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COHOMOLOGIES, EXTENSIONS AND DEFORMATIONS OF DIFFERENTIAL ALGEBRAS OF ARBITRARY WEIGHT

LI GUO, YUNNAN LI, YUNHE SHENG, AND GUODONG ZHOU

Abstract. As an algebraic structure underlying the differential calculus and differential equations, a differential algebra is an associative algebra equipped with a linear map satisfying the Leibniz rule. The subject has been studied for about a century and has become an important area of mathematics. In recent years the area has been expanded to the noncommutative associative and Lie algebra contexts and to the case when the defining operator identity has a weight in order to include difference operators. This paper provides a cohomology theory for differential algebras of arbitrary weight, via a uniform approach to cover both the zero weight case which is similar to the earlier study of differential Lie algebras, and the non-zero weight case which poses challenges. The cohomology of a differential algebra is related to the Hochschild cohomology by a type of long exact sequence for relative homology. As an application, abelian extensions of a differential algebra are classified by the second cohomology group. Furthermore, formal deformations of a differential algebra are characterized by the second cohomology group and the rigidity of a differential algebra is characterized by the vanishing of the second cohomology group.

Contents

1. Introduction

This paper studies the cohomology theory, abelian extensions and formal deformations of differential algebras of arbitrary weight.

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1.1. Differential algebras. Extracting from the first-order derivation on differentiable functions in analysis, a differential algebra is a pair consisting of an algebra R, commonly assumed to be commutative, and a linear operator d on R that satisfies the Leibniz rule

$$d(xy) = d(x)y + xd(y), \quad \forall x, y \in A.$$

The origin of differential algebra is the algebraic study of differential equations pioneered by Ritt in the 1930s [27]. Through the work of many mathematicians [13, 19, 25] in the following decades, the subject has been fully developed into a vast area of mathematics, comprised of differential Galois theory, differential algebraic geometry and differential algebraic groups, with broad connections to other areas in mathematics such as arithmetic geometry and logic, as well as computer science (mechanical proof of geometric theorems) and mathematical physics (renormalization in quantum field theory) [3, 6, 21, 30]. Another frequently usage of the differential operator is the derivation on a (co)chain complex. There the operator d is assumed to satisfy the nilpotency condition $d^2 = 0$.

In recent years, differential algebras without the commutativity or nilpotency conditions have appeared naturally, for example in the study of path algebras and enveloping algebras of differential Lie algebras [10, 23, 24]. In [17] differential associative algebras are studied from an operadic point of view, showing that the corresponding operad is Koszul. In [1], the notion of a differential algebra was generalized to non(anti)commutative superspace by deformation in the study of instantons in string theory.

On the other hand, the Leibniz rule is generalized to include the difference quotient $\frac{f(x+\lambda)-f(x)}{\lambda}$ before taking the limit $\lambda \mapsto 0$, leading to the notion of a differential algebra of *weight* λ [9]. This generalized notion of differential algebra provides a framework for a uniform approach to the differential algebra (corresponding to the case when $\lambda = 0$) and another extensively studied algebraic structure, the difference algebra (corresponding to the case when $\lambda = 1$), as an algebraic study of difference equations [2, 15, 26]. This notion also furnishes an algebraic context for the study of quantum calculus [12]. Differential operators of arbitrary weight on the Virasoro algebra were also investigated [16].

1.2. Homology and deformations of differential algebras. Cohomology and deformation theory are fundamental in the understanding of specific algebraic structures, starting with the seminal works of Gerstenhaber for associative algebras and of Nijenhuis and Richardson for Lie algebras [7, 8, 22], and of the general context of operads, culminated in the monographs [18, 20]. As a further step in this direction, studies of deformations and the related cohomology have recently emerged for algebras with linear operators, including Rota-Baxter operators and differential operators on Lie algebras [28, 29].

The importance of differential (associative) algebras makes it compelling to develop their cohomology theory, in both the zero weight case and nonzero weight case. This is the purpose of this paper, which also gives applications in the study of abelian extensions and formal deformations of differential algebras.

