* when it has a globally supported object U for which $\mathscr{E}/U$ is anisotropic.

The present line of inquiry was initiated in the predecessor [Funk and Hofstra, 2018] to this paper, where we answered the question in the case that (1) splits in the sense that ψ has an étale section $\mathscr{E}_\theta \longrightarrow \mathscr{E}$. In this situation we find that $\mathscr{E}_\theta$ is anisotropic (has trivial isotropy) and that $\mathscr{E} \simeq \mathcal{B}(\mathscr{E}_\theta; G)$ for a group G internal to $\mathscr{E}_\theta$. Loc. cit. also introduced the concept of a locally anisotropic topos; in fact, the structure theorem $\mathscr{E} \simeq \mathcal{B}(\mathscr{E}_\theta; G)$ implies that $\mathscr{E}$ is locally anisotropic, since $\mathscr{E}/yG \simeq \mathscr{E}_\theta$, where yG is the representable G-object in $\mathscr{E}$. This paper builds on its predecessor, but also extends it in the sense that no assumptions are made about the isotropy quotient being split.

Main Results. The main theorem of the paper (Theorem 5.3) asserts that a locally anisotropic topos $\mathscr{E}$ is equivalent to a topos of group actions $\mathcal{B}(\mathscr{F}; \mathbb{G})$ of a connected groupoid $\mathbb{G}$ internal to an anisotropic topos $\mathscr{F}$. In fact, this characterizes the class of locally anisotropic toposes. *A priori* it is not clear that the anisotropic topos $\mathscr{F}$ equals the isotropy quotient $\mathscr{E}_\theta$. Indeed, generally it is not the case that $\mathscr{E}_\theta$ is anisotropic [Funk et al., 2018]: we must iterate the isotropy quotient, possibly transfinitely, in order to obtain an anisotropic quotient. However, we prove that if $\mathscr{E}$ is locally anisotropic, then $\mathscr{E}_\theta$ is indeed anisotropic, and that $\mathscr{E} \simeq \mathcal{B}(\mathscr{E}_\theta; \mathbb{G})$ for a connected groupoid $\mathbb{G}$ internal to $\mathscr{E}_\theta$.

The main result is fruitfully applied to the class of toposes known as étendues: an étendue is a topos $\mathscr{E}$ with a globally supported object U for which $\mathscr{E}/U$ is localic. Such a topos is equivalently given as the topos of actions of an étale localic groupoid. An étendue is thus locally anisotropic since a localic topos is anisotropic. The isotropy quotient of an étendue is again an étendue ([Funk and Hofstra, 2018], Cor. 6.5), but with Theorem 5.3 we may deduce that in fact it is anisotropic (but generally not localic). Moreover, an étendue is recovered from its isotropy quotient as the topos of actions of a connected groupoid internal to the isotropy quotient.

As an interesting special case, we may consider the étendue of étale S-sets associated with an inverse semigroup S , denoted $\mathcal{B}(S)$. Its isotropy quotient $\mathcal{B}(S)_\theta$ can be described at the semigroup level. Indeed, if μ is the maximum idempotent-separating congruence on S , so that two elements $s, t \in S$ are μ-equivalent just when $s^*s = t^*t$ and $ses^* = tet^*$ for every idempotent e , then the quotient S/μ is an inverse semigroup, and $\mathcal{B}(S/\mu)$ is the isotropy quotient of $\mathcal{B}(S)$. Our Theorem 5.3 asserts that necessarily the topos $\mathcal{B}(S/\mu)$ is anisotropic. The semigroup S/μ is therefore *fundamental* in the semigroup sense [Lawson, 1998] (herein Def. 6.4): the only elements commuting with all idempotents are the idempotents themselves. Of course, this fact is already well known and easily verified directly. However, Theorem 5.3 says more: it says also that $\mathcal{B}(S)$ may be recovered from

$\mathcal{B}(S/\mu)$ as the topos of actions of a connected groupoid internal to $\mathcal{B}(S/\mu)$. We obtain the following corollary: an arbitrary inverse semigroup S is Morita-equivalent to an ordered groupoid of the form $\mathbb{G} \ltimes S/\mu$. We hasten to add that in principle this result, which is new as far we know, could also be derived directly as a special case of a more general fact about homomorphisms of inverse semigroups with the same idempotent set (Remark 6.8). Nevertheless, we include the result as we came upon it by topos-theoretic means, including an explicit calculation of the ordered groupoid in question.

Methods. Our methods include techniques and ideas developed in the predecessor of this paper [Funk and Hofstra, 2018] as well as in two other articles [Funk et al., 2012, Funk et al., 2018]. In particular, our proof of Theorem 5.3 relies in an essential way on the concept of higher isotropy [Funk et al., 2018]. We also make essential use of Galois theory for toposes, as developed by Bunge [Bunge, 2004, Bunge, 2008], Janelidze [Janelidze, 1990], and Borceux-Janelidze [Borceux and Janelidze, 2001]. (The main concepts and results we need from isotropy theory and from Galois theory shall be reviewed.) In fact, we shall demonstrate how isotropy theory and Galois theory meet in a natural and effective way. For example, we work out the idea of an isotropically normal object in a topos, and of the normal closure of an object in a topos.

Organization. In § 2 we review basic isotropy theory for toposes, including some special cases used heavily in the paper. This section also develops the idea of the isotropy quotient of a topos associated with a crossed sheaf. § 3 reviews Galois theory for toposes; in particular, it discusses connected groupoids in toposes, split objects and normal objects, the 'fundamental pushout' topos of split objects, and the central theorem of Galois theory in the topos setting. § 4 interprets, unpacks, and develops topos Galois theory relative to the isotropy quotient, relating the basic notions of Galois theory and isotropy theory. In § 5 we present the main result for locally anisotropic toposes (Theorem 5.3). Finally, we apply the main result to inverse semigroups in § 6.

2. Isotropy theory

Let us begin in § 2.1 with a brief review of the main ingredients of the theory of isotropy groups in toposes and their quotients. For more details we refer the reader to [Funk et al., 2012, Funk et al., 2018]. Johnstone [Johnstone, 2002] is the standard reference for topos theory. We then discuss some aspects of crossed sheaves and their isotropy quotients. Throughout, $\mathcal{B}(\mathscr{F};G)$ denotes the topos of group actions of a group G internal to a topos $\mathscr{F}$. An object (X,μ) of $\mathcal{B}(\mathscr{F};G)$ is an object X of $\mathscr{F}$ equipped with a unital and associative action $\mu : X \times G \longrightarrow X$, while morphisms of $\mathcal{B}(\mathscr{F};G)$ are the morphisms of $\mathscr{F}$ that are equivariant with respect to these actions. A *geometric morphism* $\psi : \mathscr{E} \longrightarrow \mathscr{F}$ of toposes is a pair of adjoint functors $\phi^* \dashv \phi_*$, with $\phi_* : \mathscr{E} \longrightarrow \mathscr{F}$, where ϕ^* preserves finite limits. It is standard to call ϕ_* the *direct image* and ϕ^* the *inverse image* of ϕ . If ϕ^* happens to have a further left adjoint we typically denote it by $\phi_!$. When $\phi,\psi : \mathscr{E} \longrightarrow \mathscr{F}$ are geometric morphisms, then a *geometric transformation* $\alpha : \phi \Rightarrow \psi$

is simply a natural transformation $\alpha : \phi^* \to \psi^*$. Equivalently, one may consider this as a natural transformation between the direct image functors $\psi_* \to \phi_*$ or between left adjoints $\psi_! \to \phi_!$ should they happen to exist.

2.1. Review of isotropy in toposes. Let $\mathscr{E}$ be a Grothendieck topos. Internal to $\mathscr{E}$ there is a group object $Z_{\mathscr{E}} = Z$, which we call the (étale) *isotropy group* of $\mathscr{E}$. It classifies (étale) isotropy in the sense that morphisms $X \to Z$ of $\mathscr{E}$ are in natural bijection with natural automorphisms of the geometric morphism

$$\mathscr{E}/X \longrightarrow \mathscr{E} \tag{2}$$

associated with X. Since such an automorphism is given by a natural automorphism α of the left adjoint $\Sigma_X : \mathscr{E}/X \longrightarrow \mathscr{E}$, it may be explicitly described as a family of automorphisms $\alpha_f : Y \to Y$, indexed by $f : Y \to X$, subject to the compatibility condition $\alpha_f g = g\alpha_{fg}$:

$$\begin{tikzcd} Y' \arrow[r, "\alpha_{fg}"] \arrow[d, "g"'] & Y' \arrow[d, "g"] \\ Y \arrow[r, "\alpha_f"'] & Y \end{tikzcd}$$

Equivalently, we may regard an element of isotropy $X \to Z$ as an automorphism of $X^* : \mathscr{E} \longrightarrow \mathscr{E}/X$, which is a natural family of maps $Y \times X \to Y$. In particular, the identity map on Z corresponds to an automorphism θ of

$$\mathscr{E}/Z \longrightarrow \mathscr{E}$$

interpreted as a natural action

$$\theta_X : X \times Z \to X\,.$$

We call this action the *isotropy action* of Z on X. The naturality of θ (in X) implies that every morphism of $\mathscr{E}$ is equivariant for this action. It also follows that the action θ_Z of Z on itself is given by conjugation.

