HOMOTOPY EQUIVALENT ALGEBRAIC STRUCTURES IN MULTICATEGORIES AND PERMUTATIVE CATEGORIES

NILES JOHNSON AND DONALD YAU

ABSTRACT. We show that the free construction from multicategories to permutative categories is a categorically-enriched non-symmetric multifunctor. Our main result then shows that the induced functor between categories of algebras is an equivalence of homotopy theories. We describe an application to ring categories.

Contents

1. Introduction

The endomorphism construction provides a Cat-enriched multifunctor

$$\mathsf{End} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Multicat}$$

where $\mathsf{PermCat}^{\mathsf{su}}$ is the Cat-enriched multicategory of small permutative categories, multilinear functors, and multilinear transformations, and $\mathsf{Multicat}$ is the Cat-enriched multicategory of small multicategories, multifunctors, and multinatural transformations. Thus it induces a functor on categories of algebras

$$\mathsf{End}^{\mathcal{O}} : (\mathsf{PermCat}^{\mathsf{su}})^{\mathcal{O}} \longrightarrow \mathsf{Multicat}^{\mathcal{O}}$$

for any small Cat-enriched multicategory $\mathcal{O}$.

Regarded as a 2-functor of underlying 2-categories, and restricting to the sub-2-category PermCat$^{\mathsf{st}}$ of strict symmetric monoidal functors, End has a left 2-adjoint **F** described in [EM09, Theorem 4.2] and [JY22, Section 5]. With respect to the stable equivalences created by Segal's K-theory functor [Seg74], the adjoint pair (**F**, End) is shown to be an equivalence of homotopy theories in [JY22, Theorem 7.3].

In this article we show that **F** extends to a non-symmetric Cat-enriched multifunctor taking values in PermCat$^{\mathsf{su}}$. This extension, also denoted **F**, fails to be adjoint to End; see Remark 10.4 for further discussion of this detail.

Our main result shows, nevertheless, that the induced functors between categories of $\mathcal{O}$-algebras do induce an equivalence of homotopy theories, for a non-symmetric Cat-multicategory $\mathcal{O}$. This is stated as follows.

1.1. THEOREM. *Suppose $\mathcal{O}$ is a small non-symmetric Cat-multicategory. Then **F** and* **E** = End *induce an equivalence of homotopy theories between categories of non-symmetric algebras*

$$\mathbf{F}^{\mathcal{O}} : \big(\mathsf{Multicat}^{\mathcal{O}}, \mathcal{S}\big) \xrightarrow{\;\;\longrightarrow\;\;} \big((\mathsf{PermCat}^{\mathsf{su}})^{\mathcal{O}}, \mathcal{S}\big) : \mathbf{E}^{\mathcal{O}},$$

where, in each category, $\mathcal{S}$ denotes the class of componentwise stable equivalences.

The proof of Theorem 1.1 is given in Section 10. Our main application, Corollary 11.3 focuses on ring categories. One key advantage of Multicat over PermCat$^{\mathsf{su}}$ is that the former has a closed symmetric monoidal structure provided by the Boardman-Vogt tensor product (Definition 4.4). The related smash product for pointed multicategories gives the Cat-multicategory structure on PermCat$^{\mathsf{su}}$, but does not induce a symmetric monoidal structure (see [JY∞, 5.7.23 and 10.2.17]).

OUTLINE. We begin with several sections reviewing relevant concepts. In Section 2 we review equivalences of homotopy theories from [GJO17b]. In Sections 3 through 5 we review enriched multicategories and the Cat-multicategory of permutative categories. The third of these sections recalls the Cat-multifunctor End from [JY∞]. In Section 6 we review the definition of the left adjoint **F** from [EM06, JY22].

The main results of this article are contained in the remaining sections. We show that **F** is a non-symmetric Cat-multifunctor in Sections 7 and 8. In Section 9 we develop the transformations comparing the composites **EF** and **FE** with the respective identity functors. The proof of Theorem 1.1 appears in Section 10, after a review of stable equivalences for multicategories and permutative categories. Section 11 describes the application to ring categories as E_1-algebras in Cat.

2. Equivalences of Homotopy Theories

In this section we review the theory of complete Segal spaces due to Rezk [Rez01]. An equivalence of homotopy theories (Definition 2.7) is an equivalence of fibrant replacements

in the complete Segal space model structure. For further context and development we refer the reader to [DK80, Hir03, Toë05, BK12].

COMPLETE SEGAL SPACES. For the purpose of this paper, we only need to know the existence of the complete Segal space model structure in Theorem 2.3; we will not use the specific definition of a complete Segal space. The next definition is included only for the reader's information. A *bisimplicial set* is a simplicial object in the category of simplicial sets. For a bisimplicial set $X = \{X(n)\}_{n \geq 0}$, each object $X(n)$ is a simplicial set. In the following definition, **2** denotes the nerve, also known as the classifying space, of the category consisting of two isomorphic objects. See [GJ09, Example I.1.4] or [JY∞, Definition 7.2.3] for more discussion of the nerve.

2.1. DEFINITION. A bisimplicial set X is a *complete Segal space* if it satisfies the following three conditions.

1. X is a fibrant object in the Reedy model structure on bisimplicial sets.

2. For each $n \geq 2$ the Segal map

$$X(n) \longrightarrow X(1) \times_{X(0)} \cdots \times_{X(0)} X(1)$$

   is a weak equivalence of simplicial sets.

3. The morphism
$$X(0) \cong \mathsf{Map}(\Delta[0], X) \longrightarrow \mathsf{Map}(\mathbf{2}, X), \tag{1}$$

   which is induced by the unique morphism $\mathbf{2} \longrightarrow \Delta[0]$, is a weak equivalence of simplicial sets.

2.2. REMARK. The definition of complete Segal space given above is equivalent to that given in [Rez01, Section 6] by [Rez01, 6.4].

2.3. THEOREM. [Rez01, 7.2] *There is a simplicial closed model structure on the category of bisimplicial sets, called the* complete Segal space model structure, *that is given as a left Bousfield localization of the Reedy model structure and in which the fibrant objects are precisely the complete Segal spaces.*

RELATIVE CATEGORIES.

2.4. DEFINITION. A *relative category* is a pair $(\mathsf{C}, \mathcal{W})$ consisting of a category C and a subcategory $\mathcal{W}$ containing all of the objects of C. A *relative functor*

$$F : (\mathsf{C}, \mathcal{W}) \longrightarrow (\mathsf{C}', \mathcal{W}')$$

is a functor from C to C' that sends morphisms of $\mathcal{W}$ to those of $\mathcal{W}'$.

2.5. DEFINITION. Suppose $(\mathsf{C}, \mathcal{W})$ is a relative category and A is another category. We let

$$(\mathsf{C}, \mathcal{W})^{\mathsf{A}}$$

denote the subcategory of C^{A} whose objects are functors $\mathsf{A} \longrightarrow \mathsf{C}$ and whose morphisms are those natural transformations with components in $\mathcal{W}$.

2.6. DEFINITION. Suppose $(\mathsf{C}, \mathcal{W})$ is a relative category. The *classification diagram* of $(\mathsf{C}, \mathcal{W})$ is the bisimplicial set

$$\mathsf{N}^{\mathsf{cl}}(\mathsf{C}, \mathcal{W}) = \mathsf{Ner}\left((\mathsf{C}, \mathcal{W})^{\Delta[\bullet]}\right)$$

given by

$$n \longmapsto \mathsf{Ner}\left((\mathsf{C}, \mathcal{W})^{\Delta[n]}\right)$$

where $\Delta[n]$ denotes the category consisting of n composable arrows.

2.7. DEFINITION. Suppose $(\mathsf{C}, \mathcal{W})$ is a relative category. We say that a bisimplicial set $R\,\mathsf{N}^{\mathsf{cl}}(\mathsf{C}, \mathcal{W})$ is a *homotopy theory of* $(\mathsf{C}, \mathcal{W})$ if it is a fibrant replacement of $\mathsf{N}^{\mathsf{cl}}(\mathsf{C}, \mathcal{W})$ in the complete Segal space model structure. We say that a relative functor

$$F : (\mathsf{C}, \mathcal{W}) \longrightarrow (\mathsf{C}', \mathcal{W}')$$

is an *equivalence of homotopy theories* if the induced morphism $R\,\mathsf{N}^{\mathsf{cl}}\,F$ between homotopy theories is a weak equivalence in the complete Segal space model structure.

2.8. REMARK. For readers familiar with the notions of hammock localization and DK-equivalence [DK80], Barwick and Kan have shown in [BK12, 1.8] that a relative functor

$$F : (\mathsf{C}, \mathcal{W}) \longrightarrow (\mathsf{C}', \mathcal{W}')$$

is an equivalence of homotopy theories if and only if it induces a DK-equivalence between hammock localizations. In that case, F induces equivalences between mapping simplicial sets and between categories of components. In particular, if F is an equivalence of homotopy theories then the induced functor between categorical localizations

$$\mathsf{C}[\mathcal{W}^{-1}] \longrightarrow \mathsf{C}'[(\mathcal{W}')^{-1}]$$

is an equivalence.

2.9. PROPOSITION. [GJO17b, 2.8] *Suppose*

$$F : (\mathsf{C}, \mathcal{W}) \longrightarrow (\mathsf{C}', \mathcal{W}')$$

is a relative functor and suppose that F induces a weak equivalence of simplicial sets

$$\mathsf{Ner}\left((\mathsf{C}, \mathcal{W})^{\Delta[n]}\right) \longrightarrow \mathsf{Ner}\left((\mathsf{C}', \mathcal{W}')^{\Delta[n]}\right) \tag{2}$$

for each n. Then F is an equivalence of homotopy theories.

PROOF. The assumption that (2) is a weak equivalence for each n means that

$$\mathsf{N}^{\mathsf{cl}} F : \mathsf{N}^{\mathsf{cl}}(\mathsf{C}, \mathcal{W}) \longrightarrow \mathsf{N}^{\mathsf{cl}}(\mathsf{C}', \mathcal{W}')$$

is a weak equivalence between classification diagrams in the Reedy model structure [Rez01, Section 2.4]. Thus $\mathsf{N}^{\mathsf{cl}} F$ is a weak equivalence in the complete Segal space model structure because it is a localization of the Reedy model structure. As a consequence, $\mathsf{N}^{\mathsf{cl}} F$ induces a weak equivalence between the homotopy theories given by fibrant replacements. ∎

Because a natural transformation between functors induces a simplicial homotopy on nerves, we have the following application of Proposition 2.9. Its proof is similar to that of [JY22, 2.12].

2.10. PROPOSITION. [GJO17b, 2.9] *Suppose given relative functors*

$$F : (\mathsf{C}, \mathcal{W}) \rightleftarrows (\mathsf{C}', \mathcal{W}') : E$$

such that each of the composites EF and FE is connected to the respective identity functor by a zigzag of natural transformations whose components are in $\mathcal{W}$ and $\mathcal{W}'$, respectively. Then F and E are equivalences of homotopy theories.

3. Enriched Multicategories

In this section we review the definitions of enriched multicategories, multifunctors, and multinatural transformations. Detailed discussion of enriched multicategories is in [JY∞, Chapters 5 and 6] and [Yau16].

Our application of interest will be the case $\mathsf{V} = (\mathsf{Cat}, \times, \mathbf{1}, \xi)$ of small categories and functors with the Cartesian product ×. We begin with some notation.

3.1. DEFINITION. Suppose C is a class.

- Denote by
$$\mathsf{Prof}(C) = \coprod_{n \geq 0} C^{\times n}$$

  the class of finite ordered sequences of elements in C. An element in $\mathsf{Prof}(C)$ is called a C-*profile*.

- A typical C-profile of length $n = \mathrm{len}\langle c \rangle$ is denoted by $\langle c \rangle = (c_1, \ldots, c_n) \in C^n$ or by $\langle c_i \rangle_i$ to indicate the indexing variable. The empty C-profile is denoted by $\langle \rangle$.

- We let ⊕ denote the concatenation of profiles, and note that ⊕ is an associative binary operation with unit given by the empty tuple $\langle \rangle$.

- An element in $\mathsf{Prof}(C) \times C$ is denoted as $\big(\langle c \rangle \,; c'\big)$ with $c' \in C$ and $\langle c \rangle \in \mathsf{Prof}(C)$.

3.2. Convention. [Symmetric Monoidal V] Throughout this section we assume that

$$\mathsf{V} = (\mathsf{V}, \otimes, \mathbb{1}, \alpha, \lambda, \rho, \xi)$$

is a symmetric monoidal category. Throughout the rest of this work, unless otherwise specified, we assume that each iterated monoidal product is *left normalized* with the left half of each pair of parentheses at the far left. For example, we denote

$$a \otimes b \otimes c \otimes d = \big((a \otimes b) \otimes c\big) \otimes d.$$

With this convention, we omit most of the parentheses for iterated monoidal products and tacitly insert the necessary associativity and unit isomorphisms. This is valid because there is a strong symmetric monoidal adjoint equivalence $\mathsf{V} \longrightarrow \mathsf{V}_{\mathsf{st}}$ with V_{st} a permutative category [Yau∞I, 1.3.10]. Because strictification is an equivalence, the strict diagrams commute if and only if their preimages in V commute.

3.3. Definition. A V-*multicategory* $(\mathsf{M}, \gamma, 1)$ consists of the following data.

- M is equipped with a class $\mathsf{Ob}\,\mathsf{M}$ of *objects*. We write $\mathsf{Prof}(\mathsf{M})$ for $\mathsf{Prof}(\mathsf{Ob}\,\mathsf{M})$.

- For $c' \in \mathsf{Ob}\,\mathsf{M}$ and $\langle c \rangle = (c_1, \ldots, c_n) \in \mathsf{Prof}(\mathsf{M})$, M is equipped with an object of V

$$\mathsf{M}\big(\langle c \rangle\,;\,c'\big) = \mathsf{M}\big(c_1, \ldots, c_n\,;\,c'\big) \in \mathsf{V}$$

  called the *n-ary operation object* or *multimorphism object* with *input profile* $\langle c \rangle$ and *output* c'.

- For $\big(\langle c \rangle\,;\,c'\big) \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ as above and a permutation $\sigma \in \Sigma_n$, M is equipped with an isomorphism in V

$$\mathsf{M}\big(\langle c \rangle\,;\,c'\big) \xrightarrow[\cong]{\ \sigma\ } \mathsf{M}\big(\langle c \rangle\sigma\,;\,c'\big),$$

  called the *right action* or the *symmetric group action*, in which

$$\langle c \rangle\sigma = \big(c_{\sigma(1)}, \ldots, c_{\sigma(n)}\big)$$

  is the right permutation of $\langle c \rangle$ by σ.

- For $c \in \mathsf{Ob}\,\mathsf{M}$, M is equipped with a morphism

$$1_c : \mathbb{1} \longrightarrow \mathsf{M}\big(c\,;\,c\big),$$

  called the *c-colored unit*.

- For $c'' \in \mathsf{Ob}\,\mathsf{M}$, $\langle c' \rangle = (c_1', \dots, c_n') \in \mathsf{Prof}(\mathsf{M})$, and $\langle c_j \rangle = (c_{j,1}, \dots, c_{j,k_j}) \in \mathsf{Prof}(\mathsf{M})$ for each $j \in \{1, \dots, n\}$, let $\langle c \rangle = \oplus_j \langle c_j \rangle \in \mathsf{Prof}(\mathsf{M})$ be the concatenation of the $\langle c_j \rangle$. Then M is equipped with a morphism in V

$$\mathsf{M}(\langle c' \rangle; c'') \otimes \bigotimes_{j=1}^{n} \mathsf{M}(\langle c_j \rangle; c_j') \xrightarrow{\gamma} \mathsf{M}(\langle c \rangle; c'') \tag{3}$$

called the *composition* or *multicategorical composition*.

These data are required to satisfy the following axioms.

Symmetric Group Action For $\big(\langle c \rangle; c'\big) \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ with $n = \mathrm{len}\langle c \rangle$ and $\sigma, \tau \in \Sigma_n$, the following diagram in V commutes.

$$
\begin{array}{ccc}
\mathsf{M}(\langle c \rangle; c') & \xrightarrow{\ \sigma\ } & \mathsf{M}(\langle c \rangle\sigma; c') \\
& \searrow{\scriptstyle \sigma\tau} & \downarrow{\scriptstyle \tau} \\
& & \mathsf{M}(\langle c \rangle\sigma\tau; c')
\end{array}
\tag{4}
$$

Moreover, the identity permutation in Σ_n acts as the identity morphism of $\mathsf{M}\big(\langle c \rangle; c'\big)$.

Associativity Suppose given

- $c''' \in \mathsf{Ob}\,\mathsf{M}$,
- $\langle c'' \rangle = (c_1'', \dots, c_n'') \in \mathsf{Prof}(\mathsf{M})$,
- $\langle c_j' \rangle = (c_{j,1}', \dots, c_{j,k_j}') \in \mathsf{Prof}(\mathsf{M})$ for each $j \in \{1, \dots, n\}$, and
- $\langle c_{j,i} \rangle = (c_{j,i,1}, \dots, c_{j,i,\ell_{j,i}}) \in \mathsf{Prof}(\mathsf{M})$ for each $j \in \{1, \dots, n\}$ and each $i \in \{1, \dots, k_j\}$,

such that $k_j = \mathrm{len}\langle c_j' \rangle > 0$ for at least one j. For each j, let $\langle c_j \rangle = \oplus_{i=1}^{k_j}\langle c_{j,i} \rangle$ denote the concatenation of the $\langle c_{j,i} \rangle$. Let $\langle c \rangle = \oplus_{j=1}^{n}\langle c_j \rangle$ denote the concatenation of the $\langle c_j \rangle$. Let $\langle c' \rangle = \oplus_{j=1}^{n}\langle c_j' \rangle$ denote the concatenation of the $\langle c_j' \rangle$. Then the *associativity diagram* below commutes.

$$
\tag{5}
$$

$$
\mathsf{M}(\langle c' \rangle; c''') \otimes \bigotimes_{j=1}^{n}\left[\bigotimes_{i=1}^{k_j}\mathsf{M}(\langle c_{j,i} \rangle; c_{j,i}')\right]
$$

$$
\begin{array}{c}
\mathsf{M}(\langle c'' \rangle; c''') \otimes \left[\bigotimes_{j=1}^{n}\mathsf{M}(\langle c_j' \rangle; c_j'')\right] \otimes \bigotimes_{j=1}^{n}\left[\bigotimes_{i=1}^{k_j}\mathsf{M}(\langle c_{j,i} \rangle; c_{j,i}')\right] \\
\\
\text{permute}\ \Big\downarrow \cong \\
\\
\mathsf{M}(\langle c'' \rangle; c''') \otimes \bigotimes_{j=1}^{n}\left[\mathsf{M}(\langle c_j' \rangle; c_j'') \otimes \bigotimes_{i=1}^{k_j}\mathsf{M}(\langle c_{j,i} \rangle; c_{j,i}')\right]
\end{array}
$$

with arrows $(\gamma, 1)$, γ to $\mathsf{M}(\langle c \rangle; c''')$, $(1, \otimes_j \gamma)$ to

$$\mathsf{M}(\langle c'' \rangle; c''') \otimes \bigotimes_{j=1}^{n}\mathsf{M}(\langle c_j \rangle; c_j'')$$

Unity Suppose $c' \in \mathsf{Ob}\,\mathsf{M}$.