Until now, cohomology and deformation for differential operators on various algebraic structures have been restricted to weight zero. This includes the cases for differential graded algebra in [14], for Lie algebras in [28, 29], and for associative algebras in the independently developed work [4] in the weight zero case. Our emphasis in this paper is on differential algebras of nonzero weight, which are needed in order to give an algebraic study of difference operators and difference quotients, but for which a different approach has to be taken. See the comments in the outline below and Remark 2.12. We construct a cohomology theory for differential algebras of nonzero weight in this paper.

1.3. Layout of the paper. A differential algebra is the combination of the underlying algebra and a differential operator. In this light we build in Section 2 the cohomology theory of a differential algebra by combining its components from the algebra and from the differential operator. In Section 2.1, we introduce the notion of a bimodule over a differential algebra of nonzero weight. Then in Section 2.6, we establish the cohomology theory for differential operators of arbitrary weight, which is quite different from the one for the underlying algebra unless the weight is zero. In Section 2.9, we combine the Hochschild cohomology for associative algebras and the just established cohomology for differential operators of arbitrary weight to define the cohomology of differential algebras of arbitrary weight, with the coboundary operators again posing extra challenges when the weight is not zero. Finally in Section 2.16, we establish a close relationship among these cohomologies. More precisely, we show that there is a short exact sequence of cochain complexes for the algebra, the differential operator and the differential algebra. The resulting long exact sequence gives linear maps from the cohomology groups of the differential algebra to those of the algebra, with the error terms (kernels and cokernels) controlled by the cohomology groups of the differential operator. This can be regarded as a long exact sequence for relative homology.

As an application and further justification of our cohomology theory for differential algebras, we apply the theory in Section 3 to study abelian extensions of differential algebras of arbitrary weight, and show that abelian extensions are classified by the second cohomology group of the differential algebras.

Further, in Section 4, we apply the above cohomology theory to study formal deformations of differential algebras of arbitrary weight. In particular, we show that if the second cohomology group of a differential algebra with coefficients in the regular representation is trivial, then this differential algebra is rigid.

Notation. Throughout this paper, $\mathbf{k}$ denotes a field of characteristic zero. All the vector spaces, algebras, linear maps and tensor products are taken over $\mathbf{k}$ unless otherwise specified.

2. Cohomology of differential algebras

After recalling basic notions, we define the cohomology of a differential operator on an associative algebra and apply it, together with the cohomology of the associative algebra, to define the cohomology of the differential algebra. These cohomologies fit together in an exact sequence resemble to the LES of a pair.

2.1. The category of bimodules over differential algebras.

2.2. Definition. ([9]) Let $\lambda \in \mathbf{k}$ be a fixed element. A differential algebra of weight λ (also called a λ-differential algebra) is an associative algebra A together with a linear operator $d_A : A \to A$ such that

$$d_A(xy) = d_A(x)y + xd_A(y) + \lambda d_A(x)d_A(y), \quad \forall x, y \in A. \tag{1}$$

If A is unital, it further requires that

$$d_A(1_A) = 0. \tag{2}$$

Such an operator is called a differential operator of weight λ or a derivation of weight λ. It is also called a λ-differential operator or a λ-derivation.

Given differential algebras (A, d_A) and (B, d_B) of the same weight λ, a homomorphism of differential algebras from (A, d_A) to (B, d_B) is an algebra homomorphism $\varphi : A \to B$ such that $\varphi \circ d_A = d_B \circ \varphi$. We denote by DA_λ the category of λ-differential algebras.

To simplify the notation, for all the above notions, we will often suppress the mentioning of the weight λ unless it needs to be specified.

Recall that a bimodule over an associative algebra A is a triple (V, ρ_l, ρ_r), where V is a vector space, $\rho_l : A \to \mathrm{End}_{\mathbf{k}}(V),\ x \mapsto (v \mapsto xv)$ and $\rho_r : A \to \mathrm{End}_{\mathbf{k}}(V),\ x \mapsto (v \mapsto vx)$ are homomorphism and anti-homomorphism of associative algebras respectively such that $(xv)y = x(vy)$ for all $x, y \in A$ and $v \in V$.

