

ON INFINITE MATROIDS WITH STRONG MAPS: PROTO-EXACTNESS AND FINITENESS CONDITIONS

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ABSTRACT. This paper investigates infinite matroids from a categorical perspective. We prove that the category of infinite matroids is a proto-exact category in the sense of Dyckerhoff and Kapranov, thereby generalizing our previous result on the category of finite matroids. We characterize finitary matroids as co-limits of finite matroids and show that the finitely presentable objects in this category are the finite matroids.

1. Introduction

Matroids combinatorially generalize properties of linear dependence in a finite-dimensional vector space. Finite matroids naturally appear in various fields in mathematics. Meanwhile, theories of infinite matroids have long sought to generalize finite matroids to infinite ground sets while retaining nice combinatorial properties as in the finite case, especially the multitude of equivalent definitions, i.e., cryptomorphisms, they enjoy. Our work uses the definition of infinite matroids from [Bruhn et al, 2013], which successfully generalizes five of the most common axiomatizations from the finite case.

Recent years have seen growing interest in categorical aspects of matroids, e.g., see [Heunen and Patta, 2018] and [Eur and Huh, 2020]. In a previous paper with M. Szczesny, we studied the category of finite matroids with strong maps in connection with combinatorial Hopf algebras, and initiated the study of algebraic K-theory for finite matroids. Our main observation was that the category of finite matroids has the structure of a finitary proto-exact category in the sense of [Dyckerhoff and Kapranov, 2019].

In this paper we define a category \mathbf{Mat}_\bullet of infinite matroids which contains the category of finite matroids and strong maps as a full subcategory. Theorem 3.9 shows that \mathbf{Mat}_\bullet has the structure of a proto-exact category, generalizing [Eppolito, Jun, and Szczesny, 2020, Theorem 5.11]. Corollary 4.5 characterizes finitary matroids as the co-limits of finite matroids. We obtain Corollary 4.7, yielding a “finitization” functor $\mathbf{Mat}_\bullet \rightarrow \mathbf{Mat}_\bullet^{\text{fin}}$. Proposition 4.9 shows that the finitely presentable objects of \mathbf{Mat}_\bullet are precisely the finite matroids. We end with a short discussion of some open problems.

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2. Preliminaries

We denote the powerset of E by $\text{pow}(E)$ and the set of non-negative integers by \mathbb{N} .

2.1. INFINITE MATROIDS. This section summarizes [Bruhn et al, 2013, Sections 1.3 and 1.4]. A collection $\mathcal{A} \subseteq \text{pow}(E)$ has *Property (M)* when for all $A \in \mathcal{A}$ and all $S \subseteq E$ the subcollection $\mathcal{A}(A, S) = \{X \in \mathcal{A} : A \subseteq X \subseteq S\}$ has a maximal element.

A function $\text{cl} : \text{pow}(E) \rightarrow \text{pow}(E)$ is a *matroid closure* on E when:

(CLO) cl is a closure operator, i.e., extensive, monotone, and idempotent on $\text{pow}(E)$.

(CLE) For all $Z \subseteq E$ and all $x, y \in E$, if $y \in \text{cl}(Z \cup x) \setminus \text{cl}(Z)$, then $x \in \text{cl}(Z \cup y)$.

(CLM) The collection $\mathcal{I}_{\text{cl}} := \{I \subseteq E : x \notin \text{cl}(I \setminus x) \text{ for all } x \in I\}$ has Property (M).

A collection $\mathcal{C} \subseteq \text{pow}(E)$ is the set of *circuits* of a matroid on E when:

(C0) $\emptyset \notin \mathcal{C}$.

(CI) Elements of \mathcal{C} are pairwise incomparable with respect to inclusion.

(CE) For all $X \subseteq C \in \mathcal{C}$, all families $\{C_x \in \mathcal{C} : x \in X\}$ satisfying $x \in C_y$ iff $x = y$, and for all $z \in C \setminus \bigcup_x C_x$, there is a $C' \in \mathcal{C}$ such that $z \in C' \subseteq (C \cup \bigcup_x C_x) \setminus X$.