1. If $\langle c \rangle = (c_1, \ldots, c_n) \in \mathsf{Prof}(\mathsf{M})$ has length $n \geq 1$, then the following *right unity diagram* is commutative. Here $\mathbb{1}^n$ is the n-fold monoidal product of $\mathbb{1}$ with itself.

$$
\begin{array}{ccc}
\mathsf{M}\big(\langle c \rangle\,;\,c'\big) \otimes \mathbb{1}^n & \xrightarrow{\;\;\rho\;\;} & \mathsf{M}\big(\langle c \rangle\,;\,c'\big) \\
{\scriptstyle 1 \otimes (\otimes_j 1_{c_j})} \downarrow & & \downarrow {\scriptstyle 1} \\
\mathsf{M}\big(\langle c \rangle\,;\,c'\big) \otimes \overset{n}{\underset{j=1}{\otimes}}\,\mathsf{M}\big(c_j\,;\,c_j\big) & \xrightarrow{\;\;\gamma\;\;} & \mathsf{M}\big(\langle c \rangle\,;\,c'\big)
\end{array}
\tag{6}
$$

2. For any $\langle c \rangle \in \mathsf{Prof}(\mathsf{M})$, the *left unity diagram* below is commutative.

$$
\begin{array}{ccc}
\mathbb{1} \otimes \mathsf{M}\big(\langle c \rangle\,;\,c'\big) & \xrightarrow{\;\;\lambda\;\;} & \mathsf{M}\big(\langle c \rangle\,;\,c'\big) \\
{\scriptstyle 1_{c'} \otimes 1} \downarrow & & \downarrow {\scriptstyle 1} \\
\mathsf{M}\big(c'\,;\,c'\big) \otimes \mathsf{M}\big(\langle c \rangle\,;\,c'\big) & \xrightarrow{\;\;\gamma\;\;} & \mathsf{M}\big(\langle c \rangle\,;\,c'\big)
\end{array}
\tag{7}
$$

Equivariance Suppose that in the definition of γ (3), $\mathrm{len}\langle c_j \rangle = k_j \geq 0$.

1. For each $\sigma \in \Sigma_n$, the following *top equivariance diagram* is commutative.

$$
\begin{array}{ccc}
\mathsf{M}\big(\langle c' \rangle\,;\,c''\big) \otimes \overset{n}{\underset{j=1}{\otimes}}\,\mathsf{M}\big(\langle c_j \rangle\,;\,c'_j\big) & \xrightarrow{\;(\sigma,\sigma^{-1})\;} & \mathsf{M}\big(\langle c' \rangle \sigma\,;\,c''\big) \otimes \overset{n}{\underset{j=1}{\otimes}}\,\mathsf{M}\big(\langle c_{\sigma(j)} \rangle\,;\,c'_{\sigma(j)}\big) \\
{\scriptstyle \gamma} \downarrow & & \downarrow {\scriptstyle \gamma} \\
\mathsf{M}\big(\langle c_1 \rangle, \ldots, \langle c_n \rangle\,;\,c''\big) & \xrightarrow{\;\sigma\langle k_{\sigma(1)}, \ldots, k_{\sigma(n)}\rangle\;} & \mathsf{M}\big(\langle c_{\sigma(1)} \rangle, \ldots, \langle c_{\sigma(n)} \rangle\,;\,c''\big)
\end{array}
\tag{8}
$$

Here $\sigma\langle k_{\sigma(1)}, \ldots, k_{\sigma(n)}\rangle \in \Sigma_{k_1 + \cdots + k_n}$ is right action of the block permutation that permutes the n consecutive blocks of lengths $k_{\sigma(1)}$, $\ldots$, $k_{\sigma(n)}$ as σ permutes $\{1, \ldots, n\}$, leaving the relative order within each block unchanged.

2. Given permutations $\tau_j \in \Sigma_{k_j}$ for $1 \leq j \leq n$, the following *bottom equivariance diagram* is commutative.

$$
\begin{array}{ccc}
\mathsf{M}\big(\langle c' \rangle\,;\,c''\big) \otimes \overset{n}{\underset{j=1}{\otimes}}\,\mathsf{M}\big(\langle c_j \rangle\,;\,c'_j\big) & \xrightarrow{\;(1, \otimes_j \tau_j)\;} & \mathsf{M}\big(\langle c' \rangle\,;\,c''\big) \otimes \overset{n}{\underset{j=1}{\otimes}}\,\mathsf{M}\big(\langle c_j \rangle \tau_j\,;\,c'_j\big) \\
{\scriptstyle \gamma} \downarrow & & \downarrow {\scriptstyle \gamma} \\
\mathsf{M}\big(\langle c_1 \rangle, \ldots, \langle c_n \rangle\,;\,c''\big) & \xrightarrow{\;\tau_1 \times \cdots \times \tau_n\;} & \mathsf{M}\big(\langle c_1 \rangle \tau_1, \ldots, \langle c_n \rangle \tau_n\,;\,c''\big)
\end{array}
\tag{9}
$$

Here the block sum $\tau_1 \times \cdots \times \tau_n \in \Sigma_{k_1 + \cdots + k_n}$ is the image of $(\tau_1, \ldots, \tau_n)$ under the canonical inclusion

$$
\Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \longrightarrow \Sigma_{k_1 + \cdots + k_n}.
$$

This finishes the definition of a V-multicategory.

Moreover, we define the following.

- A V-multicategory is *small* if its class of objects is a set.

- A V-*operad* is a V-multicategory with one object. If M is a V-operad, then its object of n-ary operations is denoted by $M_n \in V$.

- A *multicategory* is a Set-multicategory, where $(Set, \times, *)$ is the symmetric monoidal category of sets and functions with the Cartesian product.

- An *operad* is a Set-operad, that is, a multicategory with one object.

- A *non-symmetric* V-*multicategory* is defined as above, except it does not have a designated symmetric group action or satisfy the related action or equivariance axioms.

3.4. DEFINITION. The *initial multicategory* has an empty set of objects. The *initial operad* I consists of a single object $*$ and a single operation, which is the unit 1_*. The *terminal multicategory* T consists of a single object $*$ and a single n-ary operation ι_n for each $n \geq 0$.

3.5. EXAMPLE. [Endomorphism Operad] Suppose M is a V-multicategory and c is an object of M. Then $End(c)$ is the V-operad consisting of the single object c and n-ary operation object

$$End(c)_n = M(\langle c \rangle ; c) \in V,$$

where $\langle c \rangle$ denotes the constant n-tuple at c. The symmetric group action, unit, and composition of $End(c)$ are given by those of M.

3.6. EXAMPLE. [Underlying Enriched Category] Each V-multicategory $(M, \gamma, 1)$ has an underlying V-category with

- the same objects,

- identities given by the colored units, and

- composition given by

$$M(b ; c) \otimes M(a ; b) \xrightarrow{\quad \gamma \quad} M(a ; c)$$

for objects $a, b, c \in M$.

The V-category associativity and unity diagrams as in e.g., [JY∞, 1.2.1], are the unary special cases of, respectively, the associativity diagram (5) and the unity diagrams (6) and (7) of a V-multicategory.

3.7. DEFINITION. A V-*multifunctor* $P : \mathsf{M} \longrightarrow \mathsf{N}$ between V-multicategories M and N consists of

- an object assignment $P : \mathsf{Ob}\,\mathsf{M} \longrightarrow \mathsf{Ob}\,\mathsf{N}$ and

- for each $\big(\langle c \rangle \, ; c'\big) \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ with $\langle c \rangle = (c_1, \dots, c_n)$, a component morphism in V
$$P : \mathsf{M}\big(\langle c \rangle \, ; c'\big) \longrightarrow \mathsf{N}\big(P\langle c \rangle \, ; Pc'\big),$$
where $P\langle c \rangle = (Pc_1, \dots, Pc_n)$.

These data are required to preserve the symmetric group action, the colored units, and the composition in the following sense.

Symmetric Group Action For each $\big(\langle c \rangle \, ; c'\big)$ as above and each permutation $\sigma \in \Sigma_n$, the following diagram in V is commutative.

$$
\begin{array}{ccc}
\mathsf{M}\big(\langle c \rangle \, ; c'\big) & \xrightarrow{\ P\ } & \mathsf{N}\big(P\langle c \rangle \, ; Pc'\big) \\
{\scriptstyle \sigma}\Big\downarrow{\scriptstyle \cong} & & {\scriptstyle \sigma}\Big\downarrow{\scriptstyle \cong} \\
\mathsf{M}\big(\langle c \rangle \sigma \, ; c'\big) & \xrightarrow{\ P\ } & \mathsf{N}\big(P\langle c \rangle \sigma \, ; c'\big)
\end{array}
\tag{10}
$$

Units For each $c \in \mathsf{Ob}\,\mathsf{M}$, the following diagram in V is commutative.

$$
\begin{array}{ccc}
 & & \mathsf{M}(c \, ; c) \\
 & {\scriptstyle 1_c}\nearrow & \Big\downarrow{\scriptstyle P} \\
\mathbb{1} & \xrightarrow[{\scriptstyle 1_{Pc}}]{} & \mathsf{N}(Pc \, ; Pc)
\end{array}
\tag{11}
$$

Composition For c'', $\langle c' \rangle$, and $\langle c \rangle = \oplus_j \langle c_j \rangle$ as in the definition of γ (3), the following diagram in V is commutative.

$$
\begin{array}{ccc}
\mathsf{M}\big(\langle c' \rangle \, ; c''\big) \otimes \bigotimes_{j=1}^{n} \mathsf{M}\big(\langle c_j \rangle \, ; c'_j\big) & \xrightarrow{\ (P, \otimes_j P)\ } & \mathsf{N}\big(P\langle c' \rangle \, ; Pc''\big) \otimes \bigotimes_{j=1}^{n} \mathsf{N}\big(P\langle c_j \rangle \, ; Pc'_j\big) \\
{\scriptstyle \gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle \gamma} \\
\mathsf{M}\big(\langle c \rangle \, ; c''\big) & \xrightarrow{\qquad\qquad P \qquad\qquad} & \mathsf{N}\big(P\langle c \rangle \, ; Pc''\big)
\end{array}
\tag{12}
$$

This finishes the definition of a V-multifunctor.

Moreover, we define the following.

1. A *multifunctor* is a Set-multifunctor.

2. A V-multifunctor $\mathsf{M} \longrightarrow \mathsf{N}$ is also called an M-*algebra in* N.

3. For another V-multifunctor $Q : \mathsf{N} \longrightarrow \mathsf{L}$ between V-multicategories, where L has object class $\mathsf{Ob}\,\mathsf{L}$, the *composition* $QP : \mathsf{M} \longrightarrow \mathsf{L}$ is the V-multifunctor defined by composing the assignments on objects

$$\mathsf{Ob}\,\mathsf{M} \xrightarrow{\ P\ } \mathsf{Ob}\,\mathsf{N} \xrightarrow{\ Q\ } \mathsf{Ob}\,\mathsf{L}$$

and the morphisms on n-ary operations

$$\mathsf{M}\big(\langle c \rangle\,;\, c'\big) \xrightarrow{\ P\ } \mathsf{N}\big(P\langle c \rangle\,;\, Pc'\big) \xrightarrow{\ Q\ } \mathsf{L}\big(QP\langle c \rangle\,;\, QPc'\big).$$

4. The *identity V-multifunctor* $1_\mathsf{M} : \mathsf{M} \longrightarrow \mathsf{M}$ is defined by the identity assignment on objects and the identity morphism on n-ary operations.

5. A V-*operad morphism* is a V-multifunctor between two V-multicategories with one object.

6. A *non-symmetric* V-multifunctor consists of the same data as above, but is not required to preserve the symmetric group action of its source and target. Thus a non-symmetric V-multifunctor is only required to preserve the colored units and composition.

3.8. Definition. Suppose $P, Q : \mathsf{M} \longrightarrow \mathsf{N}$ are V-multifunctors as in Definition 3.7. A V-*multinatural transformation* $\theta : P \longrightarrow Q$ consists of component morphisms in V

$$\theta_c : \mathbb{1} \longrightarrow \mathsf{N}\big(Pc\,;\, Qc\big) \quad \text{for} \quad c \in \mathsf{Ob}\,\mathsf{M}$$

such that the following V-*naturality diagram* in V commutes for each $\big(\langle c \rangle\,;\, c'\big) \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ with $\langle c \rangle = (c_1, \ldots, c_n)$.

$$(13)$$

This finishes the definition of a V-multinatural transformation.

   Moreover, we define the following.

- The *identity V-multinatural transformation* $1_P : P \longrightarrow P$ has components

$$(1_P)_c = 1_{Pc} \quad \text{for} \quad c \in \mathsf{Ob}\,\mathsf{M}.$$

- A *multinatural transformation* is a Set-multinatural transformation.

- For non-symmetric multifunctors $P, Q : \mathsf{M} \longrightarrow \mathsf{N}$, a V-multinatural transformation $\theta : P \longrightarrow Q$ has the same definition given above, and we use the same terminology.

3.9. DEFINITION. Suppose $\theta : P \longrightarrow Q$ is a V-multinatural transformation between V-multifunctors as in Definition 3.8.

1. Suppose $\beta : Q \longrightarrow R$ is a V-multinatural transformation for a V-multifunctor $R :$ M $\longrightarrow$ N. The *vertical composition*

$$\beta\theta : P \longrightarrow R \qquad (14)$$

   is the V-multinatural transformation with components at $c \in$ Ob M given by the following composites in V.

2. Suppose $\theta' : P' \longrightarrow Q'$ is a V-multinatural transformation for V-multifunctors $P', Q' :$ N $\longrightarrow$ L. The *horizontal composition*

$$\theta' * \theta : P'P \longrightarrow Q'Q \qquad (15)$$

   is the V-multinatural transformation with components at $c \in$ Ob M given by the following composites in V.

This finishes the definition.

4. The Cat-Multicategory of Small Multicategories

The 2-category Multicat of small multicategories, multifunctors, and multinatural transformations has a closed symmetric monoidal structure given by the Boardman-Vogt tensor product. This induces a Cat-multicategory structure that we will describe below. We give only those details necessary for the arguments in the sequel, and refer the reader to [JY∞, Chapters 5 and 6] for a full treatment of related background.

We begin with preliminary notation.

4.1. DEFINITION. Given profiles $\langle c \rangle \in \mathsf{Prof}(C)$ and $\langle d \rangle \in \mathsf{Prof}(D)$ with $m = \mathrm{len}\langle c \rangle$ and $n = \mathrm{len}\langle d \rangle$, we define the following profiles.

$$\langle c \rangle \times d_j = \langle (c_i, d_j) \rangle_{i=1}^m = \big((c_1, d_j), (c_2, d_j), \dots, (c_m, d_j) \big)$$
$$c_i \times \langle d \rangle = \langle (c_i, d_j) \rangle_{j=1}^n = \big((c_i, d_1), (c_i, d_2), \dots, (c_i, d_n) \big)$$
$$\langle c \rangle \otimes \langle d \rangle = \langle \langle (c_i, d_j) \rangle_{i=1}^m \rangle_{j=1}^n = \big(\langle c \rangle \times d_1, \dots, \langle c \rangle \times d_n \big)$$
$$\langle c \rangle \otimes^{\mathsf{t}} \langle d \rangle = \langle \langle (c_i, d_j) \rangle_{j=1}^n \rangle_{i=1}^m = \big(c_1 \times \langle d \rangle, \dots, c_m \times \langle d \rangle \big)$$

Thus, $\otimes$ uses the reverse lexicographic order, and $\otimes^{\mathsf{t}}$ uses the lexicographic order. We denote by

$$\xi^{\otimes} = \xi_{m,n}^{\otimes} : \langle c \rangle \otimes \langle d \rangle \xrightarrow{\ \cong\ } \langle c \rangle \otimes^{\mathsf{t}} \langle d \rangle$$

the permutation induced by changing order of indexing.

4.2. EXAMPLE. Suppose $\langle c \rangle = \{c_1, c_2\}$ and $\langle d \rangle = \{d_1, d_2, d_3\}$. Then the profiles $\langle c \rangle \otimes \langle d \rangle$ and $\langle c \rangle \otimes^{\mathsf{t}} \langle d \rangle$ are given as follows.

$$\langle c \rangle \otimes \langle d \rangle = \big\{ (c_1, d_1), (c_2, d_1), (c_1, d_2), (c_2, d_2), (c_1, d_3), (c_2, d_3) \big\}$$
$$\langle c \rangle \otimes^{\mathsf{t}} \langle d \rangle = \big\{ (c_1, d_1), (c_1, d_2), (c_1, d_3), (c_2, d_1), (c_2, d_2), (c_2, d_3) \big\}$$

Note that $\xi_{m,1}^{\otimes}$ and $\xi_{1,n}^{\otimes}$ are identity permutations.

4.3. REMARK. [Choice of Ordering] Throughout this paper we consistently use the reverse lexicographic ordering $\otimes$. This is simply a matter of convenience. In other words, we can also consistently use the lexicographic ordering $\otimes^{\mathsf{t}}$ throughout, and our main results remain valid, as we explain further in Remark 7.11. The reason we prefer $\otimes$ over $\otimes^{\mathsf{t}}$ is that, in the profile $\langle c \rangle \otimes \langle d \rangle$, the indices i and j appear in the same left-to-right order as they do in (c_i, d_j). Some of our main constructions involve iterating the tensor product; see (54) and (58). If we use $\otimes^{\mathsf{t}}$ instead of $\otimes$, then (54) would involve the indices

$$1 \le j_n \le r_n, \quad 1 \le j_{n-1} \le r_{n-1}, \quad \dots, \quad 1 \le j_1 \le r_1,$$

further complicating the formulas.

4.4. DEFINITION. [Boardman-Vogt Tensor Product of Multicategories] Suppose given small multicategories M and N. The *tensor product* $\mathsf{M} \otimes \mathsf{N}$ is the multicategory defined as follows. Its object set is $\mathsf{Ob}\,\mathsf{M} \times \mathsf{Ob}\,\mathsf{N}$. For $c \in \mathsf{Ob}\,\mathsf{M}$ and $d \in \mathsf{Ob}\,\mathsf{N}$, the corresponding object in $\mathsf{M} \otimes \mathsf{N}$ is denoted (c, d) or $c \otimes d$.

The operations in $\mathsf{M} \otimes \mathsf{N}$ are generated by the following operations for $c \in \mathsf{Ob}\,\mathsf{M}$, $d \in \mathsf{Ob}\,\mathsf{N}$, $\phi \in \mathsf{M}\big(\langle c_i \rangle_{i=1}^m ; c' \big)$, and $\psi \in \mathsf{N}\big(\langle d_j \rangle_{j=1}^n ; d' \big)$.

$$\phi \otimes d \in (\mathsf{M} \otimes \mathsf{N})\big(\langle (c_i, d) \rangle_{i=1}^m ; (c', d) \big)$$
$$c \otimes \psi \in (\mathsf{M} \otimes \mathsf{N})\big(\langle (c, d_j) \rangle_{j=1}^n ; (c, d') \big)$$

These data are subject to relations (1) through (6) below.

1. For $c \in \mathsf{M}$ and $d \in \mathsf{N}$, we have
$$1_c \otimes d = 1_{(c,d)} = c \otimes 1_d.$$

2. For operations $\phi, \phi_1, \ldots, \phi_n$ in M such that the composite below is defined, we have
$$\gamma\big(\phi \otimes d\,;\, \phi_1 \otimes d, \ldots, \phi_n \otimes d\big) = \gamma\big(\phi\,;\, (\phi_1, \ldots, \phi_n)\big) \otimes d.$$

3. For $\sigma \in \Sigma_m$, we have
$$(\phi \otimes d) \cdot \sigma = (\phi \cdot \sigma) \otimes d.$$

4. For operations $\psi, \psi_1, \ldots, \psi_m$ in N such that the composite below is defined, we have
$$\gamma\big(c \otimes \psi\,;\, c \otimes \psi_1, \ldots, c \otimes \psi_m\big) = c \otimes \gamma\big(\psi\,;\, \psi_1, \ldots, \psi_m\big).$$

5. For $\sigma \in \Sigma_n$, we have
$$(c \otimes \psi) \cdot \sigma = c \otimes (\psi \cdot \sigma).$$

6. For operations $\phi \in \mathsf{M}\big(\langle c_i \rangle_{i=1}^m\,;\, c'\big)$ and $\psi \in \mathsf{N}\big(\langle d_j \rangle_{j=1}^n\,;\, d'\big)$, we have
$$\gamma\big(c' \otimes \psi\,;\, \phi \otimes d_1, \ldots, \phi \otimes d_n\big) = \gamma\big(\phi \otimes d'\,;\, c_1 \otimes \psi, \ldots, c_m \otimes \psi\big) \cdot \xi^{\otimes}$$

in $(\mathsf{M} \otimes \mathsf{N})\big(\langle c \rangle \otimes \langle d \rangle\,;\, (c', d')\big)$. This is called the *interchange relation.*

This finishes the definition of $\mathsf{M} \otimes \mathsf{N}$. Moreover, we define the operations
$$\begin{aligned}
\phi \otimes \psi &= \gamma\big(c' \otimes \psi\,;\, \phi \otimes d_1, \ldots, \phi \otimes d_n\big) \quad \text{and} \\
\phi \otimes^{\mathsf{t}} \psi &= \gamma\big(\phi \otimes d'\,;\, c_1 \otimes \psi, \ldots, c_m \otimes \psi\big),
\end{aligned} \tag{16}$$

which are the two composite operations in relation (6) above.

