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ERRATUM TO "EXACT SEQUENCES IN THE ENCHILADA CATEGORY"

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ABSTRACT. In this note we correct two propositions from the paper "Exact sequences in the enchilada category". Moreover, we present our further investigation on monomorphisms and epimorphisms in the enchilada category.

1. Introduction

[1] concerns the *enchilada category* of C^*-algebras, in which morphisms are isomorphism classes of (nondegenerate) C^*-correspondences. Propositions 3.6 and 4.6 in that paper state characterizations of split monomorphisms and split epimorphisms as Hilbert bimodules that are full on the left or right, respectively. However, we subsequently realized that there are split monomorphisms and split epimorphisms that do not have a bimodule structure. In this erratum we present examples of such morphisms. There is no real harm done, because the main results of [1] do not depend upon the incorrect characterizations of split mono- or epimorphisms.

2. Split Monomorphisms and Split Epimorphisms

[1, Proposition 3.6] states that a morphism $[{}_AX_B]$ is a split monomorphism in the enchilada category if and only if it is a left full Hilbert bimodule. However, here we show that there exists a split monomorphism $[{}_AX_B]$ in the enchilada category such that ${}_AX_B$ does not have a Hilbert bimodule structure.

2.1. PROPOSITION. *The Enchilada category has a split monomorphism that is not the isomorphism class of a Hilbert bimodule. However, if the isomorphism class of a Hilbert bimodule ${}_AX_B$ is a split monomorphism, then ${}_AX_B$ has to be left full.*

PROOF. Let X be the injective $\mathbb{C}-\mathbb{C}^2$ correspondence associated to the homomorphism $a \mapsto (a,a)$, and let Y be the $\mathbb{C}^2-\mathbb{C}$ correspondence associated the homomorphism $(a,b) \mapsto a$. Then, we have

$$_{\mathbb{C}}(X \otimes_{\mathbb{C}^2} Y)_{\mathbb{C}} \cong {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}}.$$

However, the correspondence ${}_{\mathbb{C}}(X)_{\mathbb{C}^2}$ is not a Hilbert bimodule.

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To prove the second half of the proposition, let ${}_AX_B$ be a Hilbert bimodule such that $[{}_AX_B]$ is a split monomorphism in the enchilada category. Then, there exists a correspondence ${}_BY_A$ such that

$$ {}_AX_B \otimes_B {}_BY_A \cong {}_AA_A. $$

Then the C^*-correspondence ${}_A(X \otimes_B Y)_A$ must be right full, i.e,

$$ \langle Y, \langle X, X\rangle_B \cdot Y\rangle_A = A. $$

Let $Z = B_XY$. Note that

$$ X \otimes_B Y = X \otimes_{B_X} Y = X \otimes_{B_X} Z $$

as $A - A$ correspondences. Thus

$$ \begin{aligned} \widetilde{X} \otimes_A X \otimes_B Y &= \widetilde{X} \otimes_A X \otimes_{B_X} Y \\ &\cong B_X \otimes_{B_X} Y \\ &\cong B_XY \\ &= Z \end{aligned} $$

as $B - A$ correspondences.

Thus

$$ \begin{aligned} A_X &\cong A_X \otimes_A A \\ &\cong X \otimes_B \widetilde{X} \otimes_A X \otimes_B Y \\ &\cong X \otimes_B Z \\ &= X \otimes_B Y \\ &\cong A \end{aligned} $$

as $A - A$ correspondences. Therefore the ideal A_X must be all of A. ∎

Proposition 2.1 has a dual counterpart:

2.2. PROPOSITION. *The Enchilada category has a split epimorphism that is not the isomorphism class of a Hilbert bimodule.*

PROOF. Let $\mathcal{K} = \mathcal{K}(H)$ for an infinite-dimensional Hilbert space H, and let $\widetilde{\mathcal{K}}$ be the (minimal) unitization, so in this case $\mathcal{K} + \mathbb{C}1_H$. Then we have a short exact sequence of C^*-algebras:

$$ 0 \longrightarrow \mathcal{K} \hookrightarrow \widetilde{\mathcal{K}} \xrightarrow{q} \mathbb{C} \longrightarrow 0. $$

Let X be the $\widetilde{\mathcal{K}} - \mathbb{C}$ correspondence given by the quotient map q in the usual way. We first show that X is a split epimorphism. Let Y be the $\mathbb{C} - \widetilde{\mathcal{K}}$ correspondence given by the inclusion map of the nondegenerate C^*-subalgebra $\mathbb{C}$ of $\widetilde{\mathcal{K}}$. Let

$$ \Phi : Y \odot X \to \mathbb{C} $$

be the unique linear map associated with the bilinear map

$$(y, \lambda) \mapsto q(y)\lambda.$$

If we can verify that Φ preserves inner products and left $\mathbb{C}$-module actions, then we will be able to conclude that it canonically determines an isomorphism $Y \otimes_{\widetilde{\mathcal{K}}} X \simeq \mathbb{C}$ of $\mathbb{C} - \mathbb{C}$ correspondences:

$$\begin{aligned}
\langle \Phi(y \otimes \lambda), \Phi(z \otimes \mu) \rangle_{\mathbb{C}} &= \langle q(y)\lambda, q(z)\mu \rangle_{\mathbb{C}} \\
&= \bar{\lambda}\, \overline{q(y)} q(z)\mu \\
&= \bar{\lambda} q(y^* z)\mu \\
&= \langle \lambda, \langle y, z \rangle_{\widetilde{\mathcal{K}}} \cdot \mu \rangle_{\mathbb{C}} \\
&= \langle y \otimes \lambda, z \otimes \mu \rangle_{\mathbb{C}},
\end{aligned}$$

and of course Φ preserves left $\mathbb{C}$-module actions because q is linear.

On the other hand, to see that X is not a Hilbert bimodule notice that $\widetilde{\mathcal{K}}$ does not have an ideal isomorphic to $\mathbb{C}$, because the only nontrivial proper ideal is $\mathcal{K}$. (Note that as a Hilbert module, X is just the standard one determined by the C^*-algebra $\mathbb{C}$, and so $\mathcal{K}(X) = \mathbb{C}$.) ∎

[1, Proposition 3.6] states that in the enchilada category every monomorphism is injective. We now show that the converse is not true in general:

2.3. Proposition. *There exists an injective C^*-correspondence that is not a monomorphism in the enchilada category.*

Proof. Let H be an infinite dimensional Hilbert space. Then we may view H as an injective C^*-correspondence over $\mathbb{C}$. Consider the usual Hilbert spaces $\mathbb{C}$ and $\mathbb{C}^2$. We have the isomorphism

$$\mathbb{C} \otimes_{\mathbb{C}} H \cong \mathbb{C}^2 \otimes_{\mathbb{C}} H,$$

but $\mathbb{C}$ and $\mathbb{C}^2$ are not isomorphic as $\mathbb{C} - \mathbb{C}$ correspondences. ∎

References

[1] M. Eryüzlü, S. Kaliszewski, and J. Quigg. *Exact sequences in the enchilada category*, **35** (2020), no. 12, 350–370.

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