(CM) The collection $\mathcal{I}_{\mathcal{C}} := \{I \subseteq E : C \not\subseteq I \text{ for all } C \in \mathcal{C}\}$ has Property (M).

These are “cryptomorphic” descriptions of a matroid on E by [Bruhn et al, 2013]. These axiomatizations agree with standard definitions of finite matroids; see [White, 1986, Appendix], noting that (CLM) and (CM) are redundant in the finite case.

A *loop* of M is an element of $\text{cl}_M(\emptyset)$ (equivalently an element $e \in E$ with $\{e\}$ a circuit of M). A *pointed matroid* is a matroid with a distinguished loop $*$. We often denote the ground set of a (pointed) matroid M by $E(M)$, or simply E when the matroid is clear.

2.2. EXAMPLE. The set $\mathcal{C}_r(E)$ of r -subsets of E is the set of circuits of $U_r(E)$, the *uniform matroid of rank r on E* , for all positive integers r . Similarly, the set $\mathcal{C}_r^*(E)$ of subsets of E with complement an r -subset of E is the set of circuits of $U_r^*(E)$, the *uniform matroid of corank r on E* , for all positive integers r .

By *matroid*, we mean *possibly infinite pointed matroid* unless otherwise stated. We explicitly state when ground sets are finite, and all of our matroids come with a distinguished loop. We often suppress the word “pointed” in our terminology.

Let M be a matroid with circuit set \mathcal{C} and let $S \subseteq E$. The *restriction* of M to S is the matroid $M|S$ on ground set S with circuit set $\mathcal{C}|S = \{C \in \mathcal{C} : C \subseteq S\}$. The *contraction* of M by S is the matroid M/S on ground set $E \setminus S$ with circuits the nonempty, inclusion-wise minimal elements of $\mathcal{C}/S = \{C \setminus S : C \in \mathcal{C}\}$.

When working with pointed matroids, we consider only pointed restrictions and pointed contractions, i.e. restrictions and contractions which preserve the distinguished loop. Thus pointed restrictions and contractions have the forms $M|(S \cup *)$ and $M/(S \setminus *)$ respectively.

2.3. **PROTO-EXACT CATEGORIES.** A *proto-exact* category is a pointed category \mathcal{C} equipped with two distinguished classes of morphisms, *admissible monomorphisms* \mathfrak{M} and *admissible epimorphisms* \mathfrak{E} , which satisfy the following conditions:

1. Every morphism $0 \rightarrow M$ is in \mathfrak{M} , and every morphism $M \rightarrow 0$ is in \mathfrak{E} .
2. The classes \mathfrak{M} and \mathfrak{E} are closed under composition and contain all isomorphisms.
3. A commutative square (1) in \mathcal{C} with $i, i' \in \mathfrak{M}$ and $j, j' \in \mathfrak{E}$ is Cartesian if and only if it is co-Cartesian.

$$\begin{array}{ccc} M & \xrightarrow{i} & N \\ j \downarrow & & \downarrow j' \\ M' & \xrightarrow{i'} & N' \end{array} \tag{1}$$

4. Every diagram $M' \xrightarrow{i'} N' \xleftarrow{j'} N$ with $i' \in \mathfrak{M}$ and $j' \in \mathfrak{E}$ can be completed to a bi-Cartesian square (1) with $i \in \mathfrak{M}$ and $j \in \mathfrak{E}$.
5. Every diagram $M' \xleftarrow{j} M \xrightarrow{i} N$ with $i \in \mathfrak{M}$ and $j \in \mathfrak{E}$ can be completed to a bi-Cartesian square (1) with $i' \in \mathfrak{M}$ and $j' \in \mathfrak{E}$.

See [Eppolito, Jun, and Szczesny, 2020] for examples and motivation regarding proto-exact categories. For proto-exact categories in connection with algebraic geometry, see [Eberhardt, Lorscheid, and Young, 2020] and [Jun and Szczesny, 20].