2.3. Definition. Let (A, d_A) be a differential algebra.

1. A bimodule over the differential algebra (A, d_A) is a quadruple (V, ρ_l, ρ_r, d_V), where $d_V \in \mathrm{End}_{\mathbf{k}}(V)$, and (V, ρ_l, ρ_r) is a bimodule over the associative algebra A, such that for all $x, y \in A, v \in V$, the following equalities hold:

$$\begin{aligned} d_V(xv) &= d_A(x)v + xd_V(v) + \lambda d_A(x)d_V(v), \\ d_V(vx) &= vd_A(x) + d_V(v)x + \lambda d_V(v)d_A(x). \end{aligned}$$

2. Given bimodules $(U, \rho_l^U, \rho_r^U, d_U)$ and $(V, \rho_l^V, \rho_r^V, d_V)$ over (A, d_A), a linear map $f : U \to V$ is called a homomorphism of bimodules, if $f \circ d_U = d_V \circ f$ and

$$f \circ \rho_l^U(x) = \rho_l^V(x) \circ f, \quad f \circ \rho_r^U(x) = \rho_r^V(x) \circ f, \quad \forall x \in A.$$

We denote by (A, d_A)-Bimod the category of bimodules over the differential algebra (A, d_A).

2.4. Example. Any differential algebra (A, d_A) is a bimodule over itself with

$$\rho_l : A \to \mathrm{End}_{\mathbf{k}}(A),\ x \mapsto (y \mapsto xy), \quad \rho_r : A \to \mathrm{End}_{\mathbf{k}}(A),\ x \mapsto (y \mapsto yx).$$

It is called the regular bimodule over the differential algebra (A, d_A).

It is straightforward to obtain

2.5. PROPOSITION. *Let (V, ρ_l, ρ_r, d_V) be a bimodule over a differential algebra (A, d_A). Then $(A \oplus V, d_A \oplus d_V)$ is a differential algebra, where the associative algebra structure on $A \oplus V$ is given by*

$$(x + u)(y + v) = xy + xv + uy, \quad \forall x, y \in A,\ u, v \in V.$$

This differential algebra is called the semi-direct product of A by V, and denoted by $(A \ltimes V, d_\ltimes)$.

2.6. COHOMOLOGY OF DIFFERENTIAL OPERATORS. In this subsection, we define the cohomology of differential operators.

Let V be a bimodule over an associative algebra A. Let

$$(C^*_{\mathsf{Alg}}(A, V) = \oplus_{n=0}^{\infty} C^n_{\mathsf{Alg}}(A, V), \partial_{\mathsf{Alg}})$$

be the Hochschild cochain complex of A with coefficients in V, with the corresponding Hochschild cohomology $\mathrm{H}^*_{\mathsf{Alg}}(A, V)$ [11]. Recall that $C^n_{\mathsf{Alg}}(A, V) = \mathrm{Hom}(A^{\otimes n}, V), \forall n \geq 0$, in particular, $C^0_{\mathsf{Alg}}(A, V) = V$, and the coboundary operator

$$\partial_{\mathsf{Alg}} : C^n_{\mathsf{Alg}}(A, V) \longrightarrow C^{n+1}_{\mathsf{Alg}}(A, V)$$

is defined by

$$\begin{aligned}\partial_{\mathsf{Alg}} f(x_1, \ldots, x_{n+1}) :=& x_1 f(x_2, \ldots, x_{n+1}) + \sum_{i=1}^{n} (-1)^i f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) \\ &+ (-1)^{n+1} f(x_1, \ldots, x_n) x_{n+1}\end{aligned}$$

for all $f \in C^n_{\mathsf{Alg}}(A, V),\ x_1, \ldots, x_{n+1} \in A$.

Let (A, d_A) be a differential algebra of weight λ and let (V, d_V) be a bimodule over (A, d_A).