As a typical example, consider the topos $\mathscr{B}(G)$ of G-sets for a (discrete) group G. Its isotropy group is G equipped with the conjugation action, and the action of this group on a G-set (X, μ) is simply μ again.

An *isotropically trivial object* is an object X for which its isotropy action θ_X is trivial in the sense that it equals the first projection. In the internal language of $\mathscr{E}$ this says that $\forall x \in X \forall z \in Z.\, xz = x$. We say that a topos $\mathscr{E}$ is *anisotropic* if its isotropy group Z is trivial, and *locally anisotropic* if $\mathscr{E}$ has a globally supported object $U \twoheadrightarrow 1$ such that $\mathscr{E}/U$ is anisotropic. An object U of $\mathscr{E}$ is called anisotropic when $\mathscr{E}/U$ is anisotropic. A localic topos is anisotropic, but the class of anisotropic toposes if significantly more inclusive. For example, a topos of presheaves on a small category is anisotropic when the category is rigid in the sense that it has no non-trivial automorphisms. The topos of a fundamental inverse semigroup is anisotropic (§ 6), but not generally localic.

We shall use the following special case of a more general calculation of the isotropy group of a topos of group actions ([Funk and Hofstra, 2018], Thm. 3.11).

2.2. Lemma. *The isotropy group of the topos of group actions $\mathcal{B}(\mathscr{F};G)$ for a group G internal to an anisotropic topos $\mathscr{F}$ is the group G itself with its conjugation action.*

We also need the description of isotropy in a slice topos. A slice topos $\mathscr{E}/U$ has its own isotropy group, denoted ζ_U in the following diagram.

$$\begin{tikzcd} Z(U) \arrow[r, "\eta_U"] \arrow[d, "\zeta_U"'] & Z \\ U & \end{tikzcd} \tag{3}$$

The object $Z(U)$ is the subobject of $U \times Z$ consisting of those (u,z) such that $uz = u$. We have $\zeta_U(u,z) = u$, and η_U is the other projection $\eta_U(u,z) = z$.

Finally, we shall need in § 4.1 a relative version of the isotropy group. When $\mathscr{F}$ is a topos, $\psi : \mathscr{E} \longrightarrow \mathscr{F}$ is a topos over $\mathscr{F}$, and X is an object of $\mathscr{E}$, then we may consider natural automorphisms α of $\mathscr{E}/X \longrightarrow \mathscr{E}$ over $\mathscr{F}$, i.e., those α whose whiskering with ψ equals the identity on ψ.

2.3. Isotropy quotients. Let $\mathscr{E}_\theta$ denote the full subcategory of $\mathscr{E}$ on the isotropically trivial objects. Then we have a geometric morphism

$$\psi : \mathscr{E} \longrightarrow \mathscr{E}_\theta \tag{4}$$

whose inverse image functor ψ^* is the full inclusion of the isotropically trivial objects. We call this geometric morphism the *isotropy quotient of $\mathscr{E}$*. For example, the isotropy quotient of a topos $\mathcal{B}(G) = \mathcal{B}(\mathsf{Set};G)$ of group actions is Set.

The geometric morphism ψ is atomic in the sense that ψ^* is logical (preserves the subobject classifier and exponentials), and connected in the sense that ψ^* is full and faithful. The left adjoint $\psi_!$ is given by the coequalizer

$$\begin{tikzcd} X \times Z \arrow[r, shift left, "\theta_X"] \arrow[r, shift right, "\mathrm{proj}"'] & X \arrow[r, two heads] & \psi_!(X) \end{tikzcd} . \tag{5}$$

Intuitively, $\psi_!(X)$ equals the object of orbits of X under the action θ_X.

In general, isotropy groups across a geometric morphism are related by a canonical span of homomorphisms. In the case where the geometric morphism $\varphi : \mathscr{E} \longrightarrow \mathscr{F}$ is locally connected (definition provided in § 3.6 - in this paper we shall need only this case) this span has just one leg $m : Z_{\mathscr{E}} \to \varphi^* Z_{\mathscr{F}}$ in $\mathscr{E}$, which we call the *canonical comparison homomorphism* associated with φ. It has the property that for every object X of $\mathscr{F}$ the diagram

$$\begin{tikzcd} \varphi^* X \times Z_{\mathscr{E}} \arrow[r, "\varphi^* X \times m"] \arrow[dr, "\theta_{\varphi^* X}"'] & \varphi^*(X \times Z_{\mathscr{F}}) \arrow[d, "\varphi^* \theta_X"] \\ & \varphi^* X \end{tikzcd}$$

commutes [Funk et al., 2012]. This immediately yields the following observation.

2.4. Proposition. *The inverse image functor of a locally connected geometric morphism preserves isotropically trivial objects. Hence, if $\varphi : \mathscr{E} \longrightarrow \mathscr{F}$ is locally connected, then there is an induced geometric morphism $\mathscr{E}_\theta \longrightarrow \mathscr{F}_\theta$ making the following diagram commute.*

$$\begin{tikzcd} \mathscr{E} \arrow[r, "\varphi"] \arrow[d] & \mathscr{F} \arrow[d] \\ \mathscr{E}_\theta \arrow[r] & \mathscr{F}_\theta \end{tikzcd}$$

Proof. If X is an object of $\mathscr{F}$ for which the isotropy action θ_X is trivial, then the action $\varphi^*\theta_X$ on φ^*X is also trivial, whence so is θ_{φ^*X} . ∎

Turning to slice toposes, let

$$\mathscr{E}/X \longrightarrow (\mathscr{E}/X)_\theta$$

denote the isotropy quotient of a slice topos $\mathscr{E}/X$, consisting of the isotropically trivial objects of $\mathscr{E}/X$, i.e., those objects of $\mathscr{E}/X$ whose action by ζ_X is trivial.

2.5. Lemma. [The fundamental lemma of isotropy [Funk and Hofstra, 2018], Lemma 4.3] *Let X be an object of a topos $\mathscr{E}$, and let $O_X = \psi_! X$ denote the orbit object of the isotropy action θ_X . Then $(\mathscr{E}/X)_\theta \simeq \mathscr{E}_\theta/O_X$ such that the two squares below are equivalent.*

$$\begin{tikzcd} \mathscr{E}/X \arrow[r] \arrow[d] & (\mathscr{E}/X)_\theta \arrow[d] \\ \mathscr{E} \arrow[r, "\psi"] & \mathscr{E}_\theta \end{tikzcd} \qquad \begin{tikzcd} \mathscr{E}/X \arrow[r] \arrow[d] & \mathscr{E}_\theta/O_X \arrow[d] \\ \mathscr{E} \arrow[r, "\psi"] & \mathscr{E}_\theta \end{tikzcd}$$

In general, the isotropy quotient of a topos is not anisotropic; however, we may iterate the construction of the isotropy quotient (transfinitely many times if necessary, taking colimits at limit ordinal stages [Funk et al., 2018]) ultimately reaching an anisotropic topos

$$\Psi : \mathscr{E} \longrightarrow \mathscr{E}_\Theta \ . \tag{6}$$

Indeed, there exists a least ordinal κ such that the κ-iteration of the isotropy quotient is anisotropic. We refer to κ as the *isotropy rank* of $\mathscr{E}$. For example, an anisotropic topos has rank zero, and if the first isotropy quotient of a topos is anisotropic, then the topos has rank at most one. This ultimate (étale) isotropy quotient Ψ has three essential properties:

(i) $\mathscr{E}_\Theta$ is anisotropic,

(ii) Ψ is connected atomic, and

(iii) the fundamental lemma of isotropy holds for Ψ: for any object U of $\mathscr{E}$ we have

$$(\mathscr{E}/U)_\Theta \simeq \mathscr{E}_\Theta/\Psi_! U \ ,$$

where $(\mathscr{E}/U)_\Theta$ denotes the ultimate isotropy quotient of $\mathscr{E}/U$.

2.6. Isotropy quotient of a crossed sheaf. We refer to an object of the slice topos $\mathscr{E}/Z$ as a crossed sheaf. (This terminology generalizes [Freyd and Yetter, 1989], which refers to a G-equivariant map $(X, \mu) \longrightarrow (G, \mathrm{conj})$ as a crossed G-set.)

2.7. Remark. An interesting aspect of $\mathscr{E}/Z$ is that it carries a canonical monoidal closed structure due to the fact that Z is a group. This monoidal structure is in fact (balanced) braided. We shall leave a full investigation of this aspect for another occasion.

A crossed sheaf $f : A \to Z$ has associated with it an isotropy quotient

$$\tau : \mathscr{E} \longrightarrow \mathscr{E}_f \tag{7}$$

such that $\mathscr{E}_f$ consists of the full subcategory of $\mathscr{E}$ on those objects X such that

$$\begin{tikzcd} X \times A \arrow[r, "X\times f"] \arrow[dr, "\mathrm{proj}"'] & X \times Z \arrow[d, "\theta_X"] \\ & X \end{tikzcd} \tag{8}$$

commutes in $\mathscr{E}$. The isotropy quotient (4) is an instance of (7) as it is the isotropy quotient of the terminal crossed sheaf, meaning the identity map $Z \longrightarrow Z$.

2.8. Remark. The isotropy quotient $\mathscr{E}_f$ is completely determined by the image of f in Z, which necessarily is a normal subobject of Z, i.e., is closed under conjugation.

The category $\mathscr{E}_f$ is a topos, and the quotient τ (7) is connected atomic. Its inverse image functor τ^* is the inclusion of the full subcategory $\mathscr{E}_f$. The left adjoint $\tau_!$ is given by the coequalizer

$$X \times A \xrightarrow{X\times f} X \times Z \overset{\theta}{\rightrightarrows} X \longrightarrow\!\!\!\!\!\rightarrow \tau_! X \quad . \tag{9}$$

First observe that $\tau_! X$ is in $\mathscr{E}_f$. Indeed, the diagram

$$\begin{tikzcd} X \times A \arrow[r, "X\times f"] \arrow[d, two heads] & X \times Z \arrow[r, shift left, "\theta"] \arrow[r, shift right] \arrow[d, two heads] & X \arrow[d, two heads] \\ \tau_! X \times A \arrow[r, "\tau_! X\times f"] & \tau_! X \times Z \arrow[r, shift left, "\theta"] \arrow[r, shift right] & \tau_! X \end{tikzcd}$$

commutes serially, so that the bottom row commutes as the left vertical map is an epimorphism. If $t : X \to Y$ is a morphism to an object Y of $\mathscr{E}_f$, then the bottom row of

$$\begin{tikzcd} X \times A \arrow[r, "X\times f"] \arrow[d, "t\times A"'] & X \times Z \arrow[r, shift left, "\theta"] \arrow[r, shift right] \arrow[d, "t\times Z"'] & X \arrow[d, "t"] \\ Y \times A \arrow[r, "Y\times f"] & Y \times Z \arrow[r, shift left, "\theta"] \arrow[r, shift right] & Y \end{tikzcd}$$

commutes, so that the top route through the diagram commutes, whence t factors through the coequalizer $\tau_! X$. It follows that (7) is equally well expressed as a coequifier

$$\begin{tikzcd} \mathscr{E}/A \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow f"] & \mathscr{E} \arrow[r, "\tau"] & \mathscr{E}_f \end{tikzcd}$$

in the sense that τ universally identifies f and the identity automorphism of $\mathscr{E}/A \longrightarrow \mathscr{E}$.

Consider now a morphism of crossed sheaves:

$$\begin{tikzcd} B \arrow[rr, "k"] \arrow[dr, "g"'] & & A \arrow[dl, "f"] \\ & Z & \end{tikzcd} .$$

If X is in $\mathscr{E}_f$, then X is in $\mathscr{E}_g$ as is immediate from the commutativity of the diagram

$$\begin{tikzcd} X \times B \arrow[r, "X\times k"] \arrow[rr, bend left, "X\times g"] \arrow[drr, "\mathrm{proj}"'] & X \times A \arrow[r, "X\times f"] \arrow[dr, "\mathrm{proj}"] & X \times Z \arrow[d, "\theta_X"] \\ & & X \end{tikzcd} .$$

In fact, the inclusion of $\mathscr{E}_f$ in $\mathscr{E}_g$ is the inverse image of a connected atomic geometric morphism depicted as the diagonal in the following commutative diagram.

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d] & \mathscr{E}_g \arrow[dl] \arrow[d] \\ \mathscr{E}_f \arrow[r] & \mathscr{E}_\theta \end{tikzcd}$$

All geometric morphisms in this diagram are connected atomic.

2.9. Remark. The assignment $f \mapsto \mathscr{E}_f$ underlies a functor from crossed sheaves on $\mathscr{E}$ to connected atomic quotients of $\mathscr{E}$ over $\mathscr{E}_\theta$.

Let us fix an object U of $\mathscr{E}$. The projection $\eta_U : Z(U) \to Z$ associated with the isotropy group ζ_U of $\mathscr{E}/U$ is a crossed sheaf (3). Let us denote its isotropy quotient $\mathscr{E}_{\eta_U}$ simply by $\mathscr{E}_U$. Thus, $\mathscr{E}_U$ is the coequifier

$$\begin{tikzcd} \mathscr{E}/Z(U) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \eta_U"] & \mathscr{E} \arrow[r, "\tau"] & \mathscr{E}_U \end{tikzcd} .$$

An object X is an object of its isotropy quotient $\mathscr{E}_U$ just when

$$\begin{tikzcd} X \times Z(U) \arrow[r, "X\times\eta_U"] & X \times Z \arrow[r, shift left, "\theta_X"] \arrow[r, shift right] & X \end{tikzcd} \tag{10}$$

commutes. By transposing under $\Sigma_U \dashv U^*$ we find that this is the case precisely when

$$\begin{tikzcd} U^*X \times \zeta_U \arrow[r, shift left, "\theta"] \arrow[r, shift right] & U^*X \end{tikzcd} \tag{11}$$

commutes in $\mathscr{E}/U$. Thus, the following holds.

2.10. Lemma. *For any object U of a topos $\mathscr{E}$ the topos $\mathscr{E}_U$ consists of those objects X of $\mathscr{E}$ for which U^*X is isotropically trivial in $\mathscr{E}/U$.*

It follows (using the fact that pushouts of toposes are constructed as pullbacks of the underlying inverse image functors) that the inside square in the following diagram is a topos pushout [Bunge, 2008].

$$\begin{tikzcd}
\mathscr{E}/U \arrow[r] \arrow[d] & (\mathscr{E}/U)_\theta \arrow[d, "p"'] \arrow[ddr, bend left] & \\
\mathscr{E} \arrow[r, "\tau"] \arrow[drr, bend right, "\psi"] & \mathscr{E}_U \arrow[dr, "\varphi"] & \\
& & \mathscr{E}_\theta
\end{tikzcd} \tag{12}$$

The geometric morphisms τ and φ are connected atomic.

3. Review of Galois theory for toposes

We review some aspects of Galois theory in toposes, including some basic observations about connected groupoids in toposes. Our sources for this information are [Bunge, 2004], [Janelidze, 1990], and [Borceux and Janelidze, 2001]. For the reader's convenience and in order to keep our account relatively self-contained we offer proofs of the facts we need.

Throughout, the reader not familiar with Galois theory for toposes may wish to keep the following motivating example in mind. When X is a sufficiently nice space, so that it has a universal covering space $p : U \to X$, then the category of locally constant sheaves on X form a topos $\mathrm{SPL}(p)$. A sheaf F on X is locally constant when there is an isomorphism $U \times F \cong \Delta(A)$ over U for some set A . That is, in the slice over U the object F is isomorphic to a constant object. Then toposes $\mathrm{SPL}(p)$ and $\mathcal{B}(G)$ are equivalent, where G is the fundamental group of X .

3.1. Connected categories and groupoids. The object of connected components of a category $\mathbb{C} = (C_1, C_0)$ internal to a topos is by definition the coequalizer

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & C_0 \arrow[r] & \pi_0(\mathbb{C}) \end{tikzcd} \quad ,$$

where d_0 and d_1 are the domain and codomain maps of $\mathbb{C}$. In particular, we say that $\mathbb{C}$ is *connected* if

$$\begin{tikzcd} C_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & C_0 \arrow[r] & 1 \end{tikzcd}$$

is a coequalizer. If $\mathbb{C}$ is connected, then C_0 has global support.

Let $\mathcal{B}(\mathscr{E};\mathbb{C})$ denote the topos of discrete fibrations on $\mathbb{C}$, internal to a topos $\mathscr{E}$, also known as presheaves internal to $\mathscr{E}$. Technically, $\mathcal{B}(\mathscr{E};\mathbb{C})$ may be defined as the category of EM-algebras for the monad on $\mathscr{E}/C_0$ whose underlying functor $\mathcal{D}$ is given by pullback along d_1 and composition with d_0, as depicted in

$$\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dd, bend right, "\mathcal{D}p"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "p"] \\
\mathbb{C}_1 \arrow[r, "d_1"'] \arrow[d, "d_0"] & \mathbb{C}_0 \\
\mathbb{C}_0 &
\end{tikzcd} \tag{13}$$

Let

$$\gamma : \mathcal{B}(\mathscr{E};\mathbb{C}) \longrightarrow \mathscr{E} \quad ; \qquad \gamma^* X = C_0 \times X \to C_0$$

denote what we call the structure geometric morphism associated with $\mathbb{C}$. It follows that $\mathbb{C}$ is connected in the above sense if and only if γ is connected in the usual sense that γ^* is full and faithful.

In a topos a connected groupoid is not generally equivalent to a group, but it is locally Morita equivalent to a group in the following sense. Let $\mathbb{G} = (G_0, G_1)$ denote a connected groupoid internal to a topos $\mathscr{F}$. The 'vertex group' equalizer

$$\begin{tikzcd}
Z \arrow[r, tail] & G_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & G_0
\end{tikzcd}$$

is a group in $\mathscr{F}/G_0$ which we denote $\zeta : Z \to G_0$. Consider the following topos pullback, where the domain map d_0 is regarded as discrete fibration in the usual way.

$$\begin{tikzcd}
\mathcal{B}(\mathscr{F};\mathbb{G})/d_0 \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& \mathcal{B}(\mathscr{F};\mathbb{G})/\gamma^* G_0 \arrow[r] \arrow[d] & \mathscr{F}/G_0 \arrow[d] \\
& \mathcal{B}(\mathscr{F};\mathbb{G}) \arrow[r, "\gamma"] & \mathscr{F}
\end{tikzcd} \tag{14}$$

The morphism $d_0 \to \gamma^* G_0$ of $\mathcal{B}(\mathscr{F};\mathbb{G})$ in (14) is given in the following diagram.

$$\begin{tikzcd}
G_1 \arrow[rr, "(d_0, d_1)"] \arrow[dr, "d_0"'] & & G_0 \times G_0 \arrow[dl, "\gamma^* G_0"] \\
& G_0 &
\end{tikzcd}$$

It follows that diagram (14) is equivalent to the following one.

$$\begin{tikzcd}
\mathscr{F}/G_0 \arrow[dr] \arrow[drr, bend left, equal] \arrow[ddr, bend right] & & \\
& \mathcal{B}(\mathscr{F}/G_0;\zeta) \arrow[r] \arrow[d] & \mathscr{F}/G_0 \arrow[d] \\
& \mathcal{B}(\mathscr{F};\mathbb{G}) \arrow[r, "\gamma"] & \mathscr{F}
\end{tikzcd} \tag{15}$$

In particular, we have an equivalence of toposes

$$\mathcal{B}(\mathscr{F}/G_0;\zeta) \;\simeq\; \mathcal{B}(\mathscr{F};\mathbb{G})/\gamma^* G_0 \tag{16}$$

showing that, after slicing by the globally supported object $\gamma^* G_0$, the topos $\mathcal{B}(\mathscr{F};\mathbb{G})$ is equivalent to a topos of group actions.

3.2. The chaotic groupoid. Any object U of a topos carries a trivial groupoid structure, which we term *chaotic*, denoted $\mathbb{U} = (U, U \times U)$. (One also finds the term *codiscrete* in the literature.) The domain and codomain maps of $\mathbb{U}$ are the two projections $U \times U \to U$, the unit map is the diagonal $U \to U \times U$, and the multiplication is also a projection:

$$(U \times U) \times_U (U \times U) \cong U \times U \times U \to U \times U \; ; \qquad (u, v, w) \mapsto (u, w) \, .$$

The inverse map is the 'switch' map $(u, v) \mapsto (v, u)$.

3.3. Remark. The map $U \mapsto \mathbb{U}$ is a functor $\mathscr{E} \longrightarrow \mathsf{Grpd}(\mathscr{E})$, which is right adjoint to the functor that maps a groupoid $\mathbb{G} = (G_0, G_1)$ to its object of objects G_0.

3.4. Proposition. *An object U has global support if and only if the groupoid $\mathbb{U}$ is connected.*

Proof. In a topos an epimorphism is the coequalizer of its kernel pair. ∎

Of course, in Set if $U \neq \emptyset$, then $\mathbb{U}$ is equivalent to the trivial groupoid consisting of one object and one morphism. However, in a general topos the groupoid $\mathbb{U}$ may not be equivalent to the trivial groupoid. Still they are Morita equivalent in the following sense.

3.5. Proposition. *If an object U of a topos $\mathscr{E}$ has global support, then its structure geometric morphism $\gamma : \mathcal{B}(\mathscr{E};\mathbb{U}) \longrightarrow \mathscr{E}$ is an equivalence.*

PROOF. A straightforward monadicity argument working directly with the definition of $\mathcal{B}(\mathscr{E};\mathbb{U})$ does the job.

$$\begin{tikzcd} \mathcal{B}(\mathscr{E};\mathbb{U}) \arrow[dr, "p^*"'] & & \mathscr{E} \arrow[ll, "\gamma^*"'] \arrow[dl, "U^*"] \\ & \mathscr{E}/U & \end{tikzcd}$$

By definition, $\mathcal{B}(\mathscr{E};\mathbb{U})$ is the category of EM-algebras for the monad on $\mathscr{E}/U$ given by pullback along codomain and compose with domain of the groupoid, as in diagram (13). In the case of $\mathbb{U}$, this monad carries an object $X \to U$ to $U \times X \to U$. The functor p^* forgets the algebra structure. (All three functors in the diagram above are inverse image functors of a geometric morphism.) The functor U^* is also monadic because U has global support (by assumption). Moreover, the two monads on $\mathscr{E}/U$ associated with U^* and p^* are precisely the same. Therefore, γ^* must be an equivalence. ∎

3.6. GALOIS THEORY FOR TOPOSES. A geometric morphism

$$\psi : \mathscr{E} \longrightarrow \mathscr{F} \tag{17}$$

is said to be *locally connected* [Barr and Paré, 1980] if its inverse image functor ψ^* has a left adjoint $\psi_!$, which is strong in the sense that if the square below (left) is a pullback, then so is the transposed one (right).

$$\begin{tikzcd} E \arrow[r] \arrow[d, "k"'] & \psi^* F \arrow[d, "\psi^* f"] \\ X \arrow[r] & \psi^* Y \end{tikzcd} \qquad \begin{tikzcd} \psi_! E \arrow[r] \arrow[d, "\psi_! k"'] & F \arrow[d, "f"] \\ \psi_! X \arrow[r] & Y \end{tikzcd}$$

If U is an object of $\mathscr{E}$, then ψ induces a geometric morphism

$$\psi/U : \mathscr{E}/U \longrightarrow \mathscr{F}/\psi_! U \tag{18}$$

whose inverse image sends $f : Y \to \psi_!(U)$ to the following pullback.

$$\begin{tikzcd} P \arrow[r] \arrow[d, "(\psi/U)^* f"'] & \psi^* Y \arrow[d, "\psi^* f"] \\ U \arrow[r, "\text{unit}"] & \psi^*\psi_!(U) \end{tikzcd} \tag{19}$$

An object X is said to be *split* by an object U relative to ψ if the adjunction square below (left) is a pullback.

$$\begin{tikzcd} X \times U \arrow[r] \arrow[d] & \psi^*\psi_!(X \times U) \arrow[d] \\ U \arrow[r] & \psi^*\psi_!(U) \end{tikzcd} \qquad \begin{tikzcd} U \times U \arrow[r] \arrow[d] & \psi^*\psi_!(U \times U) \arrow[d] \\ U \arrow[r] & \psi^*\psi_!(U) \end{tikzcd}$$

For brevity we also say that U ψ-splits X. Following [Janelidze, 1990] we shall say that an object U is *ψ-normal* if it ψ-splits itself, i.e., if the adjunction square above (right) is a pullback.

3.7. Example. In the case $\mathscr{E} = \mathcal{B}(G)$, the (transitive) G-set G/H associated with a subgroup $H \subseteq G$ is normal relative to $\mathcal{B}(G) \longrightarrow \mathsf{Set}$ in the topos sense if and only if H is a normal subgroup of G.

Let $\mathrm{SPL}_\psi(U)$ denote the full subcategory of $\mathscr{E}$ on the objects that are ψ-split by U. The category $\mathrm{SPL}_\psi(U)$ is a topos; in fact, it readily follows from the definition of U-split object and the description (19) of $(\psi/U)^*$ that $\mathrm{SPL}_\psi(U)$ fits in the following 'fundamental pushout' relative to ψ associated with an object U of $\mathscr{E}$ [Bunge, 2004, Bunge, 2008].

$$\begin{tikzcd} \mathscr{E}/U \arrow[r, "\psi/U"] \arrow[d] & \mathscr{F}/\psi_! U \arrow[d, "p"'] \arrow[ddr, bend left] & \\ \mathscr{E} \arrow[r, "\tau"] \arrow[drr, bend right, "\psi"] & \mathrm{SPL}_\psi(U) \arrow[dr, "\varphi"] & \\ & & \mathscr{F} \end{tikzcd} \tag{20}$$

Generally the leftmost adjoint $\psi_!$ of a locally connected morphism does not preserve finite limits; for example, it preserves the terminal object if and only if ψ is connected. However, the following lemma identifies a useful special case where a particular kind of pullback is preserved.

3.8. Lemma. *Suppose that an object X of $\mathscr{E}$ is ψ-split by another object U. Then $\psi_!$ preserves any pullback of the following form.*

$$\begin{tikzcd} X \times E \arrow[r, "X\times f"] \arrow[d] & X \times U \arrow[d, "p_2"] \\ E \arrow[r, "f"] & U \end{tikzcd}$$

Proof. The two squares in the diagram

$$\begin{tikzcd} X \times E \arrow[r, "X\times f"] \arrow[d] & X \times U \arrow[r] \arrow[d, "p_2"] & \psi^*\psi_!(X \times U) \arrow[d] \\ E \arrow[r, "f"] & U \arrow[r] & \psi^*\psi_!(U) \end{tikzcd}$$

are pullbacks - the right one by our assumption that X is ψ-split by U. Therefore, the outside square is one, so that the transposed square

$$\begin{tikzcd} \psi_!(X \times E) \arrow[r, "\psi_!(X\times f)"] \arrow[d] & \psi_!(X \times U) \arrow[d, "\psi_! p_2"] \\ \psi_!(E) \arrow[r, "\psi_! f"] & \psi_!(U) \end{tikzcd}$$

is also a pullback. ∎

As in § 3.2, $\mathbb{U}$ denotes what we call the chaotic groupoid associated with an object U.

3.9. Corollary. *If an object U of $\mathscr{E}$ is ψ-normal, then $\psi_!$ preserves a pullback such as the following (left).*

$$\begin{tikzcd} U \times E \arrow[r, "U\times f"] \arrow[d] & U \times U \arrow[d, "p_2"] \\ E \arrow[r, "f"] & U \end{tikzcd} \qquad \begin{tikzcd} U \times U \times U \arrow[r] \arrow[d] & U \times U \arrow[d, "p_2"] \\ U \times U \arrow[r, "p_1"] & U \end{tikzcd}$$

In particular, $\psi_!$ preserves the pullback above (right). Consequently,

$$\psi(\mathbb{U}) = (\psi_!(U), \psi_!(U \times U))$$

is a groupoid in $\mathscr{F}$. Moreover, if ψ is connected (so $\psi_!(1) = 1$) *and U has global support, then $\psi(\mathbb{U})$ is a connected groupoid.*

With these preliminaries in place we return to the fundamental pushout topos (20). We shall need the following result due to Bunge ([Bunge, 2004], Proposition 2.8). The result as stated here, which suffices for our purposes, makes the simplifying assumption that the splitting object is normal. It is thus also a special case of Janelidze ([Janelidze, 1990], Thm. 2.7).

3.10. Proposition. *Suppose that $\psi : \mathscr{E} \longrightarrow \mathscr{F}$ is locally connected and suppose that a globally supported object U of $\mathscr{E}$ is ψ-normal. Then we have an equivalence*

$$\mathrm{SPL}_\psi(U) \simeq \mathcal{B}(\mathscr{F}; \psi(\mathbb{U}))$$

over $\mathscr{F}$. Thus, $\mathrm{SPL}_\psi(U)$ *is a topos, and moreover in diagram (20) τ is connected and locally connected, φ is atomic, and p is étale.*

Proof. We have geometric morphisms as follows.

$$\begin{tikzcd} \mathscr{E}/U \arrow[r, "\psi/U"] \arrow[d] & \mathscr{F}/\psi_!U \arrow[d, "q"] \\ \mathcal{B}(\mathscr{E};\mathbb{U}) \arrow[r, "\rho"] \arrow[d, "\gamma"'] & \mathcal{B}(\mathscr{F};\psi(\mathbb{U})) \arrow[d, "\delta"] \\ \mathscr{E} \arrow[r, "\psi"] & \mathscr{F} \end{tikzcd}$$

An object of $\mathcal{B}(\mathscr{F};\psi(\mathbb{U}))$ is a discrete fibration $f:\mathbb{F}\to\psi(\mathbb{U})$: this consists of a morphism $f:F\to\psi_!(U)$ equipped with an action by $\psi(\mathbb{U})$. The geometric morphism ρ is given analogously to ψ/U: its inverse image functor ρ^* is given by applying ψ^* to a discrete fibration f, and then pulling back along the unit.

$$\begin{tikzcd} \mathbb{X} \arrow[r] \arrow[d] & \psi^*(\mathbb{F}) \arrow[d, "\psi^* f"] \\ \mathbb{U} \arrow[r, "\text{unit}"] & \psi^*\psi(\mathbb{U}) \end{tikzcd}$$

The geometric morphism ρ is locally connected. Indeed, by Cor. 3.9, observe that $\psi_!$ preserves discrete fibrations on $\mathbb{U}$. This gives a left adjoint $\rho_!$, which inherits the strength of $\psi_! \dashv \psi^*$. It follows that ρ^* is full and faithful, so that ρ is connected. By Prop. 3.5, as U is globally supported (by assumption) γ is an equivalence, identifying an object X of $\mathscr{E}$ with the discrete fibration $X\times U\to U$. Then such a discrete fibration on $\mathbb{U}$ is in the image of ρ^* precisely when X is ψ-split by U. The factor τ in diagram (20) is identified with ρ modulo the equivalence γ, and φ with δ. Thus, τ is connected and locally connected, and φ is atomic. Finally, we have

$$\mathcal{B}(\mathscr{F};\psi(\mathbb{U}))/d_0 \simeq \mathscr{F}/\psi_!U\ ,$$

where d_0 is the domain discrete fibration on the groupoid $\psi_!(\mathbb{U})$. Thus, q is étale and therefore p is as well. ∎

3.11. Remark. Under the equivalence in Prop. 3.10, the domain discrete fibration d_0 is identified with $\tau_!U$.

3.12. Corollary. *Suppose that U is a globally supported ψ-normal object for a locally connected geometric morphism ψ (17). Then the following are equivalent:*

1. *ψ is connected;*
2. *the groupoid $\psi(\mathbb{U})$ is connected;*
3. *φ is connected.*

PROOF. If ψ is connected, then $\psi_!(1) = 1$ whence $\psi(\mathbb{U})$ is connected. If $\psi(\mathbb{U})$ is connected, then δ is connected, whence φ is connected by Prop. 3.10. Finally, if φ is connected, then ψ is connected because τ is. ∎

4. Isotropic Galois theory

We turn to interpreting the Galois-theoretic machinery in the context of isotropy quotients. In particular, we interpret Prop. 3.10 in the case of the isotropy quotient (4).

4.1. CONNECTED GROUPOIDS AND ISOTROPY. We wish to generalize Lemma 2.2 to the case of a connected groupoid. We begin by observing the following.

4.2. LEMMA. *Suppose that $\mathbb{G} = (G_0, G_1)$ is a groupoid internal to a topos $\mathscr{F}$ for which $\mathscr{F}/G_0$ is anisotropic. Then $\mathcal{B}(\mathscr{F};\mathbb{G})$ is locally anisotropic.*

PROOF. The domain object d_0 is by right multiplication a discrete fibration on $\mathbb{G}$. As such it is a globally supported object of $\mathcal{B}(\mathscr{F};\mathbb{G})$. Furthermore, we have

$$\mathcal{B}(\mathscr{F};\mathbb{G})/d_0 \simeq \mathscr{F}/G_0 .$$

(The familiar fact that if G is a group in a topos $\mathscr{F}$, then $\mathcal{B}(\mathscr{F};G)/yG \simeq \mathscr{F}$, where yG is the representable G-object in $\mathscr{F}$ is a special case of this equivalence.) Thus, if the object G_0 is anisotropic, then $\mathcal{B}(\mathscr{F};\mathbb{G})$ is locally anisotropic. ∎

Recall from § 3.1 that the vertex group $\zeta : Z \twoheadrightarrow G_0$ associated with a groupoid $\mathbb{G}$ in a topos $\mathscr{F}$ is a group object in $\mathscr{F}/G_0$. On the other hand, ζ with conjugation is a group internal to $\mathcal{B}(\mathscr{F};\mathbb{G})$. The following is a straightforward verification: we omit the details.

4.3. LEMMA. *The vertex group ζ classifies isotropy of $\mathcal{B}(\mathscr{F};\mathbb{G})$ relative to $\mathscr{F}$ in the sense that for any discrete fibration $p : X \twoheadrightarrow G_0$, morphisms $p \twoheadrightarrow \zeta$ in $\mathcal{B}(\mathscr{F};\mathbb{G})$ are in natural bijection with automorphisms of*

$$\mathcal{B}(\mathscr{F};\mathbb{G})/p \longrightarrow \mathcal{B}(\mathscr{F};\mathbb{G})$$

over $\mathscr{F}$. Moreover, the isotropy quotient of $\mathcal{B}(\mathscr{F};\mathbb{G})$ relative to $\mathscr{F}$ is $\mathscr{F}/\pi_0(\mathbb{G})$.

4.4. LEMMA. *Suppose that $\mathbb{G} = (G_0, G_1)$ is a groupoid internal to a topos $\mathscr{F}$. If $\mathbb{G}$ is connected and $\mathscr{F}$ is anisotropic, then the vertex group ζ is the isotropy group of $\mathcal{B}(\mathscr{F};\mathbb{G})$, and the isotropy quotient of $\mathcal{B}(\mathscr{F};\mathbb{G})$ is $\mathscr{F}$.*

PROOF. We use the equivalence of diagrams (14) and (15) (§ 3.1). For the sake of argument let β denote the isotropy group of $\mathcal{B}(\mathscr{F};\mathbb{G})$. There is a homomorphism $\zeta \twoheadrightarrow \beta$ in $\mathcal{B}(\mathscr{F};\mathbb{G})$ because isotropy over $\mathscr{F}$, classified by ζ (Lemma 4.2), passes by composition with $\mathscr{F} \longrightarrow \mathsf{Set}$ to the usual 'absolute' isotropy of $\mathscr{F}$, classified by β. If $\mathscr{F}$ is anisotropic, then so is $\mathscr{F}/G_0$, so that by Lemma 2.2 the isotropy group of $\mathcal{B}(\mathscr{F}/G_0;\zeta)$ is $\overline{\zeta}$, meaning the vertex group ζ with its conjugation action. Every object of $\mathscr{F} = \mathscr{F}_\theta$ is isotropically trivial. In particular, G_0 is isotropically trivial. Therefore, by Prop. 2.4, $\gamma^* G_0$ is isotropically trivial, so that pulling back over $\gamma^* G_0$ preserves isotropy groups: β must go to $\overline{\zeta}$. This proves vis-a-vis the equivalence

$$\mathcal{B}(\mathscr{F}/G_0;\zeta) \;\simeq\; \mathcal{B}(\mathscr{F};\mathbb{G})/\gamma^* G_0$$

that over $\gamma^* G_0$ the homomorphism $\zeta \twoheadrightarrow \beta$ is an isomorphism because over $\gamma^* G_0$, ζ is canonically associated across the equivalence with $\overline{\zeta}$. Whence $\zeta \twoheadrightarrow \beta$ is already an isomorphism. The upshot is that ζ is therefore the isotropy group of $\mathcal{B}(\mathscr{F};\mathbb{G})$, so that $\mathscr{F}$ is the isotropy quotient. ∎

4.5. Isotropically split objects. The interpretation of Galois theory relative to the isotropy quotient geometric morphism deserves special terminology. We shall say that an object U of a topos $\mathscr{E}$ *isotropically splits an object* X, or that X *is isotropically split by* U, if the adjunction square (left)

$$\begin{tikzcd} X \times U \arrow[r] \arrow[d] & \psi^*\psi_!(X \times U) \arrow[d] \\ U \arrow[r] & \psi^*\psi_!(U) \end{tikzcd} \qquad \begin{tikzcd} X \times U \arrow[r] \arrow[d] & O_{X \times U} \arrow[d] \\ U \arrow[r] & O_U \end{tikzcd}$$

is a pullback, where ψ is the isotropy quotient of $\mathscr{E}$ (4). The square above (right) depicts the same adjunction square using our sometimes preferred and convenient notation for the orbit objects associated with the canonical action by the isotropy group.

Let $\mathrm{SPL}_\theta(U)$ denote the full subcategory of $\mathscr{E}$ on those objects that are isotropically split by U. This topos is the following 'fundamental pushout topos' [Bunge, 2008].

$$\begin{tikzcd} \mathscr{E}/U \arrow[r] \arrow[d] & \mathscr{E}_\theta/\psi_! U \arrow[d, "p"'] \arrow[ddr, bend left] & \\ \mathscr{E} \arrow[r, "\tau"] \arrow[drr, bend right, "\psi"'] & \mathrm{SPL}_\theta(U) \arrow[dr, "\varphi"] & \\ & & \mathscr{E}_\theta \end{tikzcd} \tag{21}$$

The fundamental lemma of isotropy 2.5 states that the outside squares of diagrams (12) and (21) are equivalent. Therefore, the two pushouts inside are equivalent:

$$\mathscr{E}_U \;\simeq\; \mathrm{SPL}_\theta(U)\,. \tag{22}$$

Combining this with Lemma 2.10, we have the following.

4.6. LEMMA. *For a fixed object U, the following are equivalent for an object X :*

(i) X is isotropically split by U ;

(ii) U^*X is isotropically trivial in $\mathscr{E}/U$;

(iii) X is an object of $\mathscr{E}_U$.

Thus, in $\mathscr{E}$ the adjunction square below (left)

$$\begin{tikzcd} X \times U \arrow[r] \arrow[d] & O_{X\times U} \arrow[d] \\ U \arrow[r] & O_U \end{tikzcd} \qquad \begin{tikzcd} X \times Z(U) \arrow[r, "X\times\eta_U"] \arrow[dr, "\mathrm{proj}"'] & X \times Z \arrow[d, "\theta_X"] \\ & X \end{tikzcd}$$

is a pullback if and only if $\theta_X(X \times \eta_U)$ equals the projection depicted above (right).

4.7. REMARK. The geometric morphism p in (21) is étale: one can show directly that

$$\mathscr{E}_\theta/\psi_!U \;\simeq\; \mathrm{SPL}_\theta(U)/\tau_!U\ .$$

Therefore, the equivalent p in (12) is étale, which we describe as follows. For any object X of $\mathscr{E}_U$, p^*X is the isotropically trivial object $\tau^*X \times U \twoheadrightarrow U$ of $\mathscr{E}/U$. If $E \twoheadrightarrow U$ is an isotropically trivial object of $\mathscr{E}/U$, then $p_!(E \twoheadrightarrow U)$ equals $\tau_!E$. For instance, $p_!1$ equals $\tau_!U$, so that $p^*p_!1$ equals the object $\tau^*\tau_!U \times U \twoheadrightarrow U$ of $(\mathscr{E}/U)_\theta$.

4.8. ISOTROPICALLY NORMAL OBJECTS. Ultimately, we would like to identify the groupoid $\psi(\mathbb{U})$ in Prop. 3.10 in the case of the isotropy quotient ψ (4), where $\mathbb{U}$ denotes the chaotic (or codiscrete) groupoid associated with U (§ 3.2). We thus need the following:

4.9. DEFINITION. *An object of a topos is* isotropically normal *if it isotropically splits itself. Equivalently, an object U is normal in this sense if $U^*(U)$ is isotropically trivial. We say that $\mathscr{E}/U \longrightarrow \mathscr{E}$ is* an isotropically normal covering *if U is a globally supported isotropically normal object.*

4.10. PROPOSITION. *An object U of a topos $\mathscr{E}$ is anisotropic if and only if*

$$\mathrm{SPL}_\theta(U) = \mathscr{E}\ .$$

In particular, an anisotropic object is isotropically normal.

PROOF. If U is anisotropic, then 11 commutes for any object X of $\mathscr{E}$ because ζ_U is trivial in this case. Therefore, by (22) we have $\mathrm{SPL}_\theta(U) = \mathscr{E}_U = \mathscr{E}$. Conversely, if $\mathrm{SPL}_\theta(U) = \mathscr{E}$, then again by (22) $\mathscr{E}_U = \mathscr{E}$ so that 11 commutes for any object X of $\mathscr{E}$. An arbitrary object $A \twoheadrightarrow U$ of $\mathscr{E}/U$ is a subobject of U^*A so that ζ_U must act trivially in it also. Thus, ζ_U acts trivially in all objects of $\mathscr{E}/U$, so that $\mathscr{E}/U = (\mathscr{E}/U)_\theta$. This implies that ζ_U is trivial, i.e., that U is anisotropic. As for the second statement, if U is anisotropic, then $\mathscr{E} = \mathrm{SPL}_\theta(U)$, so that U isotropically splits all objects of $\mathscr{E}$. In particular, U splits itself. ∎

Although an object U need not be normal, we may form its (isotropic) *normal closure* $\widehat{U} = \tau^*\tau_!(U)$ (for $\tau : \mathscr{E} \longrightarrow \mathrm{SPL}_\theta(U)$) so that

$$\begin{tikzcd} U \times Z(U) \arrow[r, "U\times\eta_U"] & U \times Z \arrow[r, shift left, "\theta"] \arrow[r, shift right, "\mathrm{proj}"'] & U \arrow[r, two heads] & \widehat{U} \end{tikzcd} \tag{23}$$

is a coequalizer in $\mathscr{E}$. It is not difficult to verify that isotropic normal closure is an endofunctor of a topos, but of course first we must establish that $\widehat{U}$ is indeed normal.

4.11. Proposition. *Consider an object U of a topos $\mathscr{E}$ with normal closure $\widehat{U}$. Then the following hold:*

(i) the morphism of crossed sheaves

$$\begin{tikzcd} Z(U) \arrow[r] \arrow[dr, "\eta_U"'] & Z(\widehat{U}) \arrow[d, "\eta_{\widehat{U}}"] \\ & Z \end{tikzcd}$$

induces an equality on isotropy quotients: $\mathscr{E}_U = \mathscr{E}_{\widehat{U}}$;

(ii) the objects U and $\widehat{U}$ split the same objects of $\mathscr{E}$: $\mathrm{SPL}_\theta(U) = \mathrm{SPL}_\theta(\widehat{U})$;

(iii) $\widehat{U}$ is isotropically normal.

Proof. The first two statements are equivalent, and the third is a consequence of the second because U splits $\widehat{U}$. In order to establish (i) we must show for any object X of $\mathscr{E}$ that

$$\begin{tikzcd} X \times Z(U) \arrow[r] & X \times Z(\widehat{U}) \arrow[r] & X \times Z \arrow[r, shift left, "\theta"] \arrow[r, shift right] & X \end{tikzcd}$$

commutes from $X \times Z(U)$ (U^*X is isotropically trivial) if and only if the part of the diagram starting from $X \times Z(\widehat{U})$ commutes ($\widehat{U}^*X$ is isotropically trivial). Of course, one implication is trivial. For the other assume that the diagram commutes from $X \times Z(U)$. Let $(x, \widehat{u}, z) \in X \times Z(\widehat{U})$, so that $\widehat{u}z = \widehat{u}$, where $\widehat{u} \in \widehat{U}$. We wish to show that $xz = x$. We know there is u such that $uz \sim u$, where $\sim$ is the equivalence relation on U such that $\widehat{U} = U/{\sim}$. Explicitly, this means that there is a zig-zag connecting uz and u. For instance, if the zig-zag consists of three steps

$$\begin{tikzcd}[column sep=small] & (u_0, v_0, z_0) \arrow[dl, "\theta"'] \arrow[dr] & & (u_1, v_1, z_1) \arrow[dl, "\theta"'] \arrow[dr] & & (u, v_2, z_2) \arrow[dl, "\theta"'] \arrow[dr] & \\ uz & & u_0 & & u_1 & & u \end{tikzcd}$$

then we have equations: $uz = u_0z_0$, $v_0z_0 = v_0$, $u_0 = u_1z_1$, $v_1z_1 = v_1$, $u_1 = uz_2$, and $v_2z_2 = v_2$. Therefore, we have

$$uz = u_0z_0 = u_1z_1z_0 = uz_2z_1z_0 .$$

Because the diagram commutes from $X \times Z(U)$ (by assumption) we therefore may conclude $xz = xz_2z_1z_0$, $xz_0 = x$, $xz_1 = x$, $xz_2 = x$, whence $xz = x$. Of course, many zig-zags are possible. For instance, we could have a zig-zag diagram such as

$$\begin{tikzcd}[column sep=small]
& (u_0, v_0, z_0) \arrow[dl, "\theta"'] \arrow[dr] & & (u_0, v_1, z_1) \arrow[dl] \arrow[dr, "\theta"] & & (u, v_2, z_2) \arrow[dl, "\theta"'] \arrow[dr] & \\
uz & & u_0 & & u_1 & & u
\end{tikzcd}$$

which gives equations $uz = u_0z_0$, $v_0z_0 = v_0$, $u_1 = u_0z_1$, $v_1z_1 = v_1$, $u_1 = uz_2$, and $v_2z_2 = v_2$. In this case we have

$$uz = u_0z_0 = u_1z_1^{-1}z_0 = uz_2z_1^{-1}z_0 \ .$$

Then $xz = x$ in just the same way as before. In fact, any such zig-zag diagram connecting u and uz with any finite number of steps may be argued in just the same way. ∎

4.12. Corollary. *The object $\widehat{U}$ is the isotropic normal closure of U in the sense that any morphism $U \twoheadrightarrow V$ to an isotropically normal object V factors uniquely through $\widehat{U}$.*

Proof. If we have a morphism $m : U \twoheadrightarrow V$ and V splits itself (relative to ψ), then U must split V , whence by Lemma 4.6 m must coequalize the parallel pair (23). ∎

4.13. Remark. One further thing to note about Prop. 4.11 is that although (ii) is true for any locally connected geometric morphism, in the case of the isotropy quotient ψ the left adjoint $\tau_!$ has the straightforward description as the coequalizer (9), and the equivalent statement (i) is proved in a relatively straightforward way.

5. Main results

We have assembled the tools that enable us to derive the main results. As always

$$\begin{tikzcd}
\mathscr{E} \arrow[r, two heads, "\tau"] \arrow[dr, "\psi"'] & \mathscr{E}_U \arrow[d, "\varphi"] \\
& \mathscr{E}_\theta
\end{tikzcd}$$

depicts the isotropy quotient ψ of $\mathscr{E}$, and the isotropy quotient τ of the crossed sheaf $\eta_U : Z(U) \twoheadrightarrow Z$ associated with an object U of $\mathscr{E}$. The following proposition characterizes $\mathscr{E}_U$ as a topos over $\mathscr{E}_\theta$. As before $\widehat{U}$ denotes the (isotropic) normal closure of U .

5.1. Proposition. *Let U be a globally supported object of a topos $\mathscr{E}$. Then we have an equivalence*

$$\mathscr{E}_U \simeq \mathcal{B}(\mathscr{E}_\theta; \psi(\widehat{\mathbb{U}}))$$

over $\mathscr{E}_\theta$, where $\psi(\widehat{\mathbb{U}})$ is the connected groupoid internal to $\mathscr{E}_\theta$ associated with the chaotic groupoid $\widehat{\mathbb{U}}$ of the normal closure $\widehat{U}$. Moreover, in diagram (12) or equivalently (21) the geometric morphisms τ and φ are connected atomic, and p is étale.

PROOF. This follows from Props. 3.10, 4.11, 4.6 and (22) noting that since $\widehat{U}$ is ψ-normal, $\psi(\widehat{\mathbb{U}})$ is a connected groupoid in $\mathscr{E}_\theta$. ∎

In the locally anisotropic case we have the following.

5.2. COROLLARY. *Suppose that a topos $\mathscr{E}$ is locally anisotropic: let U denote a globally supported anisotropic object of $\mathscr{E}$. Then the groupoid $\psi(\mathbb{U})$ is connected, and we have*

$$\mathscr{E} \simeq \mathcal{B}(\mathscr{E}_\theta; \psi(\mathbb{U})) .$$

In other words, the geometric morphism τ in diagram (12) is an equivalence so that the isotropy quotient ψ is recovered as the structure geometric morphism

$$\gamma : \mathcal{B}(\mathscr{E}_\theta; \psi(\mathbb{U})) \longrightarrow \mathscr{E}_\theta$$

associated with the groupoid $\psi(\mathbb{U})$.

PROOF. If a globally supported object U is anisotropic, then $\mathscr{E}_U = \mathscr{E}$. Moreover, the anisotropic U is isotropically normal, so that $\widehat{U} = U$. ∎

We are in position to state the main result of the paper. In general, the (first) isotropy quotient $\mathscr{E}_\theta$ is not anisotropic, but we may iterate the isotropy quotient, possibly transfinitely, until we reach an anisotropic quotient Ψ (6). Let us call this the *ultimate* isotropy quotient of $\mathscr{E}$.

5.3. THEOREM. *A topos is locally anisotropic if and only if it is equivalent to $\mathcal{B}(\mathscr{F}; \mathbb{G})$, where $\mathscr{F}$ is anisotropic and $\mathbb{G}$ is a connected groupoid internal to $\mathscr{F}$. Moreover, in this case $\mathscr{F}$ is the first isotropy quotient of $\mathcal{B}(\mathscr{F}; \mathbb{G})$. In particular, the first isotropy quotient of a locally anisotropic topos is anisotropic. In other words, a locally anisotropic topos has isotropy rank at most one.*

PROOF. Suppose that a topos $\mathscr{E}$ is locally anisotropic with $U \twoheadrightarrow 1$ such that $\mathscr{E}/U$ is anisotropic. We have an equivalence

$$\mathscr{E}/U = (\mathscr{E}/U)_\theta = (\mathscr{E}/U)_\Theta \simeq \mathscr{E}_\Theta/\Psi_! U , \tag{24}$$

which implies in particular that the adjunction square

$$\begin{array}{ccc} E & \longrightarrow & \Psi^*\Psi_!(E) \\ \downarrow & & \downarrow \\ U & \longrightarrow & \Psi^*\Psi_!(U) \end{array}$$

is a pullback for every object $E \to U$ of $\mathscr{E}/U$. In particular,

$$\begin{array}{ccc} U \times U & \longrightarrow & \Psi^*\Psi_!(U \times U) \\ \downarrow & & \downarrow \\ U & \longrightarrow & \Psi^*\Psi_!(U) \end{array}$$

is a pullback, so that U is Ψ-normal. The pushout

$$\tau : \mathscr{E} \longrightarrow \mathrm{SPL}_{\Psi}(U) \simeq \mathcal{B}(\mathscr{E}_{\Theta}; \Psi(\mathbb{U}))$$

of the equivalence (24) along $\mathscr{E}/U \longrightarrow \mathscr{E}$ is an equivalence, and the other equivalence is by Prop. 3.10. The topos $\mathscr{E}_{\Theta}$ is anisotropic, and the groupoid $\Psi(\mathbb{U})$ is connected for the usual reason that $\mathbb{U}$ is connected and Ψ is connected (Cor. 3.9). Therefore, by Lemma 4.4 $\mathscr{E}_{\Theta}$ is the first isotropy quotient of

$$\mathscr{E} \simeq \mathcal{B}(\mathscr{E}_{\Theta}; \Psi(\mathbb{U})) \ .$$

In other words, $\mathscr{E}$ has isotropy rank at most one. ∎

5.4. Remark. While Theorem 5.3 gives a complete answer in the locally anisotropic case it is not so that in general a topos is recovered from its (ultimate) isotropy quotient by means of a connected groupoid. Indeed, consider the case $\mathscr{E} = \mathcal{B}(\mathrm{Aut}(\mathbb{N}))$, also known as the Schanuel topos. Here $\mathrm{Aut}(\mathbb{N})$ is considered as a topological group in the usual way. It can be shown that in this case $\mathscr{E}_{\theta} = \mathsf{Set}$. However, $\mathscr{E}$ is not equivalent to $\mathcal{B}(\mathbb{G})$ for any groupoid $\mathbb{G}$. Indeed, in general if H is a topological group, then $\mathcal{B}(H)$ is equivalent to $\mathcal{B}(G)$ for a discrete group G precisely when H is itself discrete. To see why recall first that we may assume that H is nearly discrete, i.e., that the intersection of all open subgroups of H is the trivial subgroup [Johnstone, 2002]. Note that the atoms in $\mathcal{B}(H)$ correspond to open subgroups of H. For a discrete group G, the topos $\mathcal{B}(G)$ has the property that the representable G-set G is the largest atom, in the sense that any atom is a quotient of it. Thus, if $\mathcal{B}(H) \simeq \mathcal{B}(G)$, then $\mathcal{B}(H)$ also has such a largest atom, corresponding to a smallest open subgroup of H. This open subgroup is then contained in all open subgroups, hence it is contained in the intersection, and therefore must be trivial. However, if the trivial subgroup is open, then the topology on H is discrete.

6. Application to inverse semigroups

In this final section we apply our main result to the structure theory of inverse semigroups. We first review some basic notions from semigroup theory as well as the definition of the classifying topos of an inverse semigroup. We then discuss the Clifford-fundamental exact sequence of an inverse semigroup and its relation to isotropy theory. Finally, we present an explicit calculation of the semidirect product groupoid that recovers an arbitrary inverse semigroup up to Morita equivalence.

6.1. Semigroups and toposes. An *inverse semigroup* [Lawson, 1998] is a semigroup S, written $(s,t) \mapsto st$, with the property that for each $s \in S$ there is a unique s^* for which $ss^*s = s$ and $s^*ss^* = s^*$. We write $E = E(S)$ for the set of idempotents of S, which forms a meet-semilattice. An inverse semigroup is partially ordered by $s \leq t \Leftrightarrow s = te$ for some $e \in E$.

6.2. Example. The prototypical example of an inverse semigroup is the collection $I(A)$ of all partial bijections on a set A. Just as every group embeds into a permutation group, every inverse semigroup embeds into a semigroup of this form.

With every inverse semigroup S we may associate a topos $\mathcal{B}(S)$. An object of $\mathcal{B}(S)$ is *an étale S-set*, which is a set X equipped with associative action $X \times S \to X$, written $(x, s) \mapsto xs$, and a structure map $p : X \to E$ such that $xp(x) = x$ and $p(xs) = s^*p(x)s$ for all $x \in X$, $s \in S$. A morphism of étale S-sets is a function commuting with the structure maps and preserving the action. The topos $\mathcal{B}(S)$ is an étendue: the domain object $\partial_0 : S \to E$, defined by $\partial_0(s) = s^*s$, is a globally supported object, which is torsion-free in the sense that the slice topos $\mathcal{B}(S)/\partial_0$ is localic. In fact, $\mathcal{B}(S)/\partial_0$ is equivalent to presheaves on the meet-semilattice E. Thus, ∂_0 is a globally supported anisotropic object. Moreover, every étendue is a subtopos of one of the form $\mathcal{B}(S)$. A *prehomomorphism* of inverse semigroups is a function $\phi : S \to T$ for which $\phi(st) \leq \phi(s)\phi(t)$. Such a ϕ induces a geometric morphism of toposes $\mathcal{B}(S) \longrightarrow \mathcal{B}(T)$.

6.3. Extensions. Two classes of inverse semigroups that are important for understanding the structure of general inverse semigroups are Clifford semigroups and fundamental semigroups.

6.4. Definition. *Let S be an inverse semigroup with idempotent set E.*

(i) S is Clifford *when $se = es$ for all $s \in S$ and $e \in E$.*

(ii) S is fundamental *when for all $s \in S$, if $se = es$ for all $e \in E$, then $s \in E$.*

For example, any semigroup $I(X)$ is fundamental (Eg. 6.2). A Clifford semigroup is characterized as a strong semilattice of groups, i.e., as a semidirect product of a (contravariant) functor of groups on a meet-semilattice.

Every inverse semigroup S fits into a short exact sequence

$$Z(S; E) \rightarrowtail S \twoheadrightarrow S/\mu \quad , \tag{25}$$

where μ is the maximum idempotent-separating congruence on S, and where

$$Z = Z(S; E) = \{s \in S \mid \forall e \in E.\, se = es\}$$

is the *centralizer of the idempotents*. The sequence (25) has the property that all three semigroups share the same idempotent lattice E. It is well known that Z is Clifford and S/μ is fundamental, showing that every inverse semigroup is an extension of a Clifford semigroup by a fundamental one. Of course, this does not suffice to reconstruct S as a semidirect product. As an étale S-set the centralizer $\partial_0 : Z \to E$ is the isotropy group of $\mathcal{B}(S)$, and the geometric morphism

$$\psi : \mathcal{B}(S) \longrightarrow \mathcal{B}(S/\mu)$$

associated with the quotient (25) coincides with the isotropy quotient of $\mathcal{B}(S)$: we have $\mathcal{B}(S/\mu) \simeq \mathcal{B}(S)_\theta$.

6.5. Reconstruction. The semidirect product of a Clifford semigroup by a fundamental one does not pick up all inverse semigroups. Nevertheless there is a structure theorem characterizing inverse semigroups up to Morita equivalence.

6.6. Proposition. *Let S be an inverse semigroup. Then there exists a connected groupoid $\mathbb{G}$ in $\mathcal{B}(S/\mu)$ such that S is Morita equivalent to an ordered groupoid $\mathbb{G} \ltimes S/\mu$. Hence every inverse semigroup is Morita equivalent to a semidirect product of a fundamental semigroup by a connected groupoid.*

Proof. This is an immediate consequence of Theorem 5.3. ∎

Of course, we would like an explicit description of $\mathbb{G}$ and $\mathbb{G} \ltimes S/\mu$ in terms of the given S. Let us first describe $\mathbb{G}$ as an object of the topos $\mathcal{B}(S/\mu)$. We have $E(S/\mu) = E(S)$. Let us write $\overline{s}$ for an element of S/μ. Note that $s \sim_\mu t$ implies $s^*s = t^*t$ and $ss^* = tt^*$. Moreover, we have $s \leq t$ if and only if $\overline{s} \leq \overline{t}$. We calculate the groupoid $\mathbb{G} = (G_0, G_1) = (\psi_!(\partial_0), \psi_!(\partial_0 \times \partial_0))$.

Objects: the object of objects $G_0 = \psi_!(\partial_0)$ is the étale S/μ-set

$$\partial_0 : S/\mu \to E \; ; \; \overline{s} \mapsto s^*s \, .$$

The action of S/μ on G_0 is by precomposition.

Morphisms: the object of morphisms $G_1 = \psi_!(\partial_0 \times \partial_0)$ is the étale S/μ-set

$$S \times_E S/\mu = \{(a, \overline{s}) \mid a^*a = ss^*\} \to E \; ; \; (a, \overline{s}) \mapsto s^*s \, .$$

The action of S/μ on G_1 is by precomposition as well.

Domain/codomain: an element $(a, \overline{s})$ of G_1 is regarded as a morphism from $\overline{as}$ to $\overline{s}$. Thus, a morphism $\overline{t} \to \overline{s}$ of $\mathbb{G}$ corresponds to an element $a \in S$ for which $\overline{as} = \overline{t}$ and $a^*a = ss^*$.

Composition: in order to compose $(b, \overline{t}) \cdot (a, \overline{s})$ we must have $\overline{s} = \overline{bt}$, so that we may set $(b, \overline{t}) \cdot (a, \overline{s}) := (ab, \overline{t})$.

$$\begin{tikzcd} \overline{abt} = \overline{as} \arrow[r, "(a,\overline{s})"'] \arrow[rr, bend left, "(b,\overline{t})\cdot(a,\overline{s}):=(ab,\overline{t})"] & \overline{s} = \overline{bt} \arrow[r, "(b,\overline{t})"'] & \overline{t} \end{tikzcd}$$

Identities: the identity at $\overline{s}$ is $(s^*s, \overline{s})$.

Inverses: the inverse of $(a, \overline{s}) : \overline{as} \to \overline{s}$ is $(a^*, \overline{as}) : \overline{a^*as} = \overline{s} \to \overline{as}$.

It is straightforward to verify that this defines a groupoid internal to the topos $\mathcal{B}(S/\mu)$.

We may now use the Grothendieck (semidirect product) construction to turn $\mathbb{G}$ into an ordered groupoid $\mathbb{G} \ltimes S/\mu$ given explicitly as follows:

Objects: an object is a pair $(e, \overline{s})$, where $e \in E$, $\overline{s} \in S/\mu$ and $s^*s = e$.

Morphisms: a morphism $(e, \overline{s}) \to (d, \overline{t})$ is a pair $(\overline{r}, b)$ with $\overline{btr} = \overline{s}$ and $b^*b = tt^* = (tr)(tr)^*$.

$$\begin{tikzcd} e \arrow[r, "\overline{r}"] \arrow[d, "\overline{s}"'] & d \arrow[d, "\overline{t}"] \\ ss^* & tt^* \arrow[l, "b"] \end{tikzcd}$$

Composition: this is by pasting diagrams.

$$\begin{tikzcd} e \arrow[r, "\overline{r}"] \arrow[d, "\overline{s}"'] & d \arrow[r, "\overline{u}"] \arrow[d, "\overline{t}"] & c \arrow[d, "\overline{v}"] \\ ss^* & tt^* \arrow[l, "b"] & vv^* \arrow[l, "a"] \end{tikzcd}$$

Identities: the identity at $(e, \overline{s})$ is $(\overline{e}, ss^*)$.

Inverses: the inverse of $(\overline{r}, b)$ is $(\overline{r^*}, b^*)$.

Ordering: the set of objects is ordered by $(e, \overline{s}) \leq (d, \overline{t})$ if and only if $s \leq t$ in S. The set of morphisms is ordered by $(\overline{r}, b) \leq (\overline{t}, c)$ if and only if $r \leq t$ and $b \leq c$ in S.

6.7. Remark. In general the ordered groupoid $\mathbb{G} \ltimes S/\mu$ is not an inductive one since the set of objects S/μ need not be a meet-semilattice. Indeed, for this to be the case the semigroup S/μ must be a meet semigroup.

6.8. Remark. We discovered Proposition 6.6 as a matter of course during our continued investigation of locally anisotropic toposes. In any case it is an instance of the more general fact, which may be proved by means independent of isotropy theory, that a homomorphism $S \to T$ of inverse semigroups with the same idempotent semilattice is recovered by the Grothendieck construction applied to a groupoid internal to $\mathcal{B}(T)$. Moreover, the groupoid is connected if the homomorphism is surjective.

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