3. Proto Exactness of \mathbf{Mat}_\bullet and $\mathbf{Mat}_\bullet^{\text{fin}}$

This section proves that \mathbf{Mat}_\bullet is proto-exact (Theorem 3.9).

Let E be a set. For $X \subseteq E$ and $e \in E$, let $X \cup e$ denote the union $X \cup \{e\}$. The proof of the following lemma is straightforward, mimicking the proof in the finite case.

3.1. **LEMMA.** *Let M be a matroid on E . Then*

$$\text{cl}_M(X) = X \cup \{e \in E : \text{there is a circuit satisfying } e \in C \subseteq X \cup e\}.$$

We say $X \subseteq E$ is a *flat* of M when $\text{cl}_M(X) = X$.

3.2. **LEMMA.** *The set $\mathcal{L}(M)$ of flats of a matroid M under inclusion is a complete, atomic lattice. Meet and join are given by $X \wedge Y = X \cap Y$ and $X \vee Y = \text{cl}_M(X \cup Y)$ respectively.*

PROOF. Let $X, Y \in \mathcal{L}(M)$. Notice $X, Y \subseteq \text{cl}_M(X \cup Y)$; if $X, Y \leq Z$, then $X \cup Y \subseteq Z$ yields $\text{cl}_M(X \cup Y) \subseteq \text{cl}_M(Z) = Z$. Hence $X \vee Y = \text{cl}_M(X \cup Y)$ and $\mathcal{L}(M)$ is a join semilattice. If $Z \leq X, Y$, then $Z \subseteq X \cap Y \subseteq \text{cl}_M(X \cap Y)$; thus $X \wedge Y \subseteq \text{cl}_M(X \cap Y)$. Assuming to the contrary that $e \in \text{cl}_M(X \cap Y) \setminus (X \cap Y)$, there is a circuit satisfying $e \in C \subseteq X \cap Y \cup e$. Thus $e \in \text{cl}_M(X) \cap \text{cl}_M(Y) = X \cap Y$, contradicting our assumption. Completeness follows noting that $\mathcal{L}(M)$ is the collection of closed sets of a closure operator. ■

3.3. LEMMA. *Let M be a matroid. For all $S \subseteq E(M)$ we have both*

$$\mathcal{L}(M/S) = \{F \setminus S : S \subseteq F \in \mathcal{L}(M)\} \quad \text{and} \quad \mathcal{L}(M|S) = \{F \cap S : F \in \mathcal{L}(M)\}.$$

The corresponding matroid closure operators are given by

$$\text{cl}_{M/S}(T) = \text{cl}_M(T \cup S) \setminus S \quad \text{and} \quad \text{cl}_{M|S}(T) = \text{cl}_M(T) \cap S.$$

PROOF. The full claim follows from our description of the closure operator. We denote the collection of minimal nonempty sets in a family \mathcal{F} by $\min(\mathcal{F})$.

We compute the closure in the contraction for all $T \subseteq E(M) \setminus S$ via

$$\begin{aligned} \text{cl}_{M/S}(T) &= T \cup \{e \in E(M) \setminus S : \exists C \in \mathcal{C}(M/S) \text{ with } e \in C \subseteq T \cup e\} \\ &= T \cup \{e \in E(M) \setminus S : \exists C \in \min(D \setminus S : D \in \mathcal{C}(M)) \text{ with } e \in C \subseteq T \cup e\} \\ &= T \cup \{e \in E(M) \setminus S : \exists C \in \mathcal{C}(M) \text{ with } e \in (C \setminus S) \subseteq T \cup e\} \\ &= T \cup \{e \in E(M) \setminus S : \exists C \in \mathcal{C}(M) \text{ with } e \in C \subseteq T \cup S \cup e\} \\ &= T \cup (\text{cl}_M(T \cup S) \setminus S) \\ &= \text{cl}_M(T \cup S) \setminus S. \end{aligned}$$