2.7. LEMMA. *A differential algebra (A, d_A) admits a new bimodule structure on (V, d_V) given by*

$$x \vdash_\lambda v := (x + \lambda d_A(x))v, \quad v \dashv_\lambda x := v(x + \lambda d_A(x)), \quad \forall x \in A, v \in V.$$

PROOF. Given $x, y \in A$ and $v \in V$, we have

$$\begin{aligned}x \vdash_\lambda (y \vdash_\lambda v) &= (x + \lambda d_A(x))((y + \lambda d_A(y))v) \\ &= (xy + \lambda(x d_A(y) + d_A(x)y + \lambda d_A(x) d_A(y)))v \\ &= (xy + \lambda d_A(xy))v \\ &= (xy) \vdash_\lambda v.\end{aligned}$$

Similarly, $(v \dashv_\lambda x) \dashv_\lambda y = v \dashv_\lambda (xy)$. Thus $(V, \vdash_\lambda, \dashv_\lambda)$ is a bimodule over the associative algebra A.

For $x \in A$ and $v \in V$,

$$
\begin{aligned}
& d_V(x \vdash_\lambda v) \\
= & d_V((x + \lambda d_A(x))v) \\
= & d_V(xv) + \lambda d_V(d_A(x)v) \\
= & d_A(x)v + xd_V(v) + \lambda d_A(x)d_V(v) + \lambda(d_A(d_A(x))v + d_A(x)d_V(v) + \lambda d_A(d_A(x))d_V(v)) \\
= & d_A(x)v + \lambda d_A(d_A(x))v + xd_V(v) + \lambda d_A(x)d_V(v) + \lambda(d_A(x)d_V(v) + \lambda d_A(d_A(x))d_V(v)) \\
= & d_A(x) \vdash_\lambda v + x \vdash_\lambda d_V(v) + \lambda d_A(x) \vdash_\lambda d_V(v).
\end{aligned}
$$

Similarly, one obtains the equality

$$d_V(v \dashv_\lambda x) = v \dashv_\lambda d_A(x) + d_V(v) \dashv_\lambda x + \lambda d_V(v) \dashv_\lambda d_A(x).$$

Also, it is obvious that $(x \vdash_\lambda v) \dashv_\lambda y = x \vdash_\lambda (v \dashv_\lambda y)$. Thus, $(V, \vdash_\lambda, \dashv_\lambda, d_V)$ is a bimodule over the differential algebra (A, d_A). ∎

For distinction, denote by V_λ the new bimodule structure over (A, d_A) given in Lemma 2.7.

2.8. Definition. The cochain complex of the (λ-)differential operator d_A with coefficients in the bimodule (V, d_V), denoted by $(C^*_{\mathsf{DO}_\lambda}(d_A, d_V) = \oplus_{n=0}^\infty C^n_{\mathsf{DO}_\lambda}(d_A, d_V), \partial_{\mathsf{DO}_\lambda})$, is defined to be the Hochschild cochain complex of the associative algebra A with coefficients in the new bimodule V_λ, that is, $C^n_{\mathsf{DO}_\lambda}(d_A, d_V) = \mathrm{Hom}(A^{\otimes n}, V), \forall n \geq 0$, and the coboudary operator

$$\partial_{\mathsf{DO}_\lambda} : C^n_{\mathsf{DO}_\lambda}(d_A, d_V) \longrightarrow C^{n+1}_{\mathsf{DO}_\lambda}(d_A, d_V)$$

is given by

$$
\begin{aligned}
\partial_{\mathsf{DO}_\lambda} f(x_1, \ldots, x_{n+1}) \quad = \quad & x_1 \vdash_\lambda f(x_2, \ldots, x_{n+1}) + \sum_{i=1}^n (-1)^i f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) \\
& + (-1)^{n+1} f(x_1, \ldots, x_n) \dashv_\lambda x_{n+1}
\end{aligned}
$$

for all $f \in C^n_{\mathsf{DO}_\lambda}(d_A, d_V)$, $x_1, \ldots, x_{n+1} \in A$. The cohomology of this cochain complex, denoted by $\mathrm{H}^*_{\mathsf{DO}_\lambda}(d_A, d_V)$, is called the cohomology of the (λ-)differential operator d_A with coefficients in the bimodule (V, d_V),

2.9. Cohomology of differential algebras. We now combine the classical Hochschild cohomology of associative algebras and the newly defined cohomology of differential operators of weight λ to define the cohomology of the differential algebra (A, d_A) with coefficients in the bimodule (V, d_V). In fact, we will define the cochain complex of a differential algebra as, up to a shift and signs, the mapping cone of a cochain map from the Hochschild cochain complex of the associative algebra to the cochain complex of the differential operator of weight λ.