4.5. EXPLANATION. If we draw an operation as an arrow from its input profile to its output object, the interchange relation means that the two composites

$$\begin{array}{ccc}
\langle c \rangle \otimes \langle d \rangle & \xrightarrow{\ \langle \phi \otimes d_j \rangle_{j=1}^n\ } & c' \times \langle d \rangle \\
& & \quad\searrow^{c' \otimes \psi} \\
& & \qquad (c', d') \\
\langle c \rangle \otimes^{\mathsf{t}} \langle d \rangle & \xrightarrow[\ \langle c_i \otimes \psi \rangle_{i=1}^m\ ]{} & \langle c \rangle \times d' \quad \nearrow_{\phi \otimes d'}
\end{array}$$

correspond under the bijection
$$\xi^{\otimes} : (\mathsf{M} \otimes \mathsf{N})\big(\langle c \rangle \otimes^{\mathsf{t}} \langle d \rangle\,;\, (c', d')\big) \xrightarrow{\ \cong\ } (\mathsf{M} \otimes \mathsf{N})\big(\langle c \rangle \otimes \langle d \rangle\,;\, (c', d')\big).$$

A multifunctor
$$H : \mathsf{M} \otimes \mathsf{N} \longrightarrow \mathsf{L}$$
consists of an assignment on objects $H(c, d) \in \mathsf{Ob}\,\mathsf{L}$ for $(c, d) \in \mathsf{Ob}\,\mathsf{M} \times \mathsf{Ob}\,\mathsf{N}$ such that each $H(c, -)$ and $H(-, d)$ is a multifunctor and such that we have
$$H(\phi \otimes \psi) = H(\phi \otimes^{\mathsf{t}} \psi) \cdot \xi^{\otimes} \tag{17}$$
for each $\phi \in \mathsf{M}\big(\langle c \rangle\,;\, c'\big)$ and $\psi \in \mathsf{N}\big(\langle d \rangle\,;\, d'\big)$.

4.6. THEOREM. [JY∞, 5.7.14, 6.4.3] *The tensor product of Definition 4.4 is a* Cat-*enriched symmetric monoidal product for* Multicat. *Its monoidal unit is the initial operad* I *of Definition 3.4.*

4.7. EXPLANATION. The Cat-enrichment of the symmetric monoidal product for Multicat is defined as follows. Given multifunctors

$$F : \mathsf{M} \longrightarrow \mathsf{M}' \quad \text{and} \quad G : \mathsf{N} \longrightarrow \mathsf{N}',$$

we define a multifunctor

$$F \otimes G : \mathsf{M} \otimes \mathsf{N} \longrightarrow \mathsf{M}' \otimes \mathsf{N}'$$

with assignment on generating operations given by

$$(F \otimes G)(\phi \otimes d) = F\phi \otimes Gd \quad \text{and} \quad (F \otimes G)(c \otimes \psi) = Fc \otimes G\psi.$$

Moreover, with F and G as above, for multinatural transformations

$$\theta : F \longrightarrow F' \quad \text{and} \quad \omega : G \longrightarrow G',$$

we define the tensor product multinatural transformation

$$\theta \otimes \omega : F \otimes G \longrightarrow F' \otimes G'$$

with components

$$(\theta \otimes \omega)_{(c,d)} = \theta_c \otimes \omega_d$$

for $(c, d) \in \mathsf{Ob}\,\mathsf{M} \times \mathsf{Ob}\,\mathsf{N}$. The right-hand side of the above equality is defined as in (16).

The closed symmetric monoidal structure on Multicat induces a Cat-enriched multicategory structure that we now describe.

4.8. DEFINITION. Let Multicat also denote the Cat-enriched multicategory whose objects are small multicategories and whose category of n-ary operations,

$$\mathsf{Multicat}(\langle \mathsf{M} \rangle\,;\,\mathsf{N}),$$

consists of multifunctors

$$H, K : \bigotimes_{i=1}^{n} \mathsf{M}_i \longrightarrow \mathsf{N} \tag{18}$$

and multinatural transformations $\theta : H \longrightarrow K$.

For a permutation $\sigma \in \Sigma_n$, the right action

$$\mathsf{Multicat}(\langle \mathsf{M} \rangle\,;\,\mathsf{N}) \xrightarrow[\cong]{\sigma} \mathsf{Multicat}(\langle \mathsf{M} \rangle \sigma\,;\,\mathsf{N})$$
$$H \longmapsto \xi_\sigma \circ H$$
$$\theta \longmapsto \xi_\sigma * \theta$$

is given by composition and whiskering with the permutation of tensor factors

$$\bigotimes_{i=1}^{n} \mathsf{M}_{\sigma(i)} \xrightarrow[\cong]{\xi_{\sigma}} \bigotimes_{i=1}^{n} \mathsf{M}_{i}.$$

Units are given by identity multifunctors. The composition $\gamma\big(H'\,;\,\langle H\rangle\big)$ for multifunctors

$$H_i : \bigotimes_{j=1}^{n_i} \mathsf{M}_{i,j} \longrightarrow \mathsf{M}_i' \quad \text{for} \quad 1 \le i \le j, \quad \text{and}$$

$$H' : \bigotimes_{i=1}^{n} \mathsf{M}_i' \longrightarrow \mathsf{M}'',$$

is given by composition with the associativity isomorphism of $\otimes$

$$\bigotimes_{i,j} \mathsf{M}_{i,j} \xrightarrow[\cong]{\alpha} \bigotimes_{i=1}^{n} \bigotimes_{j=1}^{n_i} \mathsf{M}_{i,j} \xrightarrow{\otimes_i H_i} \bigotimes_{i=1}^{n} \mathsf{M}_i' \xrightarrow{H'} \mathsf{M}'' \tag{19}$$

where the first tensor product is the left normalized tensor product (Convention 3.2) of the concatenation of the $\langle \mathsf{M}_i \rangle = \langle \mathsf{M}_{i,j} \rangle_{j=1}^{n_i}$. The V-multicategory axioms for $\mathsf{V} = \mathsf{Cat}$ follow from the enriched symmetric monoidal axioms for $\mathsf{Multicat}$. See [JY∞, Section 6.3] for further details.

POINTED MULTICATEGORIES. Recall from Definition 3.4 the terminal multicategory T.

4.9. DEFINITION. A *pointed* multicategory is a multicategory M together with a distinguished multifunctor

$$\iota : \mathsf{T} \longrightarrow \mathsf{M}$$

called the *basepoint* of M. Pointed multifunctors and multinatural transformations are those that commute with the basepoint morphisms. The 2-category of small pointed multicategories, $\mathsf{Multicat}_*$, consists of small pointed multicategories, pointed multifunctors, and pointed multinatural transformations.

The tensor product of small multicategories induces a smash product of small pointed multicategories. For small pointed multicategories M and N, the smash product $\mathsf{M} \wedge \mathsf{N}$ is defined as the following pushout in $\mathsf{Multicat}$.

$$\begin{array}{ccc}
(\mathsf{M} \otimes \mathsf{T}) \amalg (\mathsf{T} \otimes \mathsf{N}) & \longrightarrow & \mathsf{M} \otimes \mathsf{N} \\
\downarrow & & \downarrow{\scriptstyle \varpi_{\mathsf{M},\mathsf{N}}} \\
\mathsf{T} & \longrightarrow & \mathsf{M} \wedge \mathsf{N}
\end{array} \tag{20}$$

In the diagram above, the left vertical arrow is the unique multifunctor to T. The top horizontal arrow is induced by the basepoints of M and N. The basepoint of $\mathsf{M} \wedge \mathsf{N}$ is the bottom horizontal arrow. Thus $\mathsf{M} \wedge \mathsf{N}$ has the same objects as $\mathsf{M} \otimes \mathsf{N}$, and the operations in $\mathsf{M} \wedge \mathsf{N}$ are represented by those in $\mathsf{M} \otimes \mathsf{N}$, subject to basepoint conditions.

The smash product provides the Cat-enriched multicategory structure for small permutative categories, which we will describe further in Section 5. Most details of the smash product for pointed multicategories will not be needed here, so we refer the reader to [JY∞, Chapters 4 and 5] for a complete treatment. For our purposes we need only the following result.

4.10. THEOREM. [JY∞, 6.4.4] *The smash product of pointed multicategories is a* Cat-*enriched symmetric monoidal product for* Multicat$_*$. *Its monoidal unit is the coproduct*

$$\mathsf{I}_+ = \mathsf{I} \coprod \mathsf{T}.$$

As with Multicat, the smash product of pointed multicategories induces a Cat-enriched multicategory of small pointed multicategories.

4.11. DEFINITION. Let Multicat$_*$ also denote the Cat-enriched multicategory whose objects are small pointed multicategories and whose category of n-ary operations

$$\mathsf{Multicat}_*\big(\langle \mathsf{M}\rangle\,;\,\mathsf{N}\big) = \mathsf{Multicat}_*\big(\bigwedge_i \mathsf{M}_i, \mathsf{N}\big)$$

consists of pointed multifunctors and pointed multinatural transformations out of an iterated smash product. The symmetric group action, units, and composition are given by the Cat-enriched symmetric monoidal structure of $(\mathsf{Multicat}_*, \wedge, \mathsf{I}_+)$.

5. The Cat-Multicategory of Permutative Categories

In this section we define the Cat-enriched multicategory of permutative categories, multilinear functors, and multinatural transformations. For further discussion of plain, braided, and symmetric monoidal categories, we refer the reader to [JS93, ML98, JY21, Yau∞I, Yau∞II].

5.1. DEFINITION. A *permutative category* $(\mathsf{C}, \oplus, e, \xi)$ consists of

- a category C,

- a functor $\oplus : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$, called the *monoidal sum*,

- an object $e \in \mathsf{C}$, called the *monoidal unit*, and

- a natural isomorphism ξ called the *symmetry isomorphism* with components

$$\xi_{X,Y} : X \oplus Y \longrightarrow Y \oplus X$$

for objects X and Y of C.

The monoidal sum is required to be associative and unital, with e as its unit. The symmetry isomorphism ξ is required to make the following symmetry and hexagon diagrams commute for objects $X, Y, Z \in \mathsf{C}$.

$$
\begin{array}{c}
\xymatrix{
X \oplus Y \ar[rr]^{1_{X \oplus Y}} \ar[dr]_{\xi_{X,Y}} & & X \oplus Y \ar[dl]^{\xi_{Y,X}} \\
& Y \oplus X &
}
\qquad
\xymatrix{
& (Y \oplus X) \oplus Z = Y \oplus (X \oplus Z) & \\
(X \oplus Y) \oplus Z \ar[ur]^{\xi_{X,Y} \oplus 1_Z} & & Y \oplus (Z \oplus X) \ar[ul]_{1_Y \oplus \xi_{X,Z}} \\
X \oplus (Y \oplus Z) \ar[u]^{\shortparallel} \ar[rr]_{\xi_{X,Y \oplus Z}} & & (Y \oplus Z) \oplus X \ar[u]_{\shortparallel}
}
\end{array}
\tag{21}
$$

A permutative category is also called a *strict symmetric monoidal category*. The strictness refers to the conditions that the monoidal sum be strictly associative and unital.

5.2. DEFINITION. Suppose C and D are permutative categories. A *symmetric monoidal functor*

$$ (P, P^2, P^0) : \mathsf{C} \longrightarrow \mathsf{D} $$

consists of a functor $P : \mathsf{C} \longrightarrow \mathsf{D}$ together with natural transformations

$$ PX \oplus PY \xrightarrow{P^2} P(X \oplus Y) \quad \text{and} \quad e \xrightarrow{P^0} Pe $$

for objects $X, Y \in \mathsf{C}$, called the *monoidal constraint* and *unit constraint*, respectively. These data satisfy the following associativity, unity, and symmetry axioms.

Associativity The following diagram is commutative for all objects $X, Y, Z \in \mathsf{C}$.

$$
\xymatrix{
& (PX \oplus PY) \oplus PZ = PX \oplus (PY \oplus PZ) & \\
P(X \oplus Y) \oplus PZ \ar[d]_{P^2} & & PX \oplus P(Y \oplus Z) \ar[d]^{P^2} \\
P((X \oplus Y) \oplus Z) \ar@{=}[rr] & & P(X \oplus (Y \oplus Z))
}
\begin{array}{l}
\\ \ar[ul]^{P^2 \oplus 1_{PZ}} \quad \ar[ur]_{1_{PX} \oplus P^2}
\end{array}
\tag{22}
$$

Unity The following two diagrams are commutative for all objects $X \in \mathsf{C}$.

$$
\xymatrix{
& e \oplus PX = PX \ar@{=}[d] \\
Pe \oplus PX \ar[ur]^{P^0 \oplus 1_{PX}} \ar[r]_{P^2} & P(e \oplus X)
}
\quad \text{and} \quad
\xymatrix{
& PX \oplus e = PX \ar@{=}[d] \\
PX \oplus Pe \ar[ur]^{1_{PX} \oplus P^0} \ar[r]_{P^2} & P(X \oplus e)
}
\tag{23}
$$

Symmetry The following diagram is commutative for all objects $X, Y \in \mathsf{C}$.

$$
\xymatrix{
PX \oplus PY \ar[r]^{\xi_{PX,PY}} \ar[d]_{P^2} & PY \oplus PX \ar[d]^{P^2} \\
P(X \oplus Y) \ar[r]_{P\xi_{X,Y}} & P(Y \oplus X)
}
\tag{24}
$$

This finishes the definition of a symmetric monoidal functor.

Composition of symmetric monoidal functors

$$\mathsf{C} \xrightarrow{\ P\ } \mathsf{D} \xrightarrow{\ Q\ } \mathsf{E}$$

is given by composition of underlying functors with monoidal and unit constraints given, respectively, by

$$(QP)^2 = \big(Q(P^2)\big) \circ Q^2 \quad \text{and} \quad (QP)^0 = \big(Q(P^0)\big) \circ Q^0.$$

An identity functor is symmetric monoidal with identity monoidal and unit constraints. A symmetric monoidal functor P is called

- *strong* if P^0 and P^2 are natural isomorphisms,

- *strictly unital* if P^0 is the identity natural transformation, and

- *strict* if both P^0 and P^2 are identities.

5.3. DEFINITION. Suppose $P, Q : \mathsf{C} \longrightarrow \mathsf{D}$ are symmetric monoidal functors between permutative categories. A *monoidal natural transformation*

$$\theta : P \longrightarrow Q$$

is a natural transformation between the underlying functors such that the following unity and constraint compatibility diagrams commute for all $X, Y \in \mathsf{C}$.

$$
\begin{array}{cc}
\begin{array}{c}
\xymatrix{
 & Pe \\
e \ar[ur]^{P^0} \ar[dr]_{Q^0} & \\
 & Qe
}
\quad \theta_e
\end{array}
&
\begin{array}{c}
\xymatrix{
PX \oplus PY \ar[r]^{\theta_X \oplus \theta_Y} \ar[d]_{P^2} & QX \oplus QY \ar[d]^{Q^2} \\
P(X \oplus Y) \ar[r]_{\theta_{X \oplus Y}} & Q(X \oplus Y)
}
\end{array}
\end{array}
\tag{25}
$$

This finishes the definition of a monoidal natural transformation. Identity and composites of monoidal natural transformations are given by those of the underlying natural transformations.

5.4. DEFINITION. We let $\mathsf{PermCat}$ denote the 2-category of small permutative categories, symmetric monoidal functors, and monoidal natural transformations. We define the following sub-2-categories consisting of the same objects and restricted collections of functors.

- The 1-cells of $\mathsf{PermCat}^{\mathsf{su}}$ are strictly unital symmetric monoidal functors.

- The 1-cells of $\mathsf{PermCat}^{\mathsf{sus}}$ are strictly unital strong symmetric monoidal functors.

- The 1-cells of $\mathsf{PermCat}^{\mathsf{st}}$ are strict symmetric monoidal functors.

In each case the 2-cells consist of all monoidal natural transformations between the indicated 1-cells.

Multilinear Functors and Transformations. Now we recall the definitions of multilinear functors and transformations between them. See [EM06, Definition 3.2] and [JY∞, Sections 6.5 and 6.6] for further details and discussion of these structures. Throughout this section, suppose $C_1, \ldots, C_n$, and D are permutative categories.

5.5. Notation. Suppose

$$\langle x \rangle = (x_1, \ldots, x_n)$$

is an n-tuple of symbols, and x'_k is a symbol for $k \in \{1, \ldots, n\}$. We denote by

$$\langle x \circ_k x'_k \rangle = \langle x \rangle \circ_k x'_k = \big(\underbrace{x_1, \ldots, x_{k-1}}_{\text{empty if } k=1}, x'_k, \underbrace{x_{k+1}, \ldots, x_n}_{\text{empty if } k=n} \big)$$

the n-tuple obtained from $\langle x \rangle$ by replacing its k-th entry by x'_k. Similarly, for $k \neq \ell \in \{1, \ldots, n\}$ and a symbol x'_ℓ, we denote by

$$\langle x \circ_k x'_k \circ_\ell x'_\ell \rangle = \langle x \rangle \circ_k x'_k \circ_\ell x'_\ell$$

the n-tuple obtained from $\langle x \circ_k x'_k \rangle$ by replacing its ℓ-th entry by x'_ℓ. We sometimes use the notation

$$\langle x \circ_k x_k \rangle = \langle x \rangle$$

to emphasize the k-th term, x_k, in $\langle x \rangle$. See, for example, the first term in (26).

5.6. Definition. An *n-linear functor*

$$\prod_{j=1}^{n} C_j \xrightarrow{\left(P, \{P_j^2\}_{j=1}^n\right)} D$$

consists of

- a functor $P : C_1 \times \cdots \times C_n \longrightarrow D$ and

- for each $j \in \{1, \ldots, n\}$, a natural transformation P_j^2, called the *j-th linearity constraint*, with component morphisms

$$P\langle X \circ_j X_j \rangle \oplus P\langle X \circ_j X'_j \rangle \xrightarrow{P_j^2} P\langle X \circ_j (X_j \oplus X'_j) \rangle \in D \qquad (26)$$

  for objects $\langle X \rangle \in \prod_j C_j$ and $X'_j \in C_j$.

These data are subject to the following five axioms.