We compute the closure in the restriction for all $T \subseteq S$ via

$$\begin{aligned} \text{cl}_{M|S}(T) &= T \cup \{e \in S : \exists C \in \mathcal{C}(M|S) \text{ with } e \in C \subseteq T \cup e\} \\ &= T \cup \{e \in S : \exists C \in \mathcal{C}(M) \text{ with } C \subseteq S \text{ and } e \in C \subseteq T \cup e\} \\ &= T \cup \{e \in S : \exists C \in \mathcal{C}(M) \text{ with } e \in C \subseteq T \cup e\} \\ &= T \cup (\text{cl}_M(T) \cap S) \\ &= \text{cl}_M(T) \cap S. \end{aligned}$$

The claimed formulas for $\mathcal{L}(M|S)$ and $\mathcal{L}(M/S)$ now follow. ■

We thus obtain the usual result that the deletion and contraction operations commute. We state this result below in terms of restriction and contraction (the proof is routine).

3.4. COROLLARY. *If $S, T \subseteq E$ are disjoint, then $(M|(S \cup T))/T = (M/T)|S$.*

Next we discuss the maps of our category.

3.5. PROPOSITION. *Let M and N be matroids and $f: E(M) \rightarrow E(N)$ be a map of ground sets. The following are equivalent.*

1. *For all $A \subseteq E(M)$ one has $f(\text{cl}_M(A)) \subseteq \text{cl}_N(f(A))$.*
2. *The preimage of every flat of N is a flat of M .*
3. *The map $f^\#: \mathcal{L}(M) \rightarrow \mathcal{L}(N)$ induced by f is a morphism of complete lattices which restricts to a map $A_M \rightarrow A_N \cup \{0\}$ of atoms.*

PROOF. This follows from [White, 1986, Proof of Proposition 8.1.3], noting completeness of the corresponding lattices for (3) \implies (1). ■

3.6. DEFINITION. A strong map of matroids M and N is a map $f: E(M) \rightarrow E(N)$ of sets satisfying any of the conditions of Proposition 3.5.

Let \mathbf{Mat}_\bullet be the category of pointed matroids with pointed strong maps. We obtain the following from Lemma 3.3 and Proposition 3.5.

3.7. COROLLARY. Restriction $M|S \xrightarrow{is} M$ and contraction $M \xrightarrow{c_S} M/S$ are strong maps.

The map c_S from Corollary 3.7 is a well-defined function because we consider *pointed* matroids. Indeed, this is our main motivation for using this convention. The result below follows from the same proof as [Heunen and Patta, 2018, Corollary 3.8].

3.8. LEMMA. Let $\mathbf{F}: \mathbf{Mat}_\bullet \rightarrow \mathbf{Set}_\bullet$ be the forgetful functor sending a pointed matroid $(M, *)$ to its pointed ground set $(E, *)$. Let $M \xrightarrow{f} N$ be a pointed strong map.

1. f is monic in \mathbf{Mat}_\bullet if and only if $\mathbf{F}(f)$ is monic in \mathbf{Set}_\bullet .
2. f is epic in \mathbf{Mat}_\bullet if and only if $\mathbf{F}(f)$ is an epic in \mathbf{Set}_\bullet .

Let \mathfrak{M} be the class of pointed strong maps which factor as $N \xrightarrow{\sim} M|S \xrightarrow{is} M$, and let \mathfrak{E} consist of all pointed strong maps which factor as $M \xrightarrow{c_S} M/S \xrightarrow{\sim} N$.

At this point, we have extended the following properties to infinite matroids.

1. (Lemma 3.3) For $T \subseteq S \subseteq E$, one has

$$\mathcal{L}((M|S)/T) = \{(F \cap S) \setminus T : T \subseteq F \in \mathcal{L}(M)\}.$$

2. (Corollary 3.4) For disjoint $S, T \subseteq E$, one has

$$(M|(S \cup T))/T = (M/T)|S.$$

3. (Corollary 3.7) The restriction and contraction maps are strong maps.
4. (Lemma 3.8) A strong map is monic (resp. epic) in \mathbf{Mat}_\bullet if and only if it is injective (resp. surjective) on ground sets.