Define the linear maps

$$\delta : C^n_{\mathsf{Alg}}(A, V) \to C^n_{\mathsf{DO}_\lambda}(d_A, d_V)$$

by

$$\delta f(x_1,\dots,x_n) :=$$

$$\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n} f(x_1,\dots,d_A(x_{i_1}),\dots,d_A(x_{i_k}),\dots,x_n) - d_V f(x_1,\dots,x_n),$$

for $f\in C^n_{\mathsf{Alg}}(A,V)$, $n\ge 1$ and

$$\delta v = -d_V(v),\quad \forall v\in C^0_{\mathsf{Alg}}(A,V)=V.$$

2.10. Proposition. *The linear map δ is a cochain map from the cochain complex*

$$(C^*_{\mathsf{Alg}}(A,V),\partial_{\mathsf{Alg}}) \text{ to } (C^*_{\mathsf{DO}_\lambda}(d_A,d_V),\partial_{\mathsf{DO}_\lambda}).$$

Proof. For $v\in C^0_{\mathsf{Alg}}(A,V)=V$ and $x\in A$, we have

$$\begin{aligned}
\delta(\partial_{\mathsf{Alg}}v)(x) &= \partial_{\mathsf{Alg}}v(d_A(x)) - d_V(\partial_{\mathsf{Alg}}v(x))\\
&= d_A(x)v - vd_A(x) - d_V(xv-vx)\\
&= d_A(x)v - vd_A(x) - d_A(x)v - xd_V(v)\\
&\qquad -\lambda d_A(x)d_V(v) + d_V(v)x + vd_A(x) + \lambda d_V(v)d_A(x)\\
&= -xd_V(v) - \lambda d_A(x)d_V(v) + d_V(v)x + \lambda d_V(v)d_A(x)\\
&= -x\vdash_\lambda d_V(v) + d_V(v)\dashv_\lambda x\\
&= x\vdash_\lambda \delta v - \delta v\dashv_\lambda x\\
&= \partial_{\mathsf{DO}_\lambda}(\delta v)(x).
\end{aligned}$$

To simplify notations, we use the abbreviation $x_{i,j} := x_i,\dots,x_j$, $i\le j$, with the convention $x_{i,j}=1$ if $i>j$. For $1\le i_1<\cdots<i_k\le n$ and $f\in C^n_{\mathsf{Alg}}(A,V)$, define a function $f^{(i_1,\dots,i_k)}$ by

$$\begin{aligned}
&f^{(i_1,\dots,i_k)}(x_1,\dots,x_n)\\
&:= f\big(x_1,\dots,x_{i_1-1},d_A(x_{i_1}),x_{i_1+1}\dots,x_{i_2-1},d_A(x_{i_2}),x_{i_2+1}\dots x_{i_k-1},d_A(x_{i_k}),x_{i_k+1},\dots,x_n\big).
\end{aligned}$$