Unity For objects $\langle X \rangle$ and morphisms $\langle f \rangle$ in $\prod_j C_j$, the following object and morphism unity axioms hold for each $j \in \{1, \ldots, n\}$.

$$\begin{cases} P\langle X \circ_j e \rangle = e \\ P\langle f \circ_j 1_e \rangle = 1_e \end{cases} \qquad (27)$$

Constraint Unity

$$P_j^2 = 1 \qquad \text{if any } X_i = e \text{ or if } X_j' = e. \tag{28}$$

Constraint Associativity The following diagram commutes for each $i \in \{1, \ldots, n\}$ and objects $\langle X \rangle \in \prod_j \mathsf{C}_j$, with $X_i', X_i'' \in \mathsf{C}_i$.

$$
\begin{array}{ccc}
P\langle X \circ_i X_i \rangle \oplus P\langle X \circ_i X_i' \rangle \oplus P\langle X \circ_i X_i'' \rangle & \xrightarrow{1 \oplus P_i^2} & P\langle X \circ_i X_i \rangle \oplus P\langle X \circ_i (X_i' \oplus X_i'') \rangle \\
{\scriptstyle P_i^2 \oplus 1} \Big\downarrow & & \Big\downarrow {\scriptstyle P_i^2} \\
P\langle X \circ_i (X_i \oplus X_i') \rangle \oplus P\langle X \circ_i X_i'' \rangle & \xrightarrow{\quad P_i^2 \quad} & P\langle X \circ_i (X_i \oplus X_i' \oplus X_i'') \rangle
\end{array}
\tag{29}
$$

Constraint Symmetry The following diagram commutes for each $i \in \{1, \ldots, n\}$ and objects $\langle X \rangle \in \prod_j \mathsf{C}_j$, with $X_i' \in \mathsf{C}_i$.

$$
\begin{array}{ccc}
P\langle X \circ_i X_i \rangle \oplus P\langle X \circ_i X_i' \rangle & \xrightarrow{P_i^2} & P\langle X \circ_i (X_i \oplus X_i') \rangle \\
{\scriptstyle \xi} \Big\downarrow & & \Big\downarrow {\scriptstyle P\langle 1 \circ_i \xi \rangle} \\
P\langle X \circ_i X_i' \rangle \oplus P\langle X \circ_i X_i \rangle & \xrightarrow{P_i^2} & P\langle X \circ_i (X_i' \oplus X_i) \rangle
\end{array}
\tag{30}
$$

Constraint 2-By-2 The following diagram commutes for each

$$i \neq k \in \{1, \ldots, n\}, \quad \langle X \rangle \in \prod_j \mathsf{C}_j, \quad X_i' \in \mathsf{C}_i, \quad \text{and} \quad X_k' \in \mathsf{C}_k.$$

$$
\begin{array}{c}
P\langle X \circ_i (X_i \oplus X_i') \circ_k X_k \rangle \oplus P\langle X \circ_i (X_i \oplus X_i') \circ_k X_k' \rangle \\[4pt]
\nearrow^{P_i^2 \oplus P_i^2} \qquad\qquad\qquad \searrow^{P_k^2} \\
\begin{array}{c} P\langle X \circ_i X_i \circ_k X_k \rangle \oplus P\langle X \circ_i X_i' \circ_k X_k \rangle \\ \oplus P\langle X \circ_i X_i \circ_k X_k' \rangle \oplus P\langle X \circ_i X_i' \circ_k X_k' \rangle \end{array} \qquad\qquad\qquad \\[4pt]
{\scriptstyle 1 \oplus \xi \oplus 1}\Big\downarrow \qquad\qquad P\langle X \circ_i (X_i \oplus X_i') \circ_k (X_k \oplus X_k') \rangle \\[4pt]
\begin{array}{c} P\langle X \circ_i X_i \circ_k X_k \rangle \oplus P\langle X \circ_i X_i \circ_k X_k' \rangle \\ \oplus P\langle X \circ_i X_i' \circ_k X_k \rangle \oplus P\langle X \circ_i X_i' \circ_k X_k' \rangle \end{array} \qquad\qquad \nearrow^{P_i^2} \\[4pt]
\searrow_{P_k^2 \oplus P_k^2} \qquad\qquad\qquad \\
P\langle X \circ_i X_i \circ_k (X_k \oplus X_k') \rangle \oplus P\langle X \circ_i X_i' \circ_k (X_k \oplus X_k') \rangle
\end{array}
\tag{31}
$$

This finishes the definition of an n-linear functor.

Moreover, we define the following.

- If $n = 0$, then a 0-linear functor is a choice of an object in D.

- An n-linear functor $(P, \{P_j^2\})$ is *strong* if each linearity constraint P_j^2 is a natural isomorphism. It is *strict* if each P_j^2 is an identity natural transformation.

- A *multilinear functor* is an n-linear functor for some $n \geq 0$.

Below, we express the domain of a multilinear functor P either as a product $\prod_j \mathsf{C}_j$ or as a tuple $\langle \mathsf{C} \rangle$. So, for example, we write

$$\prod_j \mathsf{C}_j \xrightarrow{\ \mathsf{P}\ } \mathsf{D} \quad \text{or} \quad \langle \mathsf{C} \rangle \xrightarrow{\ \mathsf{P}\ } \mathsf{D}$$

to denote that P is a multilinear functor from $\langle \mathsf{C} \rangle$ to D.

5.7. EXAMPLE. For permutative categories C and D, the definition of a 1-linear functor from C to D consists of the same data and axioms as the definition of a strictly unital symmetric monoidal functor from C to D.

5.8. DEFINITION. Suppose P, Q are n-linear functors as displayed below.

$$\prod_{j=1}^{n} \mathsf{C}_j \overset{(P, \{P_j^2\})}{\underset{(Q, \{Q_j^2\})}{\Downarrow \theta}} \mathsf{D} \tag{32}$$

An *n-linear transformation* $\theta : P \longrightarrow Q$ is a natural transformation of underlying functors that satisfies the following two *multilinearity conditions*.

Unity

$$\theta_{\langle X \rangle} = 1_e \qquad \text{if any } X_i = e \in \mathsf{C}_i. \tag{33}$$

Constraint Compatibility The diagram

$$\begin{array}{ccc}
P\langle X \circ_i X_i \rangle \oplus P\langle X \circ_i X_i' \rangle & \xrightarrow{\ P_i^2\ } & P\langle X \circ_i (X_i \oplus X_i') \rangle \\
{\scriptstyle \theta \oplus \theta} \downarrow & & \downarrow {\scriptstyle \theta} \\
Q\langle X \circ_i X_i \rangle \oplus Q\langle X \circ_i X_i' \rangle & \xrightarrow{\ Q_i^2\ } & Q\langle X \circ_i (X_i \oplus X_i') \rangle
\end{array} \tag{34}$$

commutes for each $i \in \{1, \ldots, n\}$, $\langle X \rangle \in \prod_j \mathsf{C}_j$, and $X_i' \in \mathsf{C}_i$.

A *multilinear transformation* is an n-linear transformation for some $n \geq 0$. Identities and composition of multilinear transformations are given componentwise.

5.9. EXAMPLE. The definition of 1-linear transformation between 1-linear functors coincides with that of monoidal natural transformation between corresponding strictly unital symmetric monoidal functors.

5.10. DEFINITION. We define the following categories of multilinear functors and transformations.

- Let $\mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C} \rangle ; \mathsf{D}\big)$ denote the category of multilinear functors from $\langle \mathsf{C} \rangle$ to D and multilinear transformations between them.

- Let $\mathsf{PermCat}^{\mathsf{sus}}\big(\langle \mathsf{C} \rangle ; \mathsf{D}\big)$ denote the full subcategory of strong multilinear functors.

- Let $\mathsf{PermCat}^{\mathsf{st}}\big(\langle \mathsf{C} \rangle ; \mathsf{D}\big)$ denote the full subcategory of strict multilinear functors.

For each of these, the 1-linear case coincides with the notation of Definition 5.4.

Now we define the rest of the multicategory data for $\mathsf{PermCat}$.

5.11. DEFINITION. [Symmetric Group Action] Suppose given multilinear functors P and Q in $\mathsf{PermCat}(\langle \mathsf{C}\rangle\,;\,\mathsf{D})$ together with a multinatural transformation θ as displayed below.

$$
\prod_{j=1}^{n} \mathsf{C}_j \underset{(Q,\{Q_j^2\})}{\overset{(P,\{P_j^2\})}{\theta \Downarrow}} \mathsf{D} \tag{35}
$$

For a permutation $\sigma \in \Sigma_n$, the symmetric group action

$$
\mathsf{PermCat}(\langle \mathsf{C}\rangle\,;\,\mathsf{D}) \xrightarrow{\ \sigma\ } \mathsf{PermCat}(\langle \mathsf{C}\rangle\sigma\,;\,\mathsf{D}) \tag{36}
$$

sends the data (35) to the following composites and whiskerings, where ξ_σ denotes the isomorphism given by permutation of terms in the product.

$$
\prod_{j=1}^{n} \mathsf{C}_{\sigma(j)} \xrightarrow{\xi_\sigma} \prod_{j=1}^{n} \mathsf{C}_j \underset{(Q,\{Q_j^2\})}{\overset{(P,\{P_j^2\})}{\theta \Downarrow}} \mathsf{D} \tag{37}
$$

The j-th linearity constraint of $P^\sigma = P \circ \xi_\sigma$ is the composite in D

$$
\begin{array}{ccc}
P^\sigma\langle A\rangle \oplus P^\sigma\langle A \circ_j A_j'\rangle & \xrightarrow{\ (P^\sigma)_j^2\ } & P^\sigma\langle A \circ_j (A_j \oplus A_j')\rangle \\
\| & & \| \\
P(\sigma\langle A\rangle) \oplus P(\sigma\langle A\rangle \circ_{\sigma(j)} A_j') & \xrightarrow{\ P_{\sigma(j)}^2\ } & P(\sigma\langle A\rangle \circ_{\sigma(j)} (A_j \oplus A_j'))
\end{array} \tag{38}
$$

for objects

$$
\langle A\rangle = (A_1, \ldots, A_n) \in \prod_{j=1}^{n} \mathsf{C}_{\sigma(j)} \quad\text{and}\quad A_j' \in \mathsf{C}_{\sigma(j)}.
$$

Note that if P is strong, respectively strict, with each P_j^2 a natural isomorphism, respectively identity, then P^σ is also strong, respectively strict.

5.12. DEFINITION. [Composition] Suppose given, for each $j \in \{1, \ldots, n\}$,

- permutative categories $\langle \mathsf{B}_j\rangle = (\mathsf{B}_{j,1}, \ldots, \mathsf{B}_{j,k_j})$,

- objects P_j' and Q_j' and a 1-cell θ_j in $\mathsf{PermCat}(\langle \mathsf{B}_j\rangle\,;\,\mathsf{C}_j)$

as follows.

$$
\prod_{i=1}^{k_j} \mathsf{B}_{j,i} \underset{Q_j'}{\overset{P_j'}{\theta_j \Downarrow}} \mathsf{C}_j \tag{39}
$$

With $\langle \mathsf{B} \rangle = \big(\langle \mathsf{B}_1 \rangle, \ldots, \langle \mathsf{B}_n \rangle \big)$, the multicategorical composition functor

$$\mathsf{PermCat}\big(\langle \mathsf{C} \rangle ; \mathsf{D}\big) \times \prod_{j=1}^{n} \mathsf{PermCat}\big(\langle \mathsf{B}_j \rangle ; \mathsf{C}_j\big) \xrightarrow{\ \gamma\ } \mathsf{PermCat}\big(\langle \mathsf{B} \rangle ; \mathsf{D}\big) \tag{40}$$

sends the data (35) and (39) to the composites

$$\prod_{j=1}^{n} \prod_{i=1}^{k_j} \mathsf{B}_{j,i} \quad \overset{\overbrace{}^{P \circ \prod_j P'_j}}{\underset{\underbrace{}_{Q \circ \prod_j Q'_j}}{\theta \otimes (\prod_j \theta_j) \Big\Vert}} \quad \mathsf{D} \tag{41}$$

defined as follows.

Composite Multilinear Functor Suppose given tuples of objects

$$\langle W_j \rangle = (W_{j,1}, \ldots, W_{j,k_j}) \in \prod_{i=1}^{k_j} \mathsf{B}_{j,i} \qquad \text{for } j \in \{1, \ldots, n\} \text{ and}$$

$$\langle W \rangle = \big(\langle W_1 \rangle, \ldots, \langle W_n \rangle \big) \in \prod_{j=1}^{n} \prod_{i=1}^{k_j} \mathsf{B}_{j,i}. \tag{42}$$

Then we have the object

$$\big(P \circ \textstyle\prod_j P'_j \big) \langle W \rangle = P\big(P'_1 \langle W_1 \rangle, \ldots, P'_n \langle W_n \rangle \big) \quad \text{in} \quad \mathsf{D}. \tag{43}$$

To describe the linearity constraints of the composite $P \circ \prod_j P'_j$ in (41), in addition to the objects in (42), consider

- an object $W'_{j,i} \in \mathsf{B}_{j,i}$ for some choice of (j, i) with $\ell = k_1 + \cdots + k_{j-1} + i$ and
- $\langle P'W \rangle = \big(P'_1 \langle W_1 \rangle, \ldots, P'_n \langle W_n \rangle \big) \in \prod_{j=1}^{n} \mathsf{C}_j$.

Note that

$$\langle W_j \circ_i W'_{j,i} \rangle = \big(\overbrace{W_{j,1}, \ldots, W_{j,i-1}}^{\text{empty if } i = 1}, W'_{j,i}, \overbrace{W_{j,i+1}, \ldots, W_{j,k_j}}^{\text{empty if } i = k_j} \big)$$

$$\langle W_j \circ_i (W_{j,i} \oplus W'_{j,i}) \rangle = \big(\underbrace{W_{j,1}, \ldots, W_{j,i-1}}_{\text{empty if } i = 1}, W_{j,i} \oplus W'_{j,i}, \underbrace{W_{j,i+1}, \ldots, W_{j,k_j}}_{\text{empty if } i = k_j} \big).$$

The ℓ-th linearity constraint $\big(P \circ \prod_j P'_j \big)^2_\ell$ is defined as the following composite in D.

$$
\begin{array}{c}
P\langle P'W\rangle \oplus P\langle P'W \circ_j P'_j\langle W_j \circ_i W'_{j,i}\rangle\rangle
\end{array}
$$

(Diagram 44)

$$(44)$$

Note that if P and each P'_j are strong, respectively, strict, then each linearity constraint $\left(P \circ \prod_j P'_j\right)^2_\ell$ is componentwise invertible, respectively, an identity, and the composite $P \circ \prod_j P'_j$ is also strong, respectively, strict.

Composite Multinatural Transformation The multinatural transformation $\theta \otimes \left(\prod_j \theta_j\right)$ in (41) is the horizontal composite of the natural transformations $\prod_j \theta_j$ and θ. The component morphism $\left(\theta \otimes \left(\prod_j \theta_j\right)\right)_{\langle W\rangle}$ is the composite

$$
P\langle P'_j\langle W_j\rangle\rangle_j \xrightarrow{P\langle (\theta_j)_{\langle W_j\rangle}\rangle_j} P\langle Q'_j\langle W_j\rangle\rangle_j \xrightarrow{\theta_{\langle Q'_j\langle W_j\rangle\rangle_j}} Q\langle Q'_j\langle W_j\rangle\rangle_j \tag{45}
$$

in D.

The following construction of pointed multicategories from permutative categories leads to an alternative description of multilinearity via the smash product of pointed multicategories.

5.13. DEFINITION. Suppose C is a small permutative category. The *endomorphism multicategory* $\mathsf{End}(\mathsf{C})$ is the small multicategory with object set $\mathsf{Ob}\,\mathsf{C}$ and with

$$
\mathsf{End}(\mathsf{C})\big(\langle X\rangle\,;\,Y\big) = \mathsf{C}(X_1 \oplus \cdots \oplus X_n, Y)
$$

for $Y \in \mathsf{Ob}\,\mathsf{C}$ and $\langle X\rangle = (X_1, \cdots, X_n) \in (\mathsf{Ob}\,\mathsf{C})^{\times n}$. An empty $\oplus$ means the unit object e.

The *canonical basepoint* of $\mathsf{End}(\mathsf{C})$ is determined by the multifunctor

$$
\mathsf{T} \longrightarrow \mathsf{End}(\mathsf{C})
$$

sending the single object of T to the monoidal unit e in C and the n-ary operation ι_n to the identity morphism of

$$
\bigoplus_{i=1}^n e = e.
$$

Strictly unital monoidal functors, and monoidal natural transformations between them, induce pointed multifunctors and pointed multinatural transformations, respectively, on endomorphism multicategories. This defines a 2-functor

$$
\mathsf{End} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Multicat}
$$

by [JY∞, 5.3.6]. Equipped with canonical basepoints, End takes values in $\mathsf{Multicat}_*$.

The following result identifies multilinear functors and transformations between permutative categories with multifunctors and multinatural transformations of endomorphism categories.

5.14. PROPOSITION. [JY∞, 6.5.10 and 6.5.13] *For permutative categories* $\mathsf{C}_1, \ldots, \mathsf{C}_n$, *and* D, *the 2-functor* End *induces an isomorphism of categories*

$$\mathsf{End} : \mathsf{PermCat}^{\mathsf{su}}\big(\langle\mathsf{C}\rangle\,;\mathsf{D}\big) \overset{\cong}{\longrightarrow} \mathsf{Multicat}_*\big(\langle\mathsf{End}(\mathsf{C})\rangle\,;\mathsf{End}(\mathsf{D})\big)$$
$$= \mathsf{Multicat}_*\big(\wedge_i\,\mathsf{End}(\mathsf{C}_i),\mathsf{End}(\mathsf{D})\big),$$

between the category of n-linear functors and transformations $\langle\mathsf{C}\rangle \longrightarrow \mathsf{D}$ *and the category of pointed multifunctors and multinatural transformations* $\wedge_i\mathsf{End}(\mathsf{C}_i) \longrightarrow \mathsf{End}(\mathsf{D})$.

5.15. DEFINITION. Let $\mathsf{PermCat}^{\mathsf{su}}$ denote the Cat-enriched multicategory whose category of n-ary operations

$$\mathsf{PermCat}^{\mathsf{su}}\big(\langle\mathsf{C}\rangle\,;\mathsf{D}\big)$$

is the category of n-linear functors and n-linear transformations

$$\langle\mathsf{C}\rangle \longrightarrow \mathsf{D}.$$

The symmetric group action and composition in $\mathsf{PermCat}^{\mathsf{su}}$ are those of Definitions 5.11 and 5.12, respectively. The multicategory axioms for $\mathsf{PermCat}^{\mathsf{su}}$ follow from Proposition 5.14 and the symmetric monoidal axioms of the smash product. Independently, a direct verification is given in [JY∞, Section 6.6].

We let $\mathsf{PermCat}^{\mathsf{sus}}$ denote the Cat-enriched sub-multicategory whose n-ary operations consist of strong n-linear functors and n-linear transformations.

5.16. REMARK. The multicategory structure on $\mathsf{PermCat}^{\mathsf{su}}$ is not induced by a symmetric monoidal structure. For example, the unit for the smash product of multicategories is not a permutative category. See [JY∞, Propositions 5.7.23 and 10.2.17] for further discussion of this point.

5.17. REMARK. [Strict Unity] In the rest of this paper, we mainly work with $\mathsf{PermCat}^{\mathsf{su}}$ instead of other variants in Definitions 5.4 and 5.10. We now briefly discuss the technical advantages of $\mathsf{PermCat}^{\mathsf{su}}$ over other variants.

1. For a symmetric monoidal functor $P : \mathsf{C} \longrightarrow \mathsf{D}$ between small permutative categories, the endomorphism multifunctor

$$\mathsf{End}(P) : \mathsf{End}(\mathsf{C}) \longrightarrow \mathsf{End}(\mathsf{D})$$

   is *not* a pointed multifunctor in general, where $\mathsf{End}(\mathsf{C})$ and $\mathsf{End}(\mathsf{D})$ are equipped with the canonical basepoints given by their respective monoidal units. The multifunctor $\mathsf{End}(P)$ is pointed precisely when P is strictly unital. The Cat-multicategory structure on $\mathsf{PermCat}^{\mathsf{su}}$ is canonically induced by the one on $\mathsf{Multicat}_*$ via End, as in Proposition 5.14. In this sense, $\mathsf{PermCat}^{\mathsf{su}}$ is more convenient than $\mathsf{PermCat}$.

2. As we recall in Proposition 6.6 below, the free permutative category 2-functor $\mathbf{F}$ has codomain the 2-category $\mathsf{PermCat^{st}}$, with strict symmetric monoidal functors as 1-cells. However, in order to extend $\mathbf{F}$ to a non-symmetric Cat-multifunctor (Theorem 8.1), we will need to precompose the assignment of $\mathbf{F}$ on 1-cells and 2-cells with a *strong* multilinear functor S; see (64). While S (Definition 7.4) is a strong multilinear functor, it is not strict because its linearity constraints, defined in (61), are generally not identity morphisms. Thus, the non-symmetric Cat-multicategorical extension of $\mathbf{F}$ should not use the codomain $\mathsf{PermCat^{st}}$, since we must allow non-identity linearity constraints.

3. Segal's K-theory functor [Seg74]

$$\mathsf{K^{Se}} : \mathsf{PermCat^{su}} \longrightarrow \mathsf{SymSp}$$

is most naturally defined on $\mathsf{PermCat^{su}}$; see [JY∞, Chapter 8] for a detailed discussion of $\mathsf{K^{Se}}$. In Section 10 we will use $\mathsf{K^{Se}}$ to define stable equivalences in $\mathsf{PermCat^{su}}$ and $\mathsf{Multicat}$. The functor $\mathsf{K^{Se}}$ is a composite of three functors, the first of which, called Segal J-theory and denoted $\mathsf{J^{Se}}$, lands in the category $\Gamma\text{-}\mathsf{Cat}$ of Γ-categories. For a small permutative category C, each level of the Γ-category $\mathsf{J^{Se}C}$ is a category in which an object is a system of objects $\{C_s\}$ in C along with some gluing morphisms, satisfying several axioms. Among these axioms is an object unity axiom—see [JY∞, (8.3.2)]—that says

$$C_\varnothing = e,$$

the monoidal unit in C. For P as in (1) above, in order for P to induce a morphism of Γ-categories

$$\mathsf{J^{Se}}P : \mathsf{J^{Se}C} \longrightarrow \mathsf{J^{Se}D},$$

we need the condition

$$P(C_\varnothing) = Pe = e,$$

the monoidal unit in D. In other words, we need P to be strictly unital.

6. Free Permutative Category on a Multicategory

In this section we recall from [JY22, Section 5] the free construction

$$\mathbf{F} : \mathsf{Multicat} \longrightarrow \mathsf{PermCat^{su}} .$$

In Section 8 we show that $\mathbf{F}$ is a non-symmetric Cat-enriched multifunctor. The definition of $\mathbf{F}$ makes use of sequences $\langle x \rangle$, indexing functions f, and permutations $\sigma^k_{g,f}$. We make the following preliminary definitions.

6.1. DEFINITION. For a natural number $r \geq 0$ we let

$$\overline{r} = \{1, \ldots, r\}$$

denote the finite set with r elements.

6.2. DEFINITION. Suppose M is a multicategory, and suppose $\langle x \rangle$ is a sequence of length r, with each $x_i \in \mathsf{M}$. Suppose

$$f : \bar{r} \longrightarrow \bar{s} \quad \text{and} \quad g : \bar{s} \longrightarrow \bar{t}$$

are functions of finite sets, for $r, s, t \geq 0$. Then we define the following.

- For $j \in \bar{s}$, let

$$\langle x \rangle_{f^{-1}(j)} = \langle x_i \rangle_{i \in f^{-1}(j)} \tag{46}$$

  denote the sequence formed by those x_i with $i \in f^{-1}(j)$, ordered as in $\langle x \rangle$. Similarly, for a length-s sequence of operations $\langle \phi \rangle$ in M and $k \in \bar{t}$, let

$$\langle \phi \rangle_{g^{-1}(k)} = \langle \phi_j \rangle_{j \in g^{-1}(k)}.$$

- For $k \in \bar{t}$, let $\sigma^k_{g,f} \in \Sigma_t$ be the unique permutation such that

$$\left[\bigoplus_{j \in g^{-1}(k)} \langle x \rangle_{f^{-1}(j)} \right] \cdot \sigma^k_{g,f} = \langle x \rangle_{(gf)^{-1}(k)}, \tag{47}$$

  where the sequence on the left hand side is the concatenation of sequences in the order specified by $g^{-1}(k)$. We will use the action of these permutations on both objects and operations.

6.3. DEFINITION. Suppose M is a multicategory. Define a permutative category $\mathbf{FM}$, called the *free permutative category on* M, as follows.

Objects The objects of $\mathbf{FM}$ are given by the $(\mathsf{Ob}\,\mathsf{M})$-profiles: finite ordered sequences $\langle x \rangle = (x_1, \ldots, x_r)$ of objects of M, with $r \geq 0$.

Morphisms Given sequences $\langle x \rangle$ and $\langle y \rangle$ with lengths r and s, respectively, the morphisms from $\langle x \rangle$ to $\langle y \rangle$ in $\mathbf{FM}$ are given by pairs $(f, \langle \phi \rangle)$ consisting of

- a function

$$f : \bar{r} \longrightarrow \bar{s}$$

  called the *index map* and

- an ordered sequence of operations

$$\langle \phi \rangle \quad \text{with} \quad \phi_j \in \mathsf{M}\big(\langle x \rangle_{f^{-1}(j)} \, ; \, y_j \big)$$

  for $j \in \bar{s}$, with $\langle x \rangle_{f^{-1}(j)}$ defined in (46).

The identity morphism on $\langle x \rangle$ is given by $1_{\bar{r}}$ and the tuple of unit operations 1_{x_i}.

Composition The composition of a pair of morphisms

$$\langle x \rangle \xrightarrow{(f,\langle \phi \rangle)} \langle y \rangle \xrightarrow{(g,\langle \psi \rangle)} \langle z \rangle$$

is the pair

$$\left(gf, \langle \theta_k \cdot \sigma_{g,f}^k \rangle_{k \in \bar{t}} \right), \tag{48}$$

where, for each $k \in \bar{t}$,

$$\theta_k = \gamma \left(\psi_k ; \langle \phi \rangle_{g^{-1}(k)} \right) \in \mathsf{M} \left(\bigoplus_{j \in g^{-1}(k)} \langle x \rangle_{f^{-1}(j)} ; z_k \right). \tag{49}$$

Note that the input profile for θ_k is the concatenation of $\langle x \rangle_{f^{-1}(j)}$ for $j \in g^{-1}(k)$. By definition (47), the right action of $\sigma_{g,f}^k$ permutes this input profile to $\langle x \rangle_{(gf)^{-1}(k)}$. Composition of morphisms is verified to be unital and associative in [JY22, Proposition 5.7].

Monoidal Sum The monoidal sum

$$\oplus : \mathsf{FM} \times \mathsf{FM} \longrightarrow \mathsf{FM}$$

is given on objects by concatenation of sequences. The monoidal sum of morphisms

$$(f, \langle \phi \rangle) : \langle x \rangle \longrightarrow \langle y \rangle \quad \text{and} \quad (f', \langle \phi' \rangle) : \langle x' \rangle \longrightarrow \langle y' \rangle,$$

is the pair

$$\left(f \oplus f', \langle \phi \rangle \oplus \langle \phi' \rangle \right)$$

where $f \oplus f'$ denotes the composite

$$\overline{r + r'} \cong \bar{r} \amalg \overline{r'} \xrightarrow{f \amalg f'} \bar{s} \amalg \overline{s'} \cong \overline{s + s'}$$

given by the disjoint union of f with f' and the canonical order-preserving isomorphisms. Functoriality of the monoidal sum follows because disjoint union of indexing functions preserves preimages and the operations in a composite (48) are determined elementwise for the indexing set of the codomain.

Monoidal Unit The monoidal unit is the empty sequence $\langle \rangle$. The unit and associativity isomorphisms for $\oplus$ are identities.

Symmetry The symmetry isomorphism for sequences $\langle x \rangle$ of length r and $\langle x' \rangle$ of length r' is

$$\xi_{\langle x \rangle, \langle x' \rangle} = \left(\tau_{r,r'}, \langle 1 \rangle \right) \tag{50}$$

where

$$\tau_{r,r'} : \overline{r + r'} \cong \bar{r} \amalg \overline{r'} \longrightarrow \overline{r'} \amalg \bar{r} \cong \overline{r' + r}$$

is induced by the block-transposition of $\bar{r}$ with $\overline{r'}$, keeping the relative order within each block fixed.

Concatenation of sequences is strictly associative and unital. The symmetry and hexagon axioms (21) follow from the corresponding equalities of block permutations.

This completes the definition of **FM**.

6.4. DEFINITION. Suppose $H : \mathsf{M} \longrightarrow \mathsf{N}$ is a multifunctor. Define a strict symmetric monoidal functor

$$\mathbf{F}H : \mathbf{F}\mathsf{M} \longrightarrow \mathbf{F}\mathsf{N}$$

via the following assignment on objects and morphisms. For a sequence $\langle x \rangle$ of length r, define

$$(\mathbf{F}H)\langle x \rangle = \langle H x_i \rangle_{i \in \overline{r}}.$$

For a morphism $(f, \langle \phi \rangle)$, define

$$(\mathbf{F}H)(f, \langle \phi \rangle) = (f, \langle H\phi_j \rangle_j). \tag{51}$$

The multifunctoriality of H shows that this assignment is functorial on morphisms. Since the monoidal sum is defined by concatenation in $\mathbf{F}\mathsf{M}$ and $\mathbf{F}\mathsf{N}$, the functor $\mathbf{F}H$ is strict monoidal. Compatibility with the symmetry of $\mathbf{F}\mathsf{M}$ and $\mathbf{F}\mathsf{N}$ follows because $\mathbf{F}H$ preserves the index map of each morphism and H preserves unit operations.

6.5. DEFINITION. Suppose $\kappa : H \longrightarrow K : \mathsf{M} \longrightarrow \mathsf{N}$ is a multinatural transformation. Define a monoidal natural transformation

$$\mathbf{F}\kappa : \mathbf{F}H \longrightarrow \mathbf{F}K$$

via components

$$(\mathbf{F}\kappa)_{\langle x \rangle} = (1, \langle \kappa_{x_i} \rangle_i) : \langle Hx \rangle \longrightarrow \langle Kx \rangle \tag{52}$$

for each sequence $\langle x \rangle$ in $\mathbf{F}\mathsf{M}$. Naturality of $\mathbf{F}\kappa$ follows from multinaturality of κ (13) because each $\sigma_{f,1}^j$ and $\sigma_{1,f}^j$ is an identity permutation and we have

$$(1, \langle \kappa_{y_j} \rangle_j)(f, \langle H\phi_j \rangle_j) = \left(f, \left\langle \gamma\big(\kappa_{y_j} \, ; \, H\phi_j\big) \right\rangle_j \right)$$
$$= \left(f, \left\langle \gamma\big(K\phi_j \, ; \, \langle \kappa_{x_i} \rangle_{i \in f^{-1}(j)}\big) \right\rangle_j \right)$$
$$= (f, \langle K\phi_j \rangle_j)(1, \langle \kappa_{x_i} \rangle_i)$$

for each morphism $(f, \langle \phi \rangle) : \langle x \rangle \longrightarrow \langle y \rangle$ in $\mathbf{F}\mathsf{M}$.

The monoidal naturality axioms (25) for $\mathbf{F}\kappa$ follow because the monoidal sum in $\mathbf{F}\mathsf{N}$ is given by concatenation of object and operation sequences. The component $(\mathbf{F}\kappa)_{\langle \rangle}$ is the identity morphism $(1_\varnothing, \langle \rangle) : \langle \rangle \longrightarrow \langle \rangle$.

6.6. PROPOSITION. [JY22, Proposition 5.13] *The free permutative category construction given in Definitions 6.3 and 6.4 provides a 2-functor*

$$\mathbf{F} : \mathsf{Multicat} \longrightarrow \mathsf{PermCat}^{\mathsf{st}}.$$

6.7. EXAMPLE. For the terminal multicategory T, the free permutative category $\mathbf{FT}$ is isomorphic to the natural number category N whose objects are given by natural numbers and morphisms are given by morphisms of finite sets

$$\mathsf{N}(r, s) = \mathsf{Set}(\bar{r}, \bar{s}).$$

The natural number $r \in \mathsf{N}$ corresponds to the length-r sequence whose terms are the unique object of T. Each morphism $f : \bar{r} \longrightarrow \bar{s}$ corresponds to the morphism

$$(f, \langle \phi \rangle) \in \mathbf{FT}$$

where ϕ_j is the unique operation in T of arity $|f^{-1}(j)|$.

6.8. EXAMPLE. For the initial operad I, the free permutative category $\mathbf{FI}$ is isomorphic to the permutation category Σ with objects given by natural numbers and morphisms given by permutations

$$\Sigma(r, s) = \begin{cases} \Sigma_r, & \text{if} \quad r = s, \\ \varnothing, & \text{if} \quad r \neq s \end{cases}$$

for each pair of natural numbers r and s.

7. Assignment on Multimorphism Categories

Throughout this section we suppose $n \geq 0$ and consider small multicategories $\mathsf{M}_1, \ldots, \mathsf{M}_n$, and N. The purpose of this section is to define the assignment on multimorphism categories

$$\mathbf{F} : \mathsf{Multicat}\big(\langle \mathsf{M} \rangle \,;\, \mathsf{N}\big) \longrightarrow \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathbf{FM} \rangle \,;\, \mathbf{FN}\big).$$

This assignment is provided by a strong n-linear functor of permutative categories

$$S : \langle \mathbf{FM} \rangle \longrightarrow \mathbf{F}\Big(\bigotimes_{i=1}^{n} \mathsf{M}_i \Big)$$

and the 2-functoriality of $\mathbf{F}$. We will use the following notation in the definition of S below, in the case $n > 0$.

7.1. DEFINITION. Suppose given objects

$$\langle x^i \rangle \in \mathbf{FM}_i \quad \text{for} \quad 1 \leq i \leq n,$$

where each $\langle x^i \rangle = (x_1^i, \ldots, x_{r_i}^i)$. So $\langle x^i \rangle$ has length r_i and each x_j^i is an object of M_i. For each n-tuple of indices $(j_1, \ldots, j_n)$ with $1 \leq j_i \leq r_i$, define

$$x_{j_1, \ldots, j_n}^{1 \cdots n} = \big(x_{j_1}^1, x_{j_2}^2, \ldots, x_{j_n}^n \big) \in \bigotimes_{i=1}^{n} \mathsf{M}_i.$$

Then define an object

$$\langle x^{1\cdots n}\rangle \in \mathbf{F}\Big(\bigotimes_{i=1}^{n}\mathsf{M}_i\Big)$$

of length $r_{1\cdots n}$, where

$$r_{1\cdots n} = \prod_{i=1}^{n} r_i \quad \text{and} \tag{53}$$

$$\langle x^{1\cdots n}\rangle = \Big\langle\cdots\Big\langle x_{j_1,\ldots,j_n}^{1\cdots n}\Big\rangle_{j_1=1}^{r_1}\cdots\Big\rangle_{j_n=1}^{r_n}. \tag{54}$$

Using the tensor product of profiles from Definition 4.1, the tuple $\langle x^{1\cdots n}\rangle$ from (54) is the iterated tensor product of the tuples $\langle x^i\rangle$, with the reverse lexicographic ordering in the subscripts.

7.2. EXAMPLE. To illustrate Definition 7.1, consider the case with $n = 3$ and the following objects.

$$\langle x^1\rangle = (x_1^1, x_2^1) \in \mathbf{F}\mathsf{M}_1$$
$$\langle x^2\rangle = (x_1^2, x_2^2, x_3^2) \in \mathbf{F}\mathsf{M}_2 \tag{55}$$
$$\langle x^3\rangle = (x_1^3, x_2^3) \in \mathbf{F}\mathsf{M}_3$$

For each triple of indices (j_1, j_2, j_3) with $j_1 \in \{1,2\}$, $j_2 \in \{1,2,3\}$, and $j_3 \in \{1,2\}$, we have the object

$$x_{j_1,j_2,j_3}^{123} = \left(x_{j_1}^1, x_{j_2}^2, x_{j_3}^3\right) \in \mathsf{M}_1 \otimes \mathsf{M}_2 \otimes \mathsf{M}_3.$$

This object is obtained from the array (55) by forming a vertical product, using the j_1-th object in the first row, the j_2-th object in the second row, and the j_3-th object in the third row. For example, for the indices $(j_1, j_2, j_3) = (2, 3, 1)$, we have the object

$$x_{231}^{123} = (x_2^1, x_3^2, x_1^3) \in \mathsf{M}_1 \otimes \mathsf{M}_2 \otimes \mathsf{M}_3.$$

The object in (54)

$$\langle x^{123}\rangle \in \mathbf{F}(\mathsf{M}_1 \otimes \mathsf{M}_2 \otimes \mathsf{M}_3)$$

is the following sequence of $2 \cdot 3 \cdot 2 = 12$ objects, with each object in $\mathsf{M}_1 \otimes \mathsf{M}_2 \otimes \mathsf{M}_3$, read left to right in the first row and then the second row.

$j_2 = 1$		$j_2 = 2$		$j_2 = 3$		
x_{111}^{123}	x_{211}^{123}	x_{121}^{123}	x_{221}^{123}	x_{131}^{123}	x_{231}^{123}	$j_3 = 1$
x_{112}^{123}	x_{212}^{123}	x_{122}^{123}	x_{222}^{123}	x_{132}^{123}	x_{232}^{123}	$j_3 = 2$
$j_2 = 1$		$j_2 = 2$		$j_2 = 3$		

In the previous display, the two rows correspond to $j_3 = 1$ and 2, as indicated by the last subscripts. In each row, the three pairs from left to right correspond to $j_2 = 1, 2$, and 3, as indicated by the middle subscripts. Within each pair, the two objects correspond to $j_1 = 1$ and 2, as indicated by the first subscript.

7.3. Definition. Suppose given objects and morphisms

$$(f^i, \langle \phi^i \rangle) : \langle x^i \rangle \longrightarrow \langle y^i \rangle \quad \text{in} \quad \mathbf{FM}_i \quad \text{for} \quad 1 \le i \le n,$$

where

- each $\langle x^i \rangle$ has length r_i,

- each $\langle y^i \rangle$ has length s_i,

- each $f^i : \overline{r_i} \longrightarrow \overline{s_i}$, and

- each $\phi^i_j \in \mathsf{M}_i\big(\langle x^i \rangle_{(f^i)^{-1}(j)} \, ; \, y^i_j\big)$.

Define $f^{1 \cdots n}$ as the composite below, where the unlabeled isomorphisms are given by the reverse lexicographic ordering of the products:

$$\overline{r_{1 \cdots n}} \xrightarrow{\ \cong\ } \textstyle\prod_i \overline{r_i} \xrightarrow{\ \prod_i f^i\ } \textstyle\prod_i \overline{s_i} \xrightarrow{\ \cong\ } \overline{s_{1 \cdots n}}. \tag{56}$$

$$\underset{f^{1 \cdots n}}{\underbrace{\hspace{6cm}}}$$

For each n-tuple of indices $(k_1, \ldots, k_n)$ with $1 \le k_i \le s_i$, let

$$\langle x^{1 \cdots n} \rangle_{f; k_1, \ldots, k_n} = \left\langle \cdots \left\langle x^{1 \cdots n}_{j_1, \ldots, j_n} \right\rangle_{j_1 \in (f^1)^{-1}(k_1)} \cdots \right\rangle_{j_n \in (f^n)^{-1}(k_n)}$$

and define

$$\phi^{1 \cdots n}_{k_1, \ldots, k_n} : \langle x^{1 \cdots n} \rangle_{f; k_1, \ldots, k_n} \longrightarrow y^{1 \cdots n}_{k_1, \ldots, k_n} \tag{57}$$

as the tensor product $\otimes_{i=1}^n \phi^i_{k_i}$. Then define

$$\langle \phi^{1 \cdots n} \rangle = \left\langle \cdots \left\langle \phi^{1 \cdots n}_{k_1, \ldots, k_n} \right\rangle^{s_1}_{k_1 = 1} \cdots \right\rangle^{s_n}_{k_n = 1}. \tag{58}$$

This defines a morphism

$$(f^{1 \cdots n}, \langle \phi^{1 \cdots n} \rangle) : \langle x^{1 \cdots n} \rangle \longrightarrow \langle y^{1 \cdots n} \rangle$$

in $\mathbf{F}\big(\otimes_{i=1}^n \mathsf{M}_i\big)$.

Recall the initial operad I (Definition 3.4), which is the monoidal unit for the tensor product of small multicategories (Theorem 4.6). Also recall from Example 6.8 that $\mathbf{F}(\mathsf{I})$ is the permutation category Σ.

7.4. DEFINITION. [Multilinear S] Suppose $n \geq 0$ and suppose given small multicategories $\mathsf{M}_1, \ldots, \mathsf{M}_n$. Define an n-linear functor

$$(S, S_b^2) : \prod_{i=1}^n \mathbf{F}\mathsf{M}_i \longrightarrow \mathbf{F}\Big(\bigotimes_{i=1}^n \mathsf{M}_i \Big)$$

as follows.

For $n = 0$, we define S by choice of object $1 \in \Sigma$. For $n > 0$, we make the following definitions. Suppose given objects and morphisms

$$(f^i, \langle \phi^i \rangle) : \langle x^i \rangle \longrightarrow \langle y^i \rangle \quad \text{in} \quad \mathbf{F}\mathsf{M}_i \quad \text{for} \quad 1 \leq i \leq n,$$

as in Definition 7.3.

Underlying Functor The underlying functor S is given by the following assignments, using (54), (56), and (58):

$$S\big(\langle x^1 \rangle, \ldots, \langle x^n \rangle \big) = \langle x^{1 \cdots n} \rangle \quad \text{and} \tag{59}$$
$$S\big((f^1, \langle \phi^1 \rangle), \ldots, (f^n, \langle \phi^n \rangle) \big) = \big(f^{1 \cdots n}, \langle \phi^{1 \cdots n} \rangle \big). \tag{60}$$

Linearity Constraints Suppose $1 \leq b \leq n$ and suppose $\langle \hat{x}^b \rangle$ is an object in $\mathbf{F}\mathsf{M}_b$ with length $\hat{r}_b$. Let

$$\tilde{r}_b = r_b + \hat{r}_b \quad \text{and} \quad \langle \tilde{x}^b \rangle = \langle x^b \rangle \oplus \langle \hat{x}^b \rangle.$$

Then define

$$\langle \hat{x}^{1 \cdots n} \rangle \quad \text{and} \quad \langle \tilde{x}^{1 \cdots n} \rangle$$

as in (54), using $\langle \hat{x}^b \rangle$ and $\langle \tilde{x}^b \rangle$, respectively, in place of $\langle x^b \rangle$.

The bth linearity constraint, S_b^2, is defined by components

$$S_b^2 = \big(\rho_{r_b, \hat{r}_b}, \langle 1 \rangle \big) : \langle x^{1 \cdots n} \rangle \oplus \langle \hat{x}^{1 \cdots n} \rangle \longrightarrow \langle \tilde{x}^{1 \cdots n} \rangle \tag{61}$$

where $\langle 1 \rangle$ is the tuple of identity operations and $\rho_{r_b, \hat{r}_b}$ is the unique permutation of entries determined by the source and target of S_b^2. Naturality of S_b^2 with respect to morphisms in $\prod_i \mathbf{F}\mathsf{M}_i$ follows from the uniqueness of $\rho_{r_b, \hat{r}_b}$.

This finishes the definition of S as an assignment on objects and morphisms, and the definition of natural transformations S_b^2. We verify that S is functorial in Proposition 7.5. We verify the multilinearity axioms of Definition 5.6 in Proposition 7.6.

7.5. PROPOSITION. *In the context of Definition 7.4, S is a functor.*

PROOF. If each $(f^i, \langle \phi^i \rangle)$ is an identity, then so are $f^{1 \cdots n}$ and each $\langle \phi^{1 \cdots n} \rangle$. Therefore S preserves identities.

Suppose given composable morphisms

$$\langle x^i \rangle \xrightarrow{(f^i, \langle \phi^i \rangle)} \langle y^i \rangle \xrightarrow{(g^i, \langle \psi^i \rangle)} \langle z^i \rangle \quad \text{in} \quad \mathbf{FM}_i$$

for $1 \leq i \leq n$. Then by (48) and (49) we have

$$(g^i, \langle \psi^i \rangle) \circ (f^i, \langle \phi^i \rangle) = (g^i f^i, \langle \theta^i_{\ell_i} \cdot \sigma^{\ell_i}_{g^i, f^i} \rangle_{\ell_i}), \quad \text{with} \quad \theta^i_{\ell_i} = \gamma \left(\psi^i_{\ell_i} ; \langle \phi^i \rangle_{(g^i)^{-1}(\ell_i)} \right).$$

Applying S to the tuple of these composites, we have

$$S \left(\prod_i (g^i f^i, \langle \theta^i_{\ell_i} \cdot \sigma^{\ell_i}_{g^i, f^i} \rangle_{\ell_i}) \right) = \left(h^{1 \cdots n}, \omega^{1 \cdots n} \right),$$

where $h^i = g^i f^i$ and

$$\omega^{1 \cdots n}_{\ell_1, \ldots, \ell_n} = \otimes_i \left(\theta^i_{\ell_i} \cdot \sigma^{\ell_i}_{g^i, f^i} \right).$$

Alternatively, applying S and then composing results in the following:

$$S \left(\prod_i (g^i, \langle \psi^i \rangle) \right) \circ S \left(\prod_i (f^i, \langle \phi^i \rangle) \right) = \left(g^{1 \cdots n}, \langle \psi^{1 \cdots n} \rangle \right) \circ \left(f^{1 \cdots n}, \langle \phi^{1 \cdots n} \rangle \right)$$

$$= \left(g^{1 \cdots n} f^{1 \cdots n}, \langle \pi^{1 \cdots n}_{\ell_1, \ldots, \ell_n} \cdot \sigma^{\ell_1, \ldots, \ell_n}_{g^{1 \cdots n}, f^{1 \cdots n}} \rangle_{(\ell_1, \ldots, \ell_n)} \right)$$

where

$$\pi^{1 \cdots n}_{\ell_1, \ldots, \ell_n} = \gamma \left(\psi^{1 \cdots n}_{\ell_1, \ldots, \ell_n} ; \langle \phi^{1 \cdots n} \rangle_{(g^{1 \cdots n})^{-1}(\ell_1, \ldots, \ell_n)} \right)$$

and $\sigma^{\ell_1, \ldots, \ell_n}_{g^{1 \cdots n}, f^{1 \cdots n}}$ is the permutation as in (47) corresponding to the index $(\ell_1, \ldots, \ell_n)$.

Functoriality of the Cartesian product for maps of sets implies that $h^{1 \cdots n} = g^{1 \cdots n} f^{1 \cdots n}$. To show that S is functorial, it remains to show

$$\otimes_i \left(\theta^i_{\ell_i} \cdot \sigma^{\ell_i}_{g^i, f^i} \right) = \gamma \left(\psi^{1 \cdots n}_{\ell_1, \ldots, \ell_n} ; \langle \phi^{1 \cdots n} \rangle_{(g^{1 \cdots n})^{-1}(\ell_1, \ldots, \ell_n)} \right) \cdot \sigma^{\ell_1, \ldots, \ell_n}_{g^{1 \cdots n}, f^{1 \cdots n}} \tag{62}$$

for each index $(\ell_1, \ldots, \ell_n)$.

Since the domain of S is a Cartesian product, it suffices to verify (62) in the cases where $(f^i, \langle \phi^i \rangle)$ is an identity for $i \neq a$ and $(g^i, \langle \phi^i \rangle)$ is an identity for $i \neq b$, for some $1 \leq a \leq n$ and $1 \leq b \leq n$.

If $a = b$, then (62) follows from relations among operations in the tensor product, Definition 4.4 (2) through (5). If $a \neq b$ then the permutations $\sigma^{\ell_i}_{g^i, f^i}$ on the left hand side of (62) are identities. For $a < b$, the permutation $\sigma^{\ell_1, \ldots, \ell_n}_{g^{1 \cdots n}, f^{1 \cdots n}}$ on the right hand side of (62) is also an identity and there is nothing to check. For $a > b$, the permutation $\sigma^{\ell_1, \ldots, \ell_n}_{g^{1 \cdots n}, f^{1 \cdots n}}$ is an instance of $\xi^{\otimes}$ and the equality holds by Definition 4.4 (6).   ∎

7.6. PROPOSITION. *In the context of Definition 7.4, (S, S_b^2) is a strong multilinear functor.*
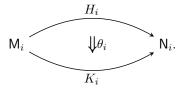
PROOF. Functoriality of S is verified in Proposition 7.5. Now we verify the multilinearity axioms of Definition 5.6. For $n = 0$ there is nothing to check. For $n > 0$, first note that, if any $\langle x^i \rangle$ is the empty tuple, then so is $\langle x^{1 \cdots n} \rangle$. Similarly, if any $(f^i, \langle \phi^i \rangle)$ is equal to

$$\left(\varnothing : \overline{0} \longrightarrow \overline{0}, \langle \rangle \right) : \langle \rangle \longrightarrow \langle \rangle,$$

the identity morphism of the empty tuple, then $f^{1 \cdots n}$ is the empty morphism and $\langle \phi^{1 \cdots n} \rangle$ is also empty. Thus S satisfies the unity axiom (27) of Definition 5.6.

The constraint unity axiom (28) for S_b^2 holds because the permutations $\rho_{0, \hat{r}_b}$ and $\rho_{r_b, 0}$ are identities. The other three constraint axioms of Definition 5.6 for S_b^2 follow from uniqueness of the permutations $\rho_{r_b, \hat{r}_b}$. Since the components S_b^2 are determined by permutations, S is a strong multilinear functor. $\blacksquare$

7.7. REMARK. In the context of Definition 7.4, for the case $b = n$, the permutations $\rho_{r_n, \hat{r}_n}$ are identities for any r_n and $\hat{r}_n$. In particular, if $n = 1$ then S is the identity monoidal functor. For $n > 1$ and $b < n$, the permutations $\rho_{r_b, \hat{r}_b}$ are generally nontrivial.

7.8. LEMMA. *The multilinear functors S are 2-natural with respect to multifunctors and multinatural transformations*

PROOF. First we verify that the following diagram of permutative categories and multilinear functors commutes.

$$
\begin{array}{ccc}
\prod_i \mathbf{F}\mathsf{M}_i & \xrightarrow{\ \Pi_i \mathbf{F} H_i\ } & \prod_i \mathbf{F}\mathsf{N}_i \\
\Big\downarrow {\scriptstyle S} & & \Big\downarrow {\scriptstyle S} \\
\mathbf{F}\left(\bigotimes_i \mathsf{M}_i \right) & \xrightarrow{\ \mathbf{F}(\otimes_i H_i)\ } & \mathbf{F}\left(\bigotimes_i \mathsf{N}_i \right)
\end{array}
\tag{63}
$$

Commutativity on objects and morphisms follows because, by Definition 6.4, the strict monoidal functor $\mathbf{F}(\otimes_i H_i)$ is given by applying $\otimes_i H_i$ componentwise to tuples of objects and operations. Therefore, both composites around the diagram above are given on objects and morphisms by the assignments

$$
\left(\langle x^1 \rangle, \ldots, \langle x^n \rangle \right) \longmapsto \langle (Hx)^{1 \cdots n} \rangle
$$
$$
\left((f^1, \langle \phi^1 \rangle), \ldots, (f^n, \langle \phi^n \rangle) \right) \longmapsto (f^{1 \cdots n}, \langle (H\phi)^{1 \cdots n} \rangle),
$$

where

$$
(Hx)^{1\cdots n}_{j_1,\ldots,j_n} = \otimes_i (H_i x^i_{j_i}) \quad \text{and}
$$
$$
(H\phi)^{1\cdots n}_{k_1,\ldots,k_n} = \otimes_i (H_i \phi^i_{k_i}).
$$

Because $\mathbf{F}(\otimes_i H_i)$ is strict monoidal, the bth linearity constraint of both composites around the diagram is given by

$$
(\rho_{r_b,\hat{r}_b}, \langle 1 \rangle) = \Big(\mathbf{F}(\otimes_i H_i)\Big)(S^2_b).
$$

Thus the two composites in (63) are equal as multilinear functors.

For multinatural transformations $\theta_i : H_i \longrightarrow K_i$, a similar analysis using Definition 6.5 shows that

$$
1_S * \Big(\textstyle\prod_i \mathbf{F}\theta_i \Big) = \mathbf{F}\Big(\otimes_i \theta_i \Big) * 1_S
$$

This completes the proof of 2-naturality for S.   ∎

Now we define $\mathbf{F}$ on multimorphism categories.

7.9. CONVENTION. To avoid ambiguity, we let

$$
\overline{\mathbf{F}} : \mathsf{Multicat}(\mathsf{M}, \mathsf{N}) \longrightarrow \mathsf{PermCat}^{\mathsf{su}}(\mathbf{F}\mathsf{M}, \mathbf{F}\mathsf{N})
$$

denote the assignment $\mathbf{F}$ on 1- and 2-cells as in Definitions 6.4 and 6.5.

7.10. DEFINITION. Define an assignment on multimorphism categories

$$
\mathbf{F} : \mathsf{Multicat}\big(\langle \mathsf{M}\rangle\,;\,\mathsf{N}\big) \longrightarrow \mathsf{PermCat}^{\mathsf{su}}\big(\langle\mathbf{F}\mathsf{M}\rangle\,;\,\mathbf{F}\mathsf{N}\big)
$$

by sending data such as the following

$$
\langle \mathsf{M}\rangle \underset{K}{\overset{H}{\rightrightarrows}} \Downarrow\theta \;\; \mathsf{N}
$$

to the composites and whiskerings

$$
\langle\mathbf{F}\mathsf{M}\rangle \xrightarrow{\ S\ } \mathbf{F}\Big(\bigotimes_{i=1}^{n} \mathsf{M}_i\Big) \underset{\overline{\mathbf{F}}K}{\overset{\overline{\mathbf{F}}H}{\rightrightarrows}} \Downarrow\overline{\mathbf{F}}\theta \;\; \mathbf{F}\mathsf{N}. \tag{64}
$$

Multilinearity of $\mathbf{F}H = (\overline{\mathbf{F}}H)\circ S$ follows from multilinearity of S and $\overline{\mathbf{F}}H$ being strict symmetric monoidal. Likewise, multinaturality of $\mathbf{F}\theta = (\overline{\mathbf{F}}\theta) * 1_S$ follows from multilinearity of S and $\overline{\mathbf{F}}\theta$ being monoidal natural.

7.11. REMARK. In our definition of the functor S, we consistently use $\otimes$ of profiles from Definition 4.1, with reverse lexicographic ordering, to define

1. $\langle x^{1\cdots n}\rangle$ in (54),

2. $f^{1\cdots n}$ in (56), and

3. $\langle\phi^{1\cdots n}\rangle$ in (58).

This is convenient because the indices $j_1,\ldots,j_n$ and $k_1,\ldots,k_n$ in those definitions iterate in left-to-right order.

However, we can also use the other choice, namely, $\otimes^{\mathsf{t}}$ in Definition 4.1, which corresponds to lexicographic ordering. In other words, in (1) above we can redefine $\langle x^{1\cdots n}\rangle$ as the iterated $\otimes^{\mathsf{t}}$-product of the tuples $\langle x^i\rangle$, and likewise for (2) and (3). With these consistent changes, the results about S in Section 7, the definition of $\mathbf{F}$ in (64), and the results in later sections are also valid.

8. Non-Symmetric Cat-Multifunctoriality of $\mathbf{F}$

In this section we verify the non-symmetric Cat-multifunctoriality axioms for $\mathbf{F}$, from Definition 3.7 (6).

8.1. THEOREM. *In the context of Definition 7.10, $\mathbf{F}$ is a non-symmetric Cat-multifunctor.*

PROOF. To verify the axiom for units, recall from Remark 7.7 that S is the identity monoidal functor if $n = 1$. Since $\overline{\mathbf{F}}$ is functorial, we have

$$\mathbf{F}(1_{\mathsf{M}}) = 1_{\mathbf{F}\mathsf{M}}$$

for each small multicategory M.

To verify the composition axiom, suppose given

$$H_a \in \mathsf{Multicat}\big(\langle\mathsf{M}_a\rangle\,;\,\mathsf{M}_a'\big) \quad \text{for} \quad 1 \le a \le n, \quad \text{and}$$
$$H' \in \mathsf{Multicat}\big(\langle\mathsf{M}'\rangle\,;\,\mathsf{M}''\big).$$

The two multilinear functors

$$\mathbf{F}\big(\gamma\big(H'\,;\,\langle H\rangle\big)\big) \quad \text{and} \quad \gamma\big(\mathbf{F}H'\,;\,\langle\mathbf{F}H\rangle\big)$$

are given by the two composites around the boundary in the following diagram, where

the unlabeled isomorphisms are given by reordering terms.

$$
\begin{array}{ccc}
\prod\limits_{a,b} \mathbf{F}\mathsf{M}_{a,b} & \xrightarrow{\ \cong\ } & \prod\limits_{a}\prod\limits_{b} \mathbf{F}\mathsf{M}_{a,b} \\[2em]
& & \Big\downarrow{\scriptstyle \Pi_a S} \\[1em]
\Big\downarrow{\scriptstyle S} & \prod\limits_{a}\mathbf{F}\big(\bigotimes\limits_{b}\mathsf{M}_{a,b}\big) \xrightarrow{\ \Pi_a\overline{\mathbf{F}}(H_a)\ } \prod\limits_{a}\mathbf{F}\mathsf{M}'_a \\[2em]
& \Big\downarrow{\scriptstyle S} \qquad\qquad \Big\downarrow{\scriptstyle S} \\[1em]
\mathbf{F}\big(\bigotimes\limits_{a,b}\mathsf{M}_{a,b}\big) \xrightarrow{\ \cong\ } \mathbf{F}\big(\bigotimes\limits_{a}\bigotimes\limits_{b}\mathsf{M}_{a,b}\big) \xrightarrow{\ \overline{\mathbf{F}}(\otimes_a H_a)\ } \mathbf{F}\big(\bigotimes\limits_{a}\mathsf{M}'_a\big) \xrightarrow{\ \overline{\mathbf{F}}H'\ } \mathbf{F}\mathsf{M}''
\end{array}
$$

In the above diagram, the two composites around the middle rectangle are equal as multilinear functors by naturality of S (Lemma 7.8) with respect to the multifunctors H_a. The rectangle at left commutes, as a diagram of underlying functors, by associativity of the products (54) and (57). The linearity constraints for the composites around the rectangle at left are $(\rho, \langle 1\rangle)$ for the same permutation ρ by the definition of S_b^2, (61), and, in the case of the top right composite, the definition of S on morphisms, (60), and the definition of composition, (48). A similar diagram for multinatural transformations $H_a \longrightarrow K_a$ and $H'_a \longrightarrow K'_a$ commutes by the 2-naturality of S. ∎

8.2. EXAMPLE. [Non-Symmetry of **F**] Suppose given a permutation $\sigma \in \Sigma_n$. The following diagram for compatibility of **F** with the action of σ generally does *not* commute for $n \geq 2$.

$$
\begin{array}{ccc}
\prod\limits_{i=1}^{n}\mathbf{F}\mathsf{M}_{\sigma(i)} & \xrightarrow{\ \ S\ \ } & \mathbf{F}\big(\bigotimes\limits_{i=1}^{n}\mathsf{M}_{\sigma(i)}\big) \\[1.5em]
{\scriptstyle \sigma}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathbf{F}(\sigma)} \\[1.5em]
\prod\limits_{i=1}^{n}\mathbf{F}\mathsf{M}_i & \xrightarrow{\ \ S\ \ } & \mathbf{F}\big(\bigotimes\limits_{i=1}^{n}\mathsf{M}_i\big)
\end{array}
\qquad (65)
$$

Indeed, suppose $n = 2$ with σ the nontrivial transposition and consider

$$
\langle x^1\rangle = (x_1^1, x_2^1, x_3^1) \in \mathbf{F}\mathsf{M}_1
$$
$$
\langle x^2\rangle = (x_1^2, x_2^2) \in \mathbf{F}\mathsf{M}_2.
$$

Then the assignments along the top and right of (65) are the following, respectively,

$$
\big(\langle x^2\rangle, \langle x^1\rangle\big) \longmapsto \big(x_{11}^{21}, x_{21}^{21}, x_{12}^{21}, x_{22}^{21}, x_{13}^{21}, x_{23}^{21}\big)
$$
$$
\longmapsto \big(x_{11}^{12}, x_{12}^{12}, x_{21}^{12}, x_{22}^{12}, x_{31}^{12}, x_{32}^{12}\big).
$$

On the other hand, the assignments along the left and bottom of (65) are the following, respectively,

$$
\big(\langle x^2\rangle, \langle x^1\rangle\big) \longmapsto \big(\langle x^1\rangle, \langle x^2\rangle\big)
$$
$$
\longmapsto \big(x_{11}^{12}, x_{21}^{12}, x_{31}^{12}, x_{12}^{12}, x_{22}^{12}, x_{32}^{12}\big).
$$

Thus the composites around (65) differ by a generally nontrivial permutation.

9. Two Transformations

Recall from Definition 5.13 that the endomorphism multicategory $\mathsf{End}(\mathsf{C})$ associated to a permutative category C defines a 2-functor

$$\mathsf{End} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Multicat}.$$

Throughout this section we let $\mathbf{E} = \mathsf{End}$. We recall from [JY22] two constructions that will be used to develop componentwise multinatural weak equivalences between the composites $\mathbf{EF}$, $\mathbf{FE}$, and the respective identity multifunctors.

Comparing $\mathbf{EF}$ and the Identity.

9.1. Definition. Suppose M is a small multicategory. Define a component

$$\eta = \eta_{\mathsf{M}} : \mathsf{M} \longrightarrow \mathbf{EFM}$$

as follows. For an object $w \in \mathsf{M}$ and an operation $\phi \in \mathsf{M}(\langle x \rangle \, ; y)$, let (w) and (ϕ) denote the corresponding length-1 sequences. For each $r \geq 0$, let $\iota_r : \overline{r} \longrightarrow \overline{1}$ be the unique map of finite sets. Then $\eta = \eta_{\mathsf{M}}$ is the following assignment:

$$\eta w = (w) \quad \text{for} \quad w \in \mathsf{M}, \quad \text{and}$$
$$\eta \phi = (\iota_r, (\phi)) : \langle x \rangle \longrightarrow (y)$$

where $\phi \in \mathsf{M}(\langle x \rangle \, ; y)$ and $|\langle x \rangle| = r$. Note that $\langle x \rangle$ is an r-fold concatenation of length-1 sequences (x_i) and the morphism $(\iota_r, (\phi))$ in $\mathbf{FM}$ is an r-ary operation in $\mathbf{EFM}$. Lemma 9.2 shows that each η_{M} is multifunctorial and that the components are Cat-multinatural.

9.2. Lemma. *The components η_{M} of Definition 9.1 define a Cat-multinatural transformation*

$$\eta : 1_{\mathsf{Multicat}} \longrightarrow \mathbf{EF}.$$

Proof. To check multifunctoriality of each component $\eta = \eta_{\mathsf{M}}$, first note that η preserves unit operations because ι_1 is the identity on $\overline{1}$. For compatibility with the symmetric group actions, first recall that the symmetry ξ in $\mathbf{FM}$, (50), has the form $(\tau, \langle 1 \rangle)$ and the symmetric group action in an endomorphism multicategory is given by permuting input objects (Definition 5.13). Thus, for an operation $\phi \in \mathsf{M}(\langle x \rangle \, ; y)$ and a permutation $\sigma \in \Sigma_r$, where $r = |\langle x \rangle|$, the operation

$$(\eta \phi) \cdot \sigma : \langle x \rangle \sigma \longrightarrow (y) \quad \text{in} \quad \mathbf{EFM}$$

is given by the composite $(\iota_r, (\phi)) \circ (\sigma, \langle 1 \rangle)$. We have

$$(\iota_r, (\phi)) \circ (\sigma, \langle 1 \rangle) = (\iota_r, (\phi \cdot \sigma)) = \eta(\phi \cdot \sigma),$$

where the first equality follows from composition (48) in $\mathbf{FM}$ and right unity (6) in M.

For compatibility with composition, suppose given

$$\psi \in \mathsf{M}\big(\langle x'\rangle\,;\, x''\big) \quad \text{where} \quad |\langle x'\rangle| = s, \quad \text{and}$$

$$\phi_j \in \mathsf{M}\big(\langle x_j\rangle\,;\, x_j'\big) \quad \text{where} \quad |\langle x_j\rangle| = r_j \quad \text{for} \quad j \in \overline{s}.$$

Let $r = \sum_j r_j$. Then the composite in $\mathbf{EFM}$ of

$$\eta\psi = \big(\iota_s, (\psi)\big) \quad \text{and} \quad \langle \eta\phi_j\rangle_{j=1}^s = \big\langle (\iota_{r_j}, (\phi_j))\big\rangle_j$$

is given by composing the morphisms

$$(\iota_s, (\psi)) \quad \text{and} \quad \bigoplus_{j \in \overline{s}}(\iota_{r_j}, \phi_j) = \big(\oplus_{j \in \overline{s}}\, \iota_{r_j}, \langle\phi\rangle\big) \tag{66}$$

in $\mathbf{FM}$. For $g = \iota_s$ and $f = \oplus_j \iota_{r_j}$, the permutation $\sigma_{g,f}^1$ of (47) is the identity on $\overline{r}$. Therefore the composite of the morphisms (66) is

$$\big(\iota_s \circ (\oplus_j \iota_{r_j}), \gamma(\psi\,;\, \langle\phi\rangle)\big) = \big(\iota_r, \gamma(\psi\,;\, \langle\phi\rangle)\big) = \eta\gamma(\psi\,;\, \langle\phi\rangle).$$

Multinaturality of η with respect to multifunctors

$$H : \bigotimes_{a=1}^{n} \mathsf{M}_a \longrightarrow \mathsf{N}$$

is given by commutativity of the following outer diagram.

$$\tag{67}$$

In the above diagram, the multifunctor ϖ is the n-variable version of the universal morphism

$$\varpi_{\mathsf{M},\mathsf{N}} : \mathsf{M} \otimes \mathsf{N} \longrightarrow \mathsf{M} \wedge \mathsf{N}$$

in (20). The multifunctor $\mathbf{E}S$ is the image of

$$S : \prod_a \mathbf{FM}_a \longrightarrow \mathbf{F}\Big(\bigotimes_a \mathsf{M}_a\Big) \quad \text{in} \quad \mathsf{PermCat}^{\mathsf{su}}\Big(\langle\mathbf{FM}\rangle\,;\, \mathbf{F}\big(\bigotimes_a \mathsf{M}_a\big)\Big)$$

under the isomorphism

$$\mathbf{E} : \mathsf{PermCat}^{\mathsf{su}}\left(\langle \mathbf{FM}\rangle\,;\, \mathbf{F}\big(\bigotimes_a \mathsf{M}_a\big)\right) \xrightarrow{\;\cong\;} \mathsf{Multicat}_*\left(\langle \mathbf{EFM}\rangle\,;\, \mathbf{EF}\big(\bigotimes_a \mathsf{M}_a\big)\right)$$

from Proposition 5.14. Thus the inner triangular region commutes by functoriality of $\mathbf{E}$ and the equality $(\mathbf{EF})H = \mathbf{E}(\mathbf{F}H)$. The remaining region commutes because both composites around its boundary have underlying assignments

$$\otimes_a x_a \longmapsto \big(H(\otimes_a x_a)\big) \quad \text{for} \quad x_a \in \mathsf{M}_a \quad \text{and}$$
$$\otimes_a \phi_a \longmapsto \big(\iota_r, H(\otimes_a \phi_a)\big) \quad \text{for} \quad \phi_a \in \mathsf{M}_a\big(\langle x_a\rangle\,;\, y_a\big),$$

where each ϕ_a has arity r_a and $r = \prod_a r_a$ is the arity of $\otimes_a \phi_a$. Multinaturality of η with respect to multinatural transformations $\kappa : H \longrightarrow K$ follows similarly. This completes the proof that

$$\eta : 1_{\mathsf{Multicat}} \longrightarrow \mathbf{EF}$$

is a Cat-enriched multinatural transformation.   ∎

9.3. REMARK. For readers familiar with Cat-enriched symmetric monoidal structure as in [JY∞, Sections 1.4 and 1.5], the diagram (67) reduces, in the case $n = 2$ and $H = 1$, to the axiom for Cat-monoidal naturality of η.

COMPARING $\mathbf{FE}$ AND THE IDENTITY.

9.4. DEFINITION. Suppose C is a small permutative category. Define a component symmetric monoidal functor

$$\rho = \rho_\mathsf{C} : \mathsf{C} \longrightarrow \mathbf{FEC}$$

by the inclusion of length-1 tuples, as follows:

$$\rho x = (x) \quad \text{and}$$
$$\rho \phi = (1_{\bar{1}}, (\phi))$$

for objects x and morphisms ϕ in C. The monoidal and unit constraints are given by the following morphisms for objects x and x' in C:

$$(\rho_\mathsf{C}^2)_{x,x'} = (\iota_2, 1_{x \oplus x'}) : (x, x') \longrightarrow (x \oplus x') \quad \text{and}$$
$$\rho_\mathsf{C}^0 = (\iota_0, 1_e) : \langle\rangle \longrightarrow (e).$$

Functoriality of ρ follows from the formula (48) for composition in $\mathbf{FEC}$. The associativity and unity axioms for $(\rho_\mathsf{C}, \rho_\mathsf{C}^2, \rho_\mathsf{C}^0)$ follow because the product in $\mathbf{FEC}$ is given by concatenation of tuples and e is a strict unit for C. The symmetry axiom follows because the symmetric group action on $\mathbf{EC}$ is given by composition with the symmetry isomorphism of C (and iterates thereof).

9.5. REMARK. One can show that the components ρ_{C} are multinatural with respect to multilinear functors and transformations. However, these components are not strictly unital because the unit constraints ρ^0 are not identities. Thus the components ρ_{C} are not 1-cells in $\mathsf{PermCat^{su}}$. To address this, we recall the following construction, replacing a general symmetric monoidal functor with a zigzag of strictly unital symmetric monoidal functors.

9.6. DEFINITION. Define a functor

$$(-)^{\dagger} : \mathsf{PermCat} \longrightarrow \mathsf{PermCat^{su}}$$

that adjoins a new monoidal unit, as follows.

Objects Suppose $(\mathsf{C}, \oplus, e)$ is a permutative category. Let $\mathsf{C}^{\dagger}$ denote the permutative category whose objects, morphisms, and symmetric monoidal structure are given by those of C, together with an additional object 0 that is a strict monoidal unit and an additional morphism $t : 0 \longrightarrow e$.

Let $I = \{0 \longrightarrow 1\}$ denote the permutative category formed by two idempotent objects and a single morphism from the monoidal unit to the other object. Then $\mathsf{C}^{\dagger}$ is the permutative category obtained by adjoining I to C by identifying the objects 1 and e.

Morphisms Suppose $P : \mathsf{C} \longrightarrow \mathsf{D}$ is a symmetric monoidal functor. Let

$$P^{\dagger} : \mathsf{C}^{\dagger} \longrightarrow \mathsf{D}^{\dagger} \tag{68}$$

be the strictly unital symmetric monoidal functor that is given by P on objects and morphisms of C, and that sends the additional morphism t in $\mathsf{C}^{\dagger}$ to the composite

$$0 \xrightarrow{\ t\ } e^{\mathsf{D}} \xrightarrow{\ P^0\ } P(e^{\mathsf{C}})$$

in $\mathsf{D}^{\dagger}$.

The monoidal constraint of $P^{\dagger}$ is given by that of P for objects $x, y \neq 0$ in C. If either x or y is 0, then $P^{\dagger}(0) = 0 \in \mathsf{D}$ and the monoidal constraint is an identity morphism.

There is a strict symmetric monoidal functor

$$\mathsf{C}^{\dagger} \xrightarrow{\ r_{\mathsf{C}}\ } \mathsf{C} \tag{69}$$

that is the identity on objects and morphisms of C and sends the additional morphism t of $\mathsf{C}^{\dagger}$ to the identity morphism of e in C. We call this the *retraction* for $\mathsf{C}^{\dagger}$. We let

$$\mathsf{C} \xrightarrow{\ j_{\mathsf{C}}\ } \mathsf{C}^{\dagger} \tag{70}$$

denote the inclusion, so that $r_{\mathsf{C}} j_{\mathsf{C}}$ is the identity on C. Note that $r_{\mathsf{C}} \dashv j_{\mathsf{C}}$ is an adjunction of underlying categories.

9.7. EXPLANATION. [Composition and Monoidal Sum in $\mathsf{C}^\dagger$] The morphisms $0 \longrightarrow x$ in $\mathsf{C}^\dagger$, for $x \neq 0$, are given by composition with t. Thus, the morphism sets in $\mathsf{C}^\dagger$ are given as follows, for $x, y \neq 0$

$$\mathsf{C}^\dagger(0,0) = \{1_0\} \qquad\qquad \mathsf{C}^\dagger(0,e) = \{t\}$$
$$\mathsf{C}^\dagger(0,x) = \mathsf{C}(e,x) \times \{t\} \qquad\qquad \mathsf{C}^\dagger(x,0) = \varnothing$$
$$\mathsf{C}^\dagger(x,y) = \mathsf{C}(x,y).$$

The composition

$$\mathsf{C}^\dagger(x,y) \times \mathsf{C}^\dagger(0,x) \longrightarrow \mathsf{C}^\dagger(0,y)$$

sends a pair of morphisms $f : x \longrightarrow y$ and $gt : 0 \longrightarrow e \longrightarrow x$ to $(fg)t$.

The monoidal sum $\oplus$ on morphisms in $\mathsf{C}^\dagger$ is defined as follows.

1. For morphisms $f : e \longrightarrow x$ and $g : e \longrightarrow y$ in C,

$$(0 \xrightarrow{t} e \xrightarrow{f} x) \oplus (0 \xrightarrow{t} e \xrightarrow{g} y) = (0 \xrightarrow{t} e \xrightarrow{1_e} e \oplus e \xrightarrow{f \oplus g} x \oplus y).$$

   In particular, $t \oplus t = t$.

2. For morphisms $f : e \longrightarrow x$ and $h : a \longrightarrow b$ in C,

$$(0 \xrightarrow{t} e \xrightarrow{f} x) \oplus (a \xrightarrow{h} b) = (0 \oplus a \xrightarrow{1_a} e \oplus a \xrightarrow{f \oplus h} x \oplus b).$$

   Similarly, $h \oplus (ft) = (h \oplus f)$.

It will be helpful to introduce additional notation for the following slight variant of the above construction $(-)^\dagger$. The variant is used in the statement of Lemma 9.9 and other discussion below.

9.8. DEFINITION. Given a symmetric monoidal functor

$$(P, P^2, P^0) : \mathsf{C} \longrightarrow \mathsf{D},$$

let $\mathsf{C}^\bullet = \mathsf{C}^\dagger$ and let $P^\bullet$ denote the composite of $P^\dagger$ with r_D shown in the following diagram.

$$
\begin{array}{ccc}
\mathsf{C}^\bullet & \xrightarrow{\quad P^\bullet \quad} & \mathsf{D} \\
\| & & \uparrow{\scriptstyle r_\mathsf{D}} \\
\mathsf{C}^\dagger & \xrightarrow{\quad P^\dagger \quad} & \mathsf{D}^\dagger
\end{array}
\qquad (71)
$$

Thus, $P^\bullet$ extends P by sending the additional morphism t of $\mathsf{C}^\bullet$ to the unit constraint P^0.

The construction $(-)^\bullet$ provides a multifunctor

$$\mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$$

determined by composition and whiskering with the canonical pointed multifunctors

$$\bigwedge_i \mathbf{E}(\mathsf{C}_i^\bullet) \longrightarrow \left(\bigwedge_i \mathbf{E}\mathsf{C}_i\right)^\bullet$$

that are strictly unital and determined by the identity on each $\mathbf{E}\mathsf{C}_i$.

9.9. LEMMA. *Using the constructions of* (69) *and Definition 9.8, the components* ρ_C *define a zigzag of* Cat*-multinatural transformations*

$$1_{\mathsf{PermCat}^{\mathsf{su}}} \xleftarrow{\;r\;} (-)^{\bullet} \xrightarrow{\;\rho^{\bullet}\;} \mathbf{FE}.$$

PROOF. For a multilinear functor

$$\langle \mathsf{C} \rangle \xrightarrow{\;P\;} \mathsf{D},$$

the left half of the diagram below commutes by definition of $(-)^{\bullet}$ on multilinear functors P.

$$
\begin{array}{ccccc}
\prod\limits_a \mathsf{C}_a & \xleftarrow{\;\prod_a r_{\mathsf{C}_a}\;} & \prod\limits_a \mathsf{C}_a^{\bullet} & \xrightarrow{\;\prod_a \rho_{\mathsf{C}_a}^{\bullet}\;} & \prod\limits_a \mathbf{FE}\mathsf{C}_a \\
{\scriptstyle P}\big\downarrow & & {\scriptstyle (P)^{\bullet}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{FE}P} \\
\mathsf{D} & \xleftarrow{\;r_{\mathsf{D}}\;} & \mathsf{D}^{\bullet} & \xrightarrow{\;\rho_{\mathsf{D}}^{\bullet}\;} & \mathbf{FE}\mathsf{D}
\end{array}
\tag{72}
$$

Away from the new unit objects $0 \in \mathsf{C}_a^{\bullet}$, the right half of the above diagram commutes because ρ is the inclusion of length-one tuples. Commutativity for unit objects follows because each arrow strictly preserves units. A similar analysis applies to multilinear transformations $\theta : P \longrightarrow Q$. ∎

10. Equivalence of Homotopy Theories

For permutative categories and for multicategories, we define stable equivalences via the stable equivalences on K-theory spectra. We let SymSp denote the Hovey-Shipley-Smith category of symmetric spectra [HSS00]. We let

$$\mathsf{K}^{\mathsf{Se}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{SymSp}$$

denote Segal's K-theory functor [Seg74] that constructs a connective symmetric spectrum from each small permutative category. See [JY∞, Chapters 7 and 8] for a review and further references.

10.1. DEFINITION. We define stable equivalences, $\mathcal{S}$, of permutative categories and multicategories via the three functors

$$
\begin{array}{c}
\mathsf{PermCat} \\
\mathsf{Multicat}
\end{array}
\quad
\begin{array}{c}
\xrightarrow{(-)^{\dagger}} \\
\xrightarrow{\;\mathbf{F}\;}
\end{array}
\quad
\mathsf{PermCat}^{\mathsf{su}} \xrightarrow{\;\mathsf{K}^{\mathsf{Se}}\;} \mathsf{SymSp}
$$

as follows.

- A strictly unital symmetric monoidal functor P is a *stable equivalence* if $\mathsf{K}^{\mathsf{Se}}P$ is a stable equivalence of connective spectra.

- A multifunctor H is a *stable equivalence* if $\mathbf{F}H$ is a stable equivalence of permutative categories.

- A general symmetric monoidal functor P is a *stable equivalence* if $P^\dagger$ is a stable equivalence in $\mathsf{PermCat}^{\mathsf{su}}$. That is, P is a stable equivalence if $\mathsf{K}^{\mathsf{se}}(P^\dagger)$ is a stable equivalence of connective spectra. Proposition 10.2 below shows that when P is strictly unital, then $\mathsf{K}^{\mathsf{se}}(P^\dagger)$ is a stable equivalence of connective spectra if and only if $\mathsf{K}^{\mathsf{se}}P$ is so.

Thus, the stable equivalences in each of $\mathsf{PermCat}^{\mathsf{su}}$, $\mathsf{PermCat}$, and $\mathsf{Multicat}$, respectively, are reflected by K^{se}, $(-)^\dagger$, and $\mathbf{F}$. We let $\mathcal{S}$ denote the class of stable equivalences in each case.

The following result shows that the homotopy theories determined by stable equivalences in $\mathsf{PermCat}$ and $\mathsf{PermCat}^{\mathsf{su}}$ are equivalent. The result is well known to experts; see e.g., [Man10, Theorem 3.9] and [GJO17a, Theorem 2.15] for discussion of the general symmetric monoidal case. For completeness, we present a short proof based on [GJO17a, Theorem 1.11].