As these were the only properties we appealed to in our proof for the finite case [Eppolito, Jun, and Szczesny, 2020, Section 5], that same proof now yields the following.

3.9. THEOREM. The triple $(\mathbf{Mat}_\bullet, \mathfrak{M}, \mathfrak{E})$ is a proto-exact category.

4. Finiteness Conditions

The propositions in this section illustrate the utility of applying categorical language to study infinite matroids. We focus primarily on two finiteness conditions for matroids in this analysis. The first condition (i.e. finitariness) is a well-studied condition in the literature on infinite matroids—indeed, this was one of the earliest definitions for infinite matroids. The second condition (i.e. finite presentability) is a well-studied categorical notion, which has not been considered in the context of infinite matroids to the best of our knowledge.

4.1. FINITARY MATROIDS. A matroid is *finitary* when all of its circuits are finite. As restrictions and contractions of finitary matroids are finitary, we obtain the following.

4.2. COROLLARY. *The full subcategory $\mathbf{Mat}_{\bullet}^{\text{fin}}$ of \mathbf{Mat}_{\bullet} with objects the finitary matroids is proto-exact with the induced proto-exact structure.*

The following is the core result of this section, and drives the remainder of our results.

4.3. PROPOSITION. *The collection \mathcal{C}^{fin} of finite circuits of matroid M is the set of circuits of a matroid M^{fin} . Moreover, the identity map on $E(M)$ is a strong map $M^{\text{fin}} \rightarrow M$.*

PROOF. The circuit elimination axiom yields that eliminations between finite circuits are again finite, and the directed union of every chain of \mathcal{C}^{fin} -independent sets is trivially \mathcal{C}^{fin} -independent; thus \mathcal{C}^{fin} is a set of matroid circuits. Hence we compute

$$\begin{aligned} \text{cl}_{\text{fin}}(S) &= S \cup \{e \in E(M) : \exists C \in \mathcal{C}(M) \text{ finite with } e \in C \subseteq S \cup e\} \\ &\subseteq S \cup \{e \in E(M) : \exists C \in \mathcal{C}(M) \text{ with } e \in C \subseteq S \cup e\} \\ &= \text{cl}_M(S), \end{aligned}$$

which verifies condition 1 of Proposition 3.5. ■

4.4. PROPOSITION. *The matroid M^{fin} is the co-limit of the diagram below in \mathbf{Mat}_{\bullet} :*

$$\{M|S \hookrightarrow M|T : S \subseteq T \text{ finite}\}.$$

PROOF. We prove that M^{fin} satisfies the universal property of co-limits. Let N be a matroid and $M|S \xrightarrow{\alpha_S} N$ be a strong map for each $S \subseteq E(M)$ such that $\alpha_T|_S = \alpha_S$ for all pairs $S \subseteq T$ of finite sets. Applying the forgetful functor $\mathbf{F}: \mathbf{Mat}_{\bullet} \rightarrow \mathbf{Set}_{\bullet}$, we obtain a unique co-limit map $E(M) \xrightarrow{\alpha} E(N)$ of pointed sets. We claim that α is a strong map $M^{\text{fin}} \xrightarrow{\alpha} N$. Let $A \subseteq E(M)$ and $e \in \text{cl}_M^{\text{fin}}(A) \setminus A$ be arbitrary. There is a finite circuit C of M with $e \in C \subseteq A \cup e$ by Lemma 3.1. Hence

$$\alpha(e) \in \alpha(C) \subseteq \alpha(\text{cl}_{M|C}(C)) \subseteq \text{cl}_N(\alpha(C))$$

yields that α is a strong map by Proposition 3.5. ■

4.5. COROLLARY. *A matroid is finitary if and only if it is a co-limit of finite matroids.*

PROOF. If M is finitary, then $M = M^{\text{fin}}$ is a co-limit of finite matroids by Proposition 4.4. Conversely, suppose that

$$M = \text{colim} \left\{ M_i \xrightarrow{f_{i,j}} M_j : i, j \in I \right\}$$

is a co-limit of finite matroids with strong maps $M_i \xrightarrow{g_i} M$. Note that $\text{cl}(A) = \text{cl}_{\text{fin}}(A)$ for all finite $A \subseteq E(M)$ by Lemma 3.1. Thus the set maps g_i are strong maps for matroids $M_i \rightarrow M^{\text{fin}}$. This induces a strong map $M \rightarrow M^{\text{fin}}$, which is necessarily the identity as a set map. Hence $M = M^{\text{fin}}$ by Proposition 4.3. ■

We obtain the following as a special case of Corollary 4.5.

4.6. COROLLARY. *A matroid on a countable set is finitary if and only if it is a direct limit of finite matroids.*

PROOF. If M is finitary on a countable ground set, then identifying the elements of M with \mathbb{N} it is easy to show that M is the co-limit of $M|_{\{0, 1, \dots, n\}} \hookrightarrow M|_{\{0, 1, \dots, n+1\}}$ where 0 represents the distinguished loop. The reverse implication follows trivially from Corollary 4.5 and the fact that a direct limit of finite sets is countable. ■

4.7. COROLLARY. *Every strong map $M \xrightarrow{f} N$ induces a strong map $M^{\text{fin}} \xrightarrow{f^{\text{fin}}} N^{\text{fin}}$. In particular, $\bullet^{\text{fin}}: \mathbf{Mat}_{\bullet} \rightarrow \mathbf{Mat}_{\bullet}^{\text{fin}}$ constitutes a full functor.*

PROOF. The map f yields strong maps $M|_S \rightarrow N|_fS \rightarrow N^{\text{fin}}$ for all $S \subseteq E(M)$. Hence we obtain a strong map $M^{\text{fin}} \rightarrow N^{\text{fin}}$ by Proposition 4.4. The second statement is clear from Proposition 4.3; indeed \bullet^{fin} is a left inverse of the inclusion functor $\mathbf{Mat}_{\bullet}^{\text{fin}} \hookrightarrow \mathbf{Mat}_{\bullet}$. ■

4.8. FINITE PRESENTABILITY. An object X in a category \mathcal{C} is *finitely presented* when the functor $\text{Hom}_{\mathcal{C}}(X, -)$ preserves filtered co-limits.

The functor $\mathbf{G}: \mathbf{Set}_{\bullet} \rightarrow \mathbf{Mat}_{\bullet}$ sending a pointed set E to the free (pointed) matroid on E has a left adjoint, namely the forgetful functor $\mathbf{F}: \mathbf{Mat}_{\bullet} \rightarrow \mathbf{Set}_{\bullet}$. In particular, there is a natural bijection between the sets $\text{Hom}_{\mathbf{Mat}_{\bullet}}(\mathbf{G}E, N)$ and $\text{Hom}_{\mathbf{Set}_{\bullet}}(E, \mathbf{F}N)$; indeed, the natural isomorphism is the identity map. On the other hand, a set X is finite if and only if $\text{Hom}(X, -)$ preserves filtered co-limits, so $\text{Hom}(\mathbf{G}E, -)$ preserves filtered co-limits if and only if E is finite. Hence the free matroid on E is finitely presented if and only if E is finite. In fact, this argument generalizes as follows.

4.9. PROPOSITION. *A matroid is finitely presented if and only if it is finite.*

PROOF. Let M be a matroid. Assuming M is finitely presented, we note by Proposition 4.3 that morphisms $M \rightarrow N$ induce $M^{\text{fin}} \rightarrow N^{\text{fin}}$. As M^{fin} is a co-limit of its finite restrictions by Proposition 4.4, we have $\text{colim}_S \text{Hom}(M, M|_S) \cong \text{Hom}(M, \text{colim}_S M|_S)$. Thus there is an element of the co-limit $f = \text{colim}_S (f_S: M \rightarrow M_S)$ and $i_S: M|_S \rightarrow M$ satisfying $i_S \circ f = \text{id}_{M^{\text{fin}}}$. As a set map, this is the identity; thus i_S is surjective, yielding M is finite.

Conversely, assuming M is finite and $\text{colim}_i N_i$ is filtered, then note $f: M \rightarrow \text{colim}_i N_i$ has image contained in the ground set of N_j for some $j \in I$; thus $f \in \text{Hom}(M, N_j)$. As $\text{colim} \text{Hom}(M, N_i) = \bigcup_i \text{Hom}(M, N_i)$ we have that the identity function is the desired bijection $\text{colim}_i \text{Hom}(M, N_i) \cong \text{Hom}(M, \text{colim} N_i)$. ■

Thus not all finitary matroids are finitely presented in \mathbf{Mat}_{\bullet} or $\mathbf{Mat}_{\bullet}^{\text{fin}}$.

4.10. REMARK. We initially hoped that the category of finitary matroids might be locally finitely presentable, but this fails spectacularly. The full subcategory of finite matroids is neither complete nor co-complete as it has neither products nor co-equalizers by [Heunen and Patta, 2018, Propositions 3.5 and 3.7].

5. Future Directions

Our initial goal for this project was to use categorical language to introduce a Tutte-Grothendieck invariant for infinite matroids. We soon discovered that this is impossible without serious modification.¹ Such an invariant should be determined by homomorphisms from some ring, as in the finite case with the general framework of [Brylawski, 1972], and should satisfy at least the following conditions.

1. There is a generator for each matroid isomorphism class.
2. For all matroids M and all elements e of M which is neither a loop nor a coloop we have $[M] = [M \setminus e] + [M/e]$.

Now letting E be any infinite set, consider $U_r(E)$, the finitary uniform matroid of rank $r \geq 1$ on E as in Example 2.2. For each $e \in E$ we compute

$$\begin{aligned} [U_r(E)] &= [U_r(E) \setminus e] + [U_r(E)/e] \\ &= [U_r(E \setminus e)] + [U_{r-1}(E \setminus e)] \\ &= [U_r(E)] + [U_{r-1}(E)]. \end{aligned}$$

Hence if our invariant is a ring satisfying both (1) and (2), then $[U_r(E)] = 0$ for all $r \geq 0$ and all infinite sets E . In particular, no *ring* can be a universal isomorphism invariant of infinite matroids which respects single-element deletion-contraction relations and evaluates nontrivially on all matroids. However, foregoing additive inverses we could instead seek a universal object satisfying (1) and (2) in the spirit of *semirings*.

5.1. **PROBLEM.** Develop a meaningful generalization of the Tutte-Grothendieck theory of finite matroids for infinite matroids using the ideas above. Can this theory be practically applied to solve other problems on infinite matroids?

5.2. **K -THEORY FOR INFINITE MATROIDS.** The categories \mathbf{Mat}_\bullet and $\mathbf{Mat}_\bullet^{\text{fin}}$ are proto-exact by Theorem 3.9 and Corollary 4.2, so we have a K -theory using the Waldhausen construction. This suggests the following problem.

5.3. **PROBLEM.** Compute the K -theory of \mathbf{Mat}_\bullet and/or $\mathbf{Mat}_\bullet^{\text{fin}}$.

We directly computed the Grothendieck group of the category of finite matroids in [Eppolito, Jun, and Szczesny, 2020, Theorem 6.3]. Our computation relies heavily on finitariness of the category of finite matroids; we showed that the class of a finite M can be identified with $(r, \#E(M) - r)$ where r is the rank of M . Neither \mathbf{Mat}_\bullet nor $\mathbf{Mat}_\bullet^{\text{fin}}$ is a finitary category, and the following *a priori* simpler problem seems like the place to start.

5.4. **PROBLEM.** Compute the class of $U_r(E)$ in $K_0(\mathbf{Mat}_\bullet^{\text{fin}})$ for all $r \geq 0$ and all sets E .

¹Our example was also discovered independently by R. Pendavingh (private communication).

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