Then for $f\in C^n_{\mathsf{Alg}}(A,V)$, $x_1,\dots,x_{n+1}\in A$ with $n\ge1$, we have

$$\begin{aligned}
\partial_{\mathsf{DO}_\lambda}(\delta f)(x_{1,n+1}) &= x_1\vdash_\lambda \delta f(x_{2,n+1}) + \sum_{j=1}^{n}(-1)^j\delta f(x_{1,j-1},x_jx_{j+1},x_{j+2,n+1})\\
&\quad +(-1)^{n+1}\delta f(x_{1,n})\dashv_\lambda x_{n+1}
\end{aligned}$$

$$U = \sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n} x_1\vdash_\lambda f^{(i_1,\dots,i_k)}(x_{2,n+1})$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n}(-1)^j f^{(i_1,\ldots,i_k)}(x_{1,j-1},x_jx_{j+1},x_{j+2,n+1})\\
&+(-1)^{n+1}\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n} f^{(i_1,\ldots,i_k)}(x_{1,n})\dashv_\lambda x_{n+1}-x_1\vdash_\lambda d_V(f(x_{2,n+1}))\\
&+\sum_{j=1}^{n}(-1)^{j-1}d_V(f(x_{1,j-1},x_jx_{j+1},x_{j+2,n+1}))+(-1)^n d_V(f(x_{1,n}))\dashv_\lambda x_{n+1}.
\end{aligned}
$$

On the other hand,

$$
\delta(\partial_{\mathsf{Alg}}f)(x_{1,n+1}) \;=\; \sum_{k=1}^{n+1}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}(\partial_{\mathsf{Alg}}f)^{(i_1,\ldots,i_k)}(x_{1,n+1})-d_V(\partial_{\mathsf{Alg}}f(x_{1,n+1})).
$$

Before completing the proof, we state a lemma whose proof is quite technical and is postponed to the Appendix.

2.11. LEMMA. *For $f\in C^n_{\mathsf{Alg}}(A,V)$, $x_1,\ldots,x_{n+1}\in A$ with $n\ge 1$,*

$$
\begin{aligned}
&\sum_{k=1}^{n+1}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}(\partial_{\mathsf{Alg}}f)^{(i_1,\ldots,i_k)}(x_{1,n+1})-d_V(\partial_{\mathsf{Alg}}f(x_{1,n+1})) \qquad (3)\\
=\;&\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n}x_1\vdash_\lambda f^{(i_1,\ldots,i_k)}(x_{2,n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n}\sum_{j=1}^{n}(-1)^j f^{(i_1,\ldots,i_k)}(x_{1,j-1},x_jx_{j+1},x_{j+2,n+1})\\
&+(-1)^{n+1}\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n} f^{(i_1,\ldots,i_k)}(x_{1,n})\dashv_\lambda x_{n+1}-x_1\vdash_\lambda d_V(f(x_{2,n+1}))\\
&+\sum_{j=1}^{n}(-1)^{j-1}d_V(f(x_{1,j-1},x_jx_{j+1},x_{j+2,n+1}))+(-1)^n d_V(f(x_{1,n}))\dashv_\lambda x_{n+1}.
\end{aligned}
$$

Now by Lemma 2.11, we conclude $\partial_{\mathsf{DO}_\lambda}\circ\delta=\delta\circ\partial_{\mathsf{Alg}}$. ■

2.12. REMARK. Note that $C^n_{\mathsf{DO}_\lambda}(d_A,d_V)$ equals to $C^n_{\mathsf{Alg}}(A,V)$ as linear spaces but they are not equal as cochain complexes unless $\lambda=0$. When λ is not zero, a new bimodule structure is needed to define $\partial_{\mathsf{DO}_\lambda}$ which eventually leads to the rather long and technical argument in order to establish the cochain map in Lemma 2.11 and Proposition 2.10.

Then we obtain

2.13. Theorem. *Define the space of n-cochains by*

$$C^n_{\mathsf{DA}_\lambda}(A,V) := \begin{cases} C^n_{\mathsf{Alg}}(A,V) \oplus C^{n-1}_{\mathsf{DO}_\lambda}(d_A,d_V), & n \geq 1, \\ C^0_{\mathsf{Alg}}(A,V) = V, & n = 0, \end{cases} \tag{4}$$

and the linear map $\partial_{\mathsf{DA}_\lambda} : C^n_{\mathsf{DA}_\lambda}(A,V) \to C^{n+1}_{\mathsf{DA}_\lambda}(A,V)$ by

$$\partial_{\mathsf{DA}_\lambda}(f,g) := (\partial_{\mathsf{Alg}} f, \partial_{\mathsf{DO}_\lambda} g + (-1)^n \delta f), \forall f \in C^n_{\mathsf{Alg}}(A,V),\ g \in C^{n-1}_{\mathsf{DO}_\lambda}(d_A,d_V), n \geq 1, \tag{5}$$

$$\partial_{\mathsf{DA}_\lambda} v := (\partial_{\mathsf{Alg}} v, \delta v), \quad \forall v \in C^0_{\mathsf{Alg}}(A,V) = V. \tag{6}$$