10.2. PROPOSITION. *There is an adjunction* $(-)^\dagger \dashv I$ *that induces an equivalence of homotopy theories*

$$(-)^\dagger : \big(\mathsf{PermCat}, \mathcal{S}\big) \;\underset{\sim}{\rightleftarrows}\; \big(\mathsf{PermCat}^{\mathsf{su}}, \mathcal{S}\big) : I \tag{73}$$

where I is the inclusion.

PROOF. The unit and counit of the adjunction are given by

$$j_{\mathsf{C}} : \mathsf{C} \longrightarrow \mathsf{C}^\dagger = I(\mathsf{C}^\dagger) \quad \text{and}$$
$$r_{\mathsf{C}} : (I\mathsf{C})^\dagger = \mathsf{C}^\dagger \longrightarrow \mathsf{C}$$

from (70) and (69), respectively. Because I is a subcategory inclusion, it is a *map extension* in the sense of [GJO17a, Definition 1.9]. Moreover, each component of the counit, r, is a stable equivalence in $\mathsf{PermCat}^{\mathsf{su}}$ because $r_{\mathsf{C}} \dashv j_{\mathsf{C}}$ is an adjunction of underlying categories. Thus, the adjunction $(-)^\dagger \dashv I$ satisfies the conditions of [GJO17a, Theorem 1.11 (3)]: the left adjoint creates stable equivalences and each component of the counit is a stable equivalence. Therefore, condition [GJO17a, Theorem 1.11 (2)] also holds: the right adjoint I creates stable equivalences and the unit is componentwise a stable equivalence in $\mathsf{PermCat}$. It follows that (73) is an adjoint stable equivalence of homotopy theories. ∎

Next we recall further properties of η and ρ from [JY22].

10.3. PROPOSITION. [JY22, 6.11, 6.13, 7.3] *The following statements hold for a small multicategory M and a small permutative category C.*

1. *The component*

$$\eta : \mathsf{M} \longrightarrow \mathbf{EFM}$$

   is a stable equivalence of multicategories.

  2. The component

$$\rho : \mathsf{C} \longrightarrow \mathbf{FE}\mathsf{C}$$

  is a right adjoint of underlying categories.

10.4. REMARK. Both statements of Proposition 10.3 are proved by using construction of strict symmetric monoidal functors [JY22, 6.4]

$$\varepsilon = \varepsilon_\mathsf{C} : \mathbf{FE}\mathsf{C} \longrightarrow \mathsf{C}$$

for each small permutative category C, given by the following assignments:

$$\varepsilon\langle x \rangle = \bigoplus_i x_i \quad \text{and} \tag{74}$$

$$\varepsilon(f, \langle \phi \rangle) = \left(\bigoplus_j \phi_j \right) \circ \xi_f \tag{75}$$

where $\langle x \rangle$ is an object of $\mathbf{FE}\mathsf{C}$, $(f, \langle \phi \rangle) : \langle x \rangle \longrightarrow \langle y \rangle$ is a morphism of $\mathbf{FE}\mathsf{C}$, and ξ_f is a certain permutation of summands [JY22, 9.2] determined by f. The results in [JY22] show that η and ε are the unit and counit of a 2-adjunction between the 2-categories Multicat and PermCat$^{\mathsf{st}}$, where the 1-cells are *strict* monoidal functors. Moreover, for each C the components $(\varepsilon_\mathsf{C}, \rho_\mathsf{C})$ are an adjunction of underlying categories.

  However, the components ε_C are *not* natural with respect to general strictly unital symmetric monoidal functors. More generally, the following multinaturality diagram for ε with respect to a multilinear functor

$$\langle \mathsf{C} \rangle \xrightarrow{\;P\;} \mathsf{D}$$

fails to commute unless each of the linearity constraints P_a^2 is an identity.

$$\tag{76}$$

Thus the components ε do not provide a counit for $\mathbf{F}$ and $\mathbf{E}$ to be adjunction between Multicat and PermCat$^{\mathsf{su}}$, either as 2-categories or as Cat-multicategories.

10.5. DEFINITION. Suppose $\mathcal{O}$ is a small non-symmetric Cat-multicategory. A *non-symmetric $\mathcal{O}$-algebra* in a Cat-multicategory M is a non-symmetric Cat-multifunctor

$$\mathcal{O} \longrightarrow \mathsf{M}.$$

The morphisms of non-symmetric $\mathcal{O}$-algebras are Cat-multinatural transformations. The category of non-symmetric $\mathcal{O}$-algebras and their morphisms in M is denoted $\mathsf{M}^{\mathcal{O}}$. Considering the underlying 1-category of M, suppose $(\mathsf{M}, \mathcal{W})$ is a relative category. We define

$\mathcal{W}^{\mathcal{O}}$ as the subcategory of $\mathsf{M}^{\mathcal{O}}$ that contains all the non-symmetric $\mathcal{O}$-algebras and the morphisms that are componentwise in $\mathcal{W}$. We consider the pair $(\mathsf{M}^{\mathcal{O}}, \mathcal{W}^{\mathcal{O}})$ as a relative category.

Now we come to the proof of Theorem 1.1.

PROOF PROOF OF THEOREM 1.1. The Cat-multifunctoriality of $\mathbf{E}$ and $\mathbf{F}$ (non-symmetric in the latter case) is described in Sections 5 and 8, respectively. By Lemma 9.2 and Proposition 10.3 (1), η provides a natural stable equivalence

$$\eta^{\mathcal{O}} : 1 \longrightarrow (\mathbf{E}^{\mathcal{O}})(\mathbf{F}^{\mathcal{O}}). \tag{77}$$

Naturality of $\eta^{\mathcal{O}}$ with respect to algebra morphisms follows from Cat-multinaturality of η in Lemma 9.2.

For the other composite, Cat-multinaturality of the zigzag

$$1 \xleftarrow{\ r\ } (-)^{\bullet} \xrightarrow{\ \rho^{\bullet}\ } \mathbf{FE}$$

in Lemma 9.9 induces a zigzag of natural transformations between endofunctors of $\left(\mathsf{PermCat}^{\mathsf{su}} \right)^{\mathcal{O}}$

$$1 \longleftarrow (-)^{\bullet} \longrightarrow (\mathbf{F}^{\mathcal{O}})(\mathbf{E}^{\mathcal{O}}). \tag{78}$$

For each permutative category C, the adjunction of underlying categories $r_{\mathsf{C}} \dashv j_{\mathsf{C}}$ implies that r_{C} is a stable equivalence in $\mathsf{PermCat}^{\mathsf{su}}$ and j_{C} is a stable equivalence in $\mathsf{PermCat}$. Furthermore, each ρ_{C} is a stable equivalence in $\mathsf{PermCat}$ because it is a right adjoint of underlying categories, by Proposition 10.3 (2). Therefore, by the 2-out-of-3 property for stable equivalences and the factorization

$$\rho_{\mathsf{C}} = \rho_{\mathsf{C}}^{\bullet} \circ j_{\mathsf{C}} : \mathsf{C} \longrightarrow \mathbf{FEC},$$

each $\rho_{\mathsf{C}}^{\bullet}$ is a stable equivalence in $\mathsf{PermCat}$. By Proposition 10.2, each $\rho_{\mathsf{C}}^{\bullet}$ is therefore also a stable equivalence in $\mathsf{PermCat}^{\mathsf{su}}$. This implies that (78) is a zigzag of stable equivalences.

Therefore, by Proposition 2.10, $\mathbf{F}^{\mathcal{O}}$ and $\mathbf{E}^{\mathcal{O}}$ are equivalences of homotopy theories. This completes the proof. ∎

11. Application to Ring Categories

As a consequence of Theorem 1.1, the functors $\mathbf{F}^{\mathcal{O}}$ and $\mathbf{E}^{\mathcal{O}}$ induce equivalences of homotopy theories of non-symmetric $\mathcal{O}$-algebras. This section gives an application to ring categories, which are non-symmetric algebras over the Cat-enriched associative operad, As, by [JY∞, 11.2.16]. We recall the essential definitions.

11.1. Definition. [EM06] A *ring category* is a tuple

$$\Big(\mathsf{C}, (\oplus, 0, \xi^{\oplus}), (\otimes, \mathbb{1}), (\partial^l, \partial^r) \Big)$$

consisting of the following data.

The Additive Structure $(\mathsf{C}, \oplus, 0, \xi^{\oplus})$ is a permutative category.

The Multiplicative Structure $(\mathsf{C}, \otimes, \mathbb{1})$ is a strict monoidal category.

The Factorization Morphisms ∂^l and ∂^r are natural transformations

$$
\begin{array}{ll}
(A \otimes C) \oplus (B \otimes C) \xrightarrow{\ \partial^l_{A,B,C}\ } (A \oplus B) \otimes C \\[2mm]
(A \otimes B) \oplus (A \otimes C) \xrightarrow{\ \partial^r_{A,B,C}\ } A \otimes (B \oplus C)
\end{array}
\tag{79}
$$

for objects $A, B, C \in \mathsf{C}$, which are called the *left factorization morphism* and the *right factorization morphism*, respectively.

We often abbreviate $\otimes$ to concatenation, with $\otimes$ always taking precedence over $\oplus$ in the absence of clarifying parentheses. The subscripts in $\xi^{\oplus}$, ∂^l, and ∂^r are sometimes omitted.

The above data are required to satisfy the following seven axioms for all objects A, A', A'', B, B', B'', C, and C' in C. Each diagram is required to be commutative.

The Multiplicative Zero Axiom

$$
\begin{array}{ccccc}
\mathbf{1} \times \mathsf{C} & \xrightarrow{\ \cong\ } & \mathsf{C} & \xleftarrow{\ \cong\ } & \mathsf{C} \times \mathbf{1} \\
{\scriptstyle 0 \times 1_{\mathsf{C}}} \downarrow & & \downarrow {\scriptstyle 0} & & \downarrow {\scriptstyle 1_{\mathsf{C}} \times 0} \\
\mathsf{C} \times \mathsf{C} & \xrightarrow{\ \otimes\ } & \mathsf{C} & \xleftarrow{\ \otimes\ } & \mathsf{C} \times \mathsf{C}
\end{array}
$$

In this diagram, the top horizontal isomorphisms drop the $\mathbf{1}$ argument. Each 0 denotes the constant functor at $0 \in \mathsf{C}$ and 1_0.

The Zero Factorization Axiom

$$
\begin{array}{ll}
\partial^l_{0,B,C} = 1_{B \otimes C} & \qquad \partial^r_{0,B,C} = 1_0 \\[1mm]
\partial^l_{A,0,C} = 1_{A \otimes C} & \qquad \partial^r_{A,0,C} = 1_{A \otimes C} \\[1mm]
\partial^l_{A,B,0} = 1_0 & \qquad \partial^r_{A,B,0} = 1_{A \otimes B}
\end{array}
$$

The Unit Factorization Axiom

$$
\begin{array}{l}
\partial^l_{A,B,\mathbb{1}} = 1_{A \oplus B} \\[1mm]
\partial^r_{\mathbb{1},B,C} = 1_{B \oplus C}
\end{array}
$$

The Symmetry Factorization Axiom

$$
\begin{array}{ccc}
AC \oplus BC & \xrightarrow{\partial^l} & (A \oplus B)C \\
{\scriptstyle \xi^\oplus} \downarrow & & \downarrow {\scriptstyle \xi^\oplus 1_C} \\
BC \oplus AC & \xrightarrow{\partial^l} & (B \oplus A)C
\end{array}
\qquad
\begin{array}{ccc}
AB \oplus AC & \xrightarrow{\partial^r} & A(B \oplus C) \\
{\scriptstyle \xi^\oplus} \downarrow & & \downarrow {\scriptstyle 1_A \xi^\oplus} \\
AC \oplus AB & \xrightarrow{\partial^r} & A(C \oplus B)
\end{array}
$$

The Internal Factorization Axiom

$$
\begin{array}{ccc}
AB \oplus A'B \oplus A''B & \xrightarrow{\partial^l \oplus 1} & (A \oplus A')B \oplus A''B \\
{\scriptstyle 1 \oplus \partial^l} \downarrow & & \downarrow {\scriptstyle \partial^l} \\
AB \oplus (A' \oplus A'')B & \xrightarrow{\partial^l} & (A \oplus A' \oplus A'')B
\end{array}
\qquad
\begin{array}{ccc}
AB \oplus AB' \oplus AB'' & \xrightarrow{\partial^r \oplus 1} & A(B \oplus B') \oplus AB'' \\
{\scriptstyle 1 \oplus \partial^r} \downarrow & & \downarrow {\scriptstyle \partial^r} \\
AB \oplus A(B' \oplus B'') & \xrightarrow{\partial^r} & A(B \oplus B' \oplus B'')
\end{array}
$$

The External Factorization Axiom

$$
\begin{array}{ccc}
ABC \oplus A'BC & \xrightarrow{\partial^l_{A,A',BC}} & (A \oplus A')BC \\
{\scriptstyle \partial^l_{AB,A'B,C}} \downarrow & & \| \\
(AB \oplus A'B)C & \xrightarrow{\partial^l_{A,A',B} 1_C} & (A \oplus A')BC
\end{array}
\qquad
\begin{array}{ccc}
ABC \oplus AB'C & \xrightarrow{\partial^l_{AB,AB',C}} & (AB \oplus AB')C \\
{\scriptstyle \partial^r_{A,BC,B'C}} \downarrow & & \downarrow {\scriptstyle \partial^r 1_C} \\
A(BC \oplus B'C) & \xrightarrow{1_A \partial^l_{B,B',C}} & A(B \oplus B')C
\end{array}
$$

$$
\begin{array}{ccc}
ABC \oplus ABC' & \xrightarrow{\partial^r_{AB,C,C'}} & AB(C \oplus C') \\
{\scriptstyle \partial^r_{A,BC,BC'}} \downarrow & & \| \\
A(BC \oplus BC') & \xrightarrow{1_A \partial^r_{B,C,C'}} & AB(C \oplus C')
\end{array}
$$

The 2-By-2 Factorization Axiom

This finishes the definition of a ring category.

Recall, e.g., from [JY∞, Section 11.1] the Cat-enriched associative operad As detects monoid structures. Since As is the free symmetric operad on the terminal non-symmetric operad, which we denote As′, monoid structures are also detected by non-symmetric algebras over As′. This yields the following equivalent formulation of [JY∞, 11.2.16].

11.2. Theorem. [JY∞, 11.2.16] *For each small permutative category* C*, there is a canonical bijective correspondence between*

- *ring category structures on* C *and*

- *non-symmetric* Cat*-enriched multifunctors*

$$H : \mathsf{As}' \longrightarrow \mathsf{PermCat}^{\mathsf{su}} \quad such \ that \quad H(*) = \mathsf{C}.$$

Defining the morphisms of ring categories via morphisms of non-symmetric As'-algebras, Theorem 11.2 yields the following application of Theorem 1.1.

11.3. Corollary. *The* Cat*-multifunctors (non-symmetric in the case of* **F***)*

$$\mathbf{F} : \mathsf{Multicat} \xrightleftharpoons{\hspace{1cm}} \mathsf{PermCat}^{\mathsf{su}} : \mathbf{E},$$

induce an equivalence of homotopy theories between associative monoids in Multicat *and ring categories (Definition 11.1).*

References

[BK12]   C. Barwick and D. M. Kan, *A characterization of simplicial localization functors and a discussion of DK equivalences*, Indag. Math. (N.S.) **23** (2012), no. 1-2, 69–79. doi:10.1016/j.indag.2011.10.001 (cit. on pp. 1158, 1159).

[DK80]   W. G. Dwyer and D. M. Kan, *Calculating simplicial localizations*, J. Pure Appl. Algebra **18** (1980), no. 1, 17–35. doi:10.1016/0022-4049(80)90113-9 (cit. on pp. 1158, 1159).

[EM06]   A. D. Elmendorf and M. A. Mandell, *Rings, modules, and algebras in infinite loop space theory*, Adv. Math. **205** (2006), no. 1, 163–228. doi:10.1016/j.aim.2005.07.007 (cit. on pp. 1157, 1175, 1204).

[EM09]   _____, *Permutative categories, multicategories and algebraic K-theory*, Algebr. Geom. Topol. **9** (2009), no. 4, 2391–2441. doi:10.2140/agt.2009.9.2391 (cit. on p. 1157).

[GJ09]   P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Modern Birkhäuser Classics, Birkhäuser Verlag, Basel, 2009, Reprint of the 1999 edition [MR1711612]. doi:10.1007/978-3-0346-0189-4 (cit. on p. 1158).

[GJO17a]   N. Gurski, N. Johnson, and A. M. Osorno, *Extending homotopy theories across adjunctions*, Homology Homotopy Appl. **19** (2017), no. 2, 89–110. doi:10.4310/HHA.2017.v19.n2.a6 (cit. on p. 1201).

[GJO17b] _____, *K-theory for 2-categories*, Adv. Math. **322** (2017), 378–472. doi:`10.1016/j.aim.2017.10.011` (cit. on pp. 1157, 1159, 1160).

[Hir03] P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. (cit. on p. 1158).

[HSS00] M. Hovey, B. Shipley, and J. Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), no. 1, 149–208. doi:`10.1090/S0894-0347-99-00320-3` (cit. on p. 1200).

[JY∞] N. Johnson and D. Yau, *Bimonoidal Categories, E_n-Monoidal Categories, and Algebraic K-Theory. Volume III: From Categories to Structured Ring Spectra*, available at `https://nilesjohnson.net`. (cit. on pp. 1157, 1158, 1160, 1164, 1167, 1170, 1171, 1172, 1175, 1181, 1182, 1197, 1200, 1203, 1205, 1206).

[JY21] _____, *2-Dimensional Categories*, Oxford University Press, New York, 2021. doi:`10.1093/oso/9780198871378.001.0001` (cit. on p. 1172).

[JY22] _____, *Multicategories Model All Connective Spectra*, To appear in Homology, Homotopy and Applications (2022). arXiv:2111.08653 (cit. on pp. 1157, 1160, 1182, 1184, 1185, 1195, 1201, 1202).

[JS93] A. Joyal and R. Street, *Braided tensor categories*, Adv. Math. **102** (1993), no. 1, 20–78. doi:`10.1006/aima.1993.1055` (cit. on p. 1172).

[ML98] S. Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. (cit. on p. 1172).

[Man10] M. A. Mandell, *An inverse K-theory functor*, Doc. Math. **15** (2010), 765–791. (cit. on p. 1201).

[Rez01] C. Rezk, *A model for the homotopy theory of homotopy theory*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 973–1007 (electronic). doi:`10.1090/S0002-9947-00-02653-2` (cit. on pp. 1157, 1158, 1160).

[Seg74] G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312. doi:`10.1016/0040-9383(74)90022-6` (cit. on pp. 1157, 1182, 1200).

[Toë05] B. Toën, *Vers une axiomatisation de la théorie des catégories supérieures*, *K*-Theory **34** (2005), no. 3, 233–263. doi:`10.1007/s10977-005-4556-6` (cit. on p. 1158).

[Yau∞I]   D. Yau, *Bimonoidal Categories, E_n-Monoidal Categories, and Algebraic K-Theory. Volume I: Symmetric Bimonoidal Categories and Monoidal Bicategories*, available at https://u.osu.edu/yau.22/main/. (cit. on pp. 1161, 1172).

[Yau∞II]   _____, *Bimonoidal Categories, E_n-Monoidal Categories, and Algebraic K-Theory. Volume II: Braided Bimonoidal Categories with Applications*, available at https://u.osu.edu/yau.22/main/. (cit. on p. 1172).

[Yau16]   _____, *Colored operads*, Graduate Studies in Mathematics, vol. 170, American Mathematical Society, Providence, RI, 2016. (cit. on p. 1160).

Department of Mathematics
The Ohio State University at Newark
1179 University Drive
Newark, OH 43055, USA
Email: johnson.5320@osu.edu
        yau.22@osu.edu