A NOTE ON IMAGES OF COVER RELATIONS

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ABSTRACT.   For a category $\mathbb{C}$, a small category $\mathbb{I}$, and a pre-cover relation $\sqsubset$ on $\mathbb{C}$ we prove, under certain completeness assumptions on $\mathbb{C}$, that a morphism $g : B \to C$ in the functor category $\mathbb{C}^{\mathbb{I}}$ has an $\sqsubset^{\mathbb{I}}$-image, where $\sqsubset^{\mathbb{I}}$ is the pre-cover relation on $\mathbb{C}^{\mathbb{I}}$ induced by $\sqsubset$, as soon as each component of g has an $\sqsubset$-image. We then apply this to show that if a pointed category $\mathbb{C}$ is: (i) algebraically cartesian closed; (ii) exact protomodular and action accessible; or (iii) admits normalizers, then the same is true of each functor category $\mathbb{C}^{\mathbb{I}}$ with $\mathbb{I}$ finite. In addition, our results give explicit constructions of $\sqsubset^{\mathbb{I}}$-images in functor categories using limits and $\sqsubset$-images in the underlying category. In particular, they can be used to give explicit constructions of both centralizers and normalizers in functor categories using limits and centralizers or normalizers (respectively) in the underlying category.

Introduction

Perhaps the most natural way to extend the definition of commuting elements of a group to homomorphisms into a group, is to say that a pair of group homomorphisms $f : A \to C$ and $g : B \to C$ commute if each element in the image of f commutes with each element in the image of g. Moreover, one can show, directly or as a special case of S. Mac Lane's characterization of bifunctors (see [15]), that this condition is equivalent to the existence of a morphism $\varphi : A \times B \to C$ making the diagram

$$A \xrightarrow{\langle 1,0 \rangle} A \times B \xleftarrow{\langle 0,1 \rangle} B$$

$$\downarrow{\varphi}$$

$$f \searrow \quad C \quad \swarrow g$$

(1)

in which $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$ are the homomorphisms defined by $\langle 1, 0 \rangle(a) = (a, 1)$ and $\langle 0, 1 \rangle(b) = (1, b)$ respectively, commute. This last formulation was used by S. Huq in [11] to study commutativity and other closely related notions in a categorical context, close to the more recent semi-abelian context introduced by G. Janelidze, L. Márki, and W. Tholen in [12]. Later, Z. Janelidze [14] introduced and studied relations on the morphisms of a category called cover relations and of which the *commutes relation* described above is an example. It was shown that other similar kinds of cover relations

arise from special kinds of monoidal structures which are called monoidal sum structures. In addition, it was shown that cover relations also arise from factorization systems on categories, from which some of terminology and notation is derived.

Let us recall briefly what a cover relation is and how both these kinds of cover relations arise. A pre-cover relation $\sqsubset$ is a relation on the class of morphisms of a category $\mathbb{C}$ such that if $f \sqsubset g$ then the codomain of f and g are equal. A pre-cover relation $\sqsubset$ is a cover relation if it satisfies: (i) if $f \sqsubset g$ and the composite hf is defined, then $hf \sqsubset hg$; (ii) if $f \sqsubset g$ and the composite fe is defined, then $fe \sqsubset g$. Let $(\mathbb{C}, \otimes, I, \alpha, \rho, \lambda)$ be a monoidal category, such that I is an initial object in $\mathbb{C}$ and for each A and B the morphisms

$$A \otimes I \xrightarrow{1_A \otimes !_B} A \otimes B \xleftarrow{!_A \otimes 1_B} I \otimes B,$$

where $!_A$ and $!_B$ are the unique morphism from I to A and B respectively, are jointly epimorphic. The induced relation $\sqsubset$ is defined by requiring, for a pair of morphisms $f : A \to C$ and $g : B \to D$, that $f \sqsubset g$ whenever $C = D$ and there exists a morphism $\varphi : A \otimes B \to C$ making the diagram

$$A \xrightarrow{\rho_A} A \otimes I \xrightarrow{1_A \otimes !_B} A \otimes B \xleftarrow{!_A \otimes 1_B} I \otimes B, \xleftarrow{\lambda_B} B$$

commute. On the other hand given a factorization system $(\mathcal{E}, \mathcal{M})$ on a category $\mathbb{C}$ the induced relation $\sqsubset$ on the morphisms of $\mathbb{C}$ is defined by requiring $f \sqsubset g$ whenever f and g have the same codomain and if $g = mv$ with m in $\mathcal{M}$, then there exists u such that $f = mu$. One of the aims of [14] was to show that under suitable conditions, both factorization systems and monoidal structures can be recovered from their induced cover relations.

Let us also recall that, given a cover relation $\sqsubset$ on a category $\mathbb{C}$, the $\sqsubset$-image of a morphism $g : B \to C$ can be defined as the terminal object in the full subcategory of $(\mathbb{C} \downarrow C)$ with objects (A, f) such that $f \sqsubset g$. A simple observation in recovering a factorization system $(\mathcal{E}, \mathcal{M})$ on a category $\mathbb{C}$ from its induced cover relation $\sqsubset$ is to note that if the class $\mathcal{M}$ of a factorization system $(\mathcal{E}, \mathcal{M})$ consists of monomorphisms, then $\mathcal{M}$ is the class of all morphisms $f : A \to C$ such that (A, f) is the $\sqsubset$-image of some morphism g. In this case it turns out that if (A, f) is the $\sqsubset$-image of a morphism $g : B \to C$, then f is the image of g with respect to the factorization system. Given a factorization system $(\mathcal{E}, \mathcal{M})$ on a category $\mathbb{C}$, such that $\mathcal{M}$ consists of monomorphisms, it is easy to observe that for each category $\mathbb{I}$ the induced factorization system $(\mathcal{E}^{\mathbb{I}}, \mathcal{M}^{\mathbb{I}})$ on the functor category $\mathbb{C}^{\mathbb{I}}$, defined componentwise, has $\mathcal{M}^{\mathbb{I}}$ consisting of monomorphisms. This means that the induced cover relation $\sqsubset^{\mathbb{I}}$ on $\mathbb{C}^{\mathbb{I}}$, defined componentwise, admits images since it is also the cover relation induced by $(\mathcal{E}^{\mathbb{I}}, \mathcal{M}^{\mathbb{I}})$, and furthermore, its images are *computed componentwise*. However, it is not the case that if the underlying cover relation admits images, then images necessarily exist for the induced cover relations on each functor category see Example 2.4

below, nor that when they do exist they are computed componentwise. Our aims here are: (a) to prove that if $\mathbb{C}$ and $\mathbb{I}$ are categories and $\sqsubset$ is a cover relation (more generally a *pre-cover relation* satisfying condition (ii) of Definition 1.1 below) on $\mathbb{C}$ admitting images, then, under certain completeness assumptions on $\mathbb{C}$, the induced cover relation (pre-cover relation) $\sqsubset^{\mathbb{I}}$ on $\mathbb{C}^{\mathbb{I}}$ admits images; (b) to apply this result to prove that several categorical algebraic conditions lift from a category to its functor categories. In particular, we show that for a *semi-abelian category* $\mathbb{C}$ *action accessibility*, *algebraically-cartesian closedness*, and the existence of normalizers, *lift* to functor categories $\mathbb{C}^{\mathbb{I}}$ with $\mathbb{I}$ finite. In fact our results hold more generally see Corollary 2.3 for a precise formulation.

1. Preliminaries and examples

In this section we recall the basic background on (pre-)cover relations that we will need in the next section. In addition, we give examples of (pre-)cover relations (one of which is new), to which we will apply our main theorem of the following section, to obtain the above mentioned results showing that certain categorical algebraic conditions lift to functor categories.

We begin by recalling the following two definitions and proposition which essentially appear in [14].

1.1. DEFINITION. *A pre-cover relation $\sqsubset$ is a relation on the class of morphisms of a category $\mathbb{C}$, such that if $f \sqsubset g$ then the codomains of f and g are equal. A pre-cover relation $\sqsubset$ is a cover relation if it satisfies:*

(i) *if $f \sqsubset g$ and the composite hf is defined, then $hf \sqsubset hg$;*

(ii) *if $f \sqsubset g$ and the composite fe is defined, then $fe \sqsubset g$.*

1.2. DEFINITION. *Let $\sqsubset$ be a pre-cover relation on a category $\mathbb{C}$. The image of a morphism $g : B \to C$ is terminal object in the full subcategory of $(\mathbb{C} \downarrow C)$ with objects (A, f) such that $f \sqsubset g$.*

We will denote the $\sqsubset$-image of g by $(\mathrm{Im}_{\sqsubset}(g), \mathrm{im}_{\sqsubset}(g))$.

1.3. PROPOSITION. *Let $\sqsubset$ be a pre-cover relation on $\mathbb{C}$ satisfying condition (ii) of Definition 1.1 and let $g : B \to C$ be a morphism in $\mathbb{C}$. If g admits an image, then the morphism $\mathrm{im}_{\sqsubset}(g)$ is a monomorphism.*

Let us recall the necessary background in order to give the two examples of cover relations mentioned above. For pointed category $\mathbb{C}$ we write 0 for the zero object as well as for each zero morphism between each pair of objects. For objects A and B we write $\pi_1 : A \times B \to A$ and $\pi_2 : A \times B \to B$ for the first and second product projections (whenever they exist), and for a pair of morphisms $f : W \to A$ and $g : W \to B$ we write $\langle f, g \rangle : W \to A \times B$ for the unique morphism with $\pi_1 \langle f, g \rangle = f$ and $\pi_2 \langle f, g \rangle = g$. Recall that a pointed category with finite limits is called unital [2], if for objects A and B the

morphisms $\langle 1, 0 \rangle : A \to A \times B$ and $\langle 0, 1 \rangle : B \to A \times B$ are jointly extremal-epimorphic. A pair of morphisms $f : A \to C$ and $g : B \to C$ are said to (Huq) commute if there exists a (necessarily unique) morphism $\varphi : A \times B \to C$ making the diagram (1) commute.

1.4. EXAMPLE. *The commutes relation on a unital category is a cover relation. (In fact it is the cover relation induced by the cartesian monoidal structure on $\mathbb{C}$). The image of a morphism g with respect to this cover relation is called the centralizer of g.*

1.5. EXAMPLE. *Let $\mathbb{C}$ be a pointed category. For morphisms $f : A \to X$ and $g : B \to Y$ in $\mathbb{C}$ we say that f normalizes g (and write $f \triangleright g$) if $X = Y$, and there exists a morphism $u : A \to C$, a normal monomorphism $v : B \to C$, and monomorphism $h : C \to X$ making the diagram*

$$A \xrightarrow{\;u\;} C \xleftarrow{\;v\;} B$$
$$f \searrow \quad \downarrow h \quad \swarrow g$$
$$X$$

commute. Note that in this case, it follows that g is a monomorphism and that h normalizes g too since $h = h1_C$ and $g = hv$. The normalizes relation is a pre-cover relation satisfying condition (ii) of Definition 1.1 and the image of a monomorphism $g : B \to C$ is the normalizer of g (in the sense of [9]). Indeed, if $g : B \to X$ is a monomorphism and $(\mathrm{Im}_\triangleright(g), \mathrm{im}_\triangleright(g))$ is the image of g, then there exists a morphism $u : \mathrm{Im}_\triangleright(g) \to C$, a monomorphism $h : C \to X$ and a normal monomorphism $v : B \to C$ such that $\mathrm{im}_\triangleright(g) = hu$ and $g = hv$. But, since h normalizes g it follows that $u : (\mathrm{Im}_\triangleright(g), \mathrm{im}_\triangleright(g)) \to (C, h)$ being a monomorphism with domain a terminal object is an isomorphism.

1.6. REMARK. *The pre-cover relation in the previous example can be enlarged to become a cover relation by dropping the requirement that h be a monomorphism, but requiring that g be a monomorphism. The idea of considering this cover relation (in the ideal determined context [13]) arose in discussions with Z. Janelidze and led to considering the above pre-cover relation. Images with respect to this larger cover relation still produce normalizers (but with a stronger universal property). The difference disappears as soon as the regular image of a normal monomorphism (exists) and is normal. By further replacing the requirement that v be a normal monomorphism by that it is a Bourn-normal monomorphism, one produces a cover relations whose images recover, in the protomodular context [3], the normalizer defined in [5].*

2. The main results

Let $\sqsubset$ be a pre-cover relation on a category $\mathbb{C}$. For a category $\mathbb{I}$, the pre-cover relation $\sqsubset$ induces a pre-cover relation $\sqsubset^{\mathbb{I}}$ on $\mathbb{C}^{\mathbb{I}}$, which is defined componentwise. Note that if $\sqsubset$ satisfies condition (i) or (ii) of Definition 1.1 then so does $\sqsubset^{\mathbb{I}}$. We have:

2.1. THEOREM. *Let $\mathbb{I}$ be a category and let $\mathbb{C}$ be a category with pullbacks and with wide pullbacks of families of monomorphisms indexed by the morphisms of $\mathbb{I}$. Let $\sqsubset$ be a pre-cover relation on $\mathbb{C}$ satisfying condition (ii) of Definition 1.1. A morphism $g : B \to C$ in the functor category $\mathbb{C}^{\mathbb{I}}$, admits an image with respect to $\sqsubset^{\mathbb{I}}$ on $\mathbb{C}^{\mathbb{I}}$ if each component of g admits an image with respect to $\sqsubset$. Moreover, when this is the case, for each X in $\mathbb{I}$ the object $(\mathrm{Im}_{\sqsubset^{\mathbb{I}}}(g)(X), \mathrm{im}_{\sqsubset^{\mathbb{I}}}(g)_X)$ is the product in the comma category $(\mathbb{C} \downarrow C(X))$ of a certain family $(W_i, w_i)_{i \in \mathcal{I}}$ where $\mathcal{I}$ is the collection of all morphisms with domain X. This family consists of monomorphisms $w_i : W_i \to C(X)$ obtained for each $i : X \to Y$ in $\mathcal{I}$ by pulling back $\mathrm{im}_{\sqsubset}(g_Y)$ along $C(i)$, as displayed in the lower square of the diagram*

$$
\begin{array}{ccc}
\mathrm{Im}_{\sqsubset^{\mathbb{I}}}(g)(X) & & \\
 & \searrow^{\ \mathrm{im}_{\sqsubset^{\mathbb{I}}}(g)_X} & \\
 \ \ \searrow_{v_i} & & \\
 & W_i \xrightarrow{\ w_i\ } C(X) & \\
 & \ \ \ \downarrow^{\tilde{i}} \qquad \downarrow^{C(i)} & \\
 & \mathrm{Im}_{\sqsubset}(g_Y) \xrightarrow[\mathrm{im}_{\sqsubset}(g_Y)]{} C(Y). &
\end{array}
\tag{2}
$$

PROOF. Let $g : B \to C$ be morphism in $\mathbb{C}^{\mathbb{I}}$. We begin by showing that the above mentioned construction produces a morphism $f : A \to C$ which we then show to be the image of g. For each $i : X \to Y$ in $\mathbb{I}$, let $w_i : W_i \to C(X)$ be the preimage of $\mathrm{im}_{\sqsubset}(g_Y)$ along $C(i)$ as displayed in (2). For each object X in $\mathbb{I}$, let $v_i : A(X) \to W_i$ be the ith projection of the wide pullback of all w_i where i is a morphism with domain X, and let $f_X = w_{1_X} v_{1_X}$. Now let $i : X \to Y$ be a morphism in $\mathbb{I}$. It is easy to check that for each morphism $j : Y \to Z$ there exists a unique morphism $\bar{i} : W_{ji} \to W_j$ making the right hand square in the diagram

$$
\begin{array}{ccc}
A(X) \xrightarrow{\ v_{ji}\ } W_{ji} \xrightarrow{\ w_{ji}\ } C(X) \\
\ \ \downarrow^{A(i)} \qquad \ \downarrow^{\bar{i}} \qquad \ \ \downarrow^{C(i)} \\
A(Y) \xrightarrow{\ v_j\ } W_j \xrightarrow{\ w_j\ } C(Y)
\end{array}
$$

commute (in fact making it a pullback). It follows that there exists a unique morphism $A(i) : A(X) \to A(Y)$ such that the left hand diagram above commutes for each such j. These assignments make A an object and $f : A \to C$ a morphism in $\mathbb{C}^{\mathbb{I}}$. Since, by definition, for each X in $\mathbb{I}$ the diagram

$$
\begin{array}{ccc}
 & \xrightarrow{\ \ f_X\ \ } & \\
A(X) \xrightarrow[v_{1_X}]{} W_{1_X} \xrightarrow[w_{1_X}]{} C(X) \\
 \ \ \ \downarrow^{\tilde{1}_X} \qquad \ \downarrow^{C(1_X)} \\
 \mathrm{Im}_{\sqsubset}(g_X) \xrightarrow[\mathrm{im}_{\sqsubset}(g_X)]{} C(X).
\end{array}
$$

commutes, we see that $f_X = \text{im}_{\sqsubset}(g_X)\tilde{1}_X v_{1_X}$ and so $f_X \sqsubset g_X$ and $f \sqsubset g$. Let $f' : A' \to C$ be a morphism in $\mathbb{C}^{\mathbb{I}}$ such that $f' \sqsubset g$, we need to show that there exists a unique morphism $u : A' \to A$ such that $fu = f'$. Since $f'_Y \sqsubset g_Y$, for each morphism $i : X \to Y$, there exists a unique morphism $\underline{u}_Y : A'(Y) \to \text{Im}_{\sqsubset}(g_Y)$ such that the solid arrows in the diagram

$$
\begin{array}{ccc}
A'(X) \overset{f'_X}{\underset{v'_i}{\dashrightarrow}} W_i \xrightarrow{w_i} C(X) \\
{\scriptstyle A'(i)}\Big\downarrow \quad {\scriptstyle \tilde{i}}\Big\downarrow \quad \Big\downarrow{\scriptstyle C(i)} \\
A'(Y) \xrightarrow{\underline{u}_Y} \text{Im}_{\sqsubset}(g_Y) \overset{\text{im}_{\sqsubset}(g_Y)}{\underset{f'_Y}{\longrightarrow}} C(Y)
\end{array}
$$

commute, and hence there exists a unique morphism $v'_i : A'(X) \to W_i$ making the entire diagram commute. It now follows that there exists a unique morphism $u_X : A'(X) \to A(X)$ such that $v_i u_X = v'_i$ for each i with domain X and hence such that $f_X u_X = f'_X$. Noting that the components of f are monomorphisms it follows that the morphisms u_X are components of a (unique) natural transformation $u : A' \to A$ with $fu = f'$, as required. ∎

2.2. REMARK. *It seems worth while pointing out that the existence of induced images in functor categories can be deduced, under additional assumptions, from standard categorical constructions.*

Suppose that $\mathbb{I}$ and $\mathbb{C}$ are categories and let $U : \mathbb{I}_0 \to \mathbb{I}$ be the functor including the objects of $\mathbb{I}$ as a discrete category in $\mathbb{I}$. Recall that the induced functor $\mathbb{C}^U : \mathbb{C}^{\mathbb{I}} \to \mathbb{C}^{\mathbb{I}_0}$ has right adjoint given by taking right Kan extensions, and these Kan extensions are computed pointwise when $\mathbb{C}$ admits certain products (for instance when $\mathbb{C}$ admits products of families whose indexing set is bounded by the morphisms of $\mathbb{I}$). Recall also that if $\mathbb{C}$ has pullbacks and the functor $\mathbb{C}^U$ has a right adjoint, then for each C in $\mathbb{C}^{\mathbb{I}}$ the induced functor $\mathbb{C}^U_C : (\mathbb{C}^{\mathbb{I}} \downarrow C) \to (\mathbb{C}^{\mathbb{I}_0} \downarrow CU)$ also has a right adjoint. Note that each functor $\mathbb{C}^U_C$ also has a right adjoint (although the functor $\mathbb{C}^U$ may not) if $\mathbb{C}$ admits wide pullbacks of families whose indexing set is bounded by the morphisms of $\mathbb{I}$. Now suppose that $g : B \to C$ is a morphism in $\mathbb{C}^{\mathbb{I}}$ such that g_X admits an image for each X in $\mathbb{I}$, and $\mathbb{C}^U_C$ has right adjoint R. One easily checks that the image of gU in $\mathbb{C}^{\mathbb{I}_0}$ is computed componentwise and that $(\text{Im}_{\sqsubset^{\mathbb{I}}}(g), \text{im}_{\sqsubset^{\mathbb{I}}}(g)) = R(\text{Im}_{\sqsubset^{\mathbb{I}_0}}(gU), \text{im}_{\sqsubset^{\mathbb{I}_0}}(gU))$.

Recall that a category $\mathbb{C}$ is semi-abelian [12] if it is pointed, exact [1], protomodular [3] and has binary coproducts.

2.3. COROLLARY. *Let $\mathbb{C}$ be a finitely complete category and let $\mathbb{I}$ be a finite category. If $\mathbb{C}$ is*

- *unital and algebraically cartesian closed [4] (see also [7, 8] where this notion is first considered but unnamed), or*

- *semi-abelian (more generally pointed exact protomodular) and action accessible [6], or*

- *admits normalizers [9],*

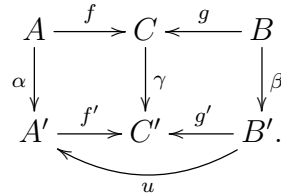*then the same is true of the functor category $\mathbb{C}^{\mathbb{I}}$.*

PROOF. Recall that: (i) a unital category is algebraically cartesian closed if and only if it admits centralizers (in the sense above), Proposition 1.2 of [4]; (ii) a pointed exact protomodular category is action accessible if and only if for each normal monomorphism $n : S \to C$ the normalizer of $\langle n, n \rangle : S \to C \times C$ exists, Theorem 3.1 of [10], (see also [5]). The claim now follows from the previous theorem applied to the pre-cover relations in Examples 1.4 and 1.5. Just note that the conditions of being pointed, unital, exact, and protomodular easily lift to functor categories. ■

We end the paper by giving a simple example showing that images don't always lift to functor categories.

2.4. EXAMPLE. *Let $\mathbb{C}$ be the poset with underlying set with distinct elements $\{A, A', B, B', C, C'\}$ and with partial order generated from*

$$\{(A, A'), (A, C), (B, B'), (B, C), (A', C'), (B', A'), (C, C')\},$$

considered as a category. For convenience, we introduce labels for some of the morphisms as shown in the diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{f} & C & \xleftarrow{g} & B \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} \\
A' & \xrightarrow{f'} & C' & \xleftarrow{g'} & B'.
\end{array}
$$

Now let $\sqsubset = \{(\theta, \phi) \in \mathbb{C}_1^2 \mid \mathrm{cod}(\theta) = \mathrm{cod}(\phi)\} \setminus \{(\gamma, g'), (1_{C'}, g')\}$ where $\mathbb{C}_1$ is the set of morphisms of $\mathbb{C}$, and cod is the codomain map. It is easy to check that $\sqsubset$ is a cover relation. The main point is to note that if (θ, ϕ) is in $\sqsubset$, $h\phi = g'$ and $h \neq 1_{C'}$, then either $h = f'$ and $\phi = u$, or $h = g'$ and $\phi = 1_{B'}$ and hence in either case $h\theta = f'\sigma$ for some σ. It straightforward to check that the $\sqsubset$-image of a morphism ϕ is $(\mathrm{cod}(\phi), 1_{\mathrm{cod}\phi})$, unless $\phi = g'$ in which case its $\sqsubset$-image is (A', f'). However, the morphism $(g, g') : (B, B', \beta) \to (C, C', \gamma)$ in $\mathbb{C}^2$, the category of morphisms of $\mathbb{C}$, has no image with respect to $\sqsubset^2$. To see why, just note that the full subcategory of $(\mathbb{C}^2 \downarrow (C, C', \gamma))$, with objects all objects $((X, X', \chi), (p, p'))$ in $(\mathbb{C}^2 \downarrow (C, C', \gamma))$ such that $((p, p'), (g, g')) \in \sqsubset^2$, considered as poset, has two maximal elements $((A, A', \alpha), (f, f'))$ and $((B, A', u\beta), (g, f'))$ (and hence no largest element).

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