Then the pair $(C^*_{\mathsf{DA}_\lambda}(A,V), \partial_{\mathsf{DA}_\lambda})$ is a cochain complex.

Proof. It is easy to verify that $(C^*_{\mathsf{DA}_\lambda}(A,V), \partial_{\mathsf{DA}_\lambda})$ is actually, up to signs, the negative shift of the mapping cone of the cochain map $\delta : (C^*_{\mathsf{Alg}}(A,V), \hat{\partial}_{\mathsf{Alg}}) \to (C^*_{\mathsf{DO}_\lambda}(d_A,d_V), \hat{\partial}_{\mathsf{DO}_\lambda})$. ∎

2.14. Definition. The cohomology $\mathrm{H}^*_{\mathsf{DA}_\lambda}(A,V)$ of the cochain complex $(C^*_{\mathsf{DA}_\lambda}(A,V), \partial_{\mathsf{DA}_\lambda})$ is called the cohomology of the differential algebra (A,d_A) with coefficients in the bimodule (V,d_V).

2.15. Remark. Our cochain complexes are constructed from Hochschild cochain complexes by using Lemma 2.7 which is a descendent property for differential bimodules. When $\lambda = 0$, this cochain complex can be obtained from the minimal model introduced by Loday [17].

We interpret the 0-cocycles and 1-cocycles of the cochain complex $(C^*_{\mathsf{DA}_\lambda}(A,V), \partial_{\mathsf{DA}_\lambda})$.

It is obvious that for all $v \in V$, $\partial_{\mathsf{DA}_\lambda} v = 0$ if and only if $\partial_{\mathsf{Alg}} v = 0$ and $d_V(v) = 0$. This means that the space of 0-cocycles is given by the space

$$\{v \in V \mid xv = vx, \forall x \in A \text{ and } d_V(v) = 0\}.$$

For all $(f,v) \in \mathrm{Hom}(A,V) \oplus V$, $\partial_{\mathsf{DA}_\lambda}(f,v) = 0$ if and only if $\partial_{\mathsf{Alg}} f = 0$, and

$$x \vdash_\lambda v - v \dashv_\lambda x = f(d_A(x)) - d_V(f(x)), \quad \forall x \in A.$$

So $f : A \to V$ is a derivation in the usual sense, but f satisfies further the equation

$$f(d_A(x)) - d_V(f(x)) = (x - \lambda d_A(x))v - v(x - \lambda d_A(x)), \quad \forall x \in A.$$

2.16. Relationship between the cohomologies. To relate the different cohomologies, note that we have the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C^n_{\mathsf{Alg}}(A,V) \arrow[r, "\partial_{\mathsf{Alg}}"] \arrow[dr, "(-1)^n\delta"] & C^{n+1}_{\mathsf{Alg}}(A,V) \arrow[r, "\partial_{\mathsf{Alg}}"] \arrow[dr, "(-1)^{n+1}\delta"] & C^{n+2}_{\mathsf{Alg}}(A,V) \arrow[r] & \cdots \\
\cdots \arrow[r] & C^{n-1}_{\mathsf{DO}_\lambda}(d_A,d_V) \arrow[r, "\partial_{\mathsf{DO}_\lambda}"] & C^n_{\mathsf{DO}_\lambda}(d_A,d_V) \arrow[r, "\partial_{\mathsf{DO}_\lambda}"] & C^{n+1}_{\mathsf{DO}_\lambda}(d_A,d_V) \arrow[r] & \cdots .
\end{tikzcd}$$

Then we obtain

2.17. Proposition. *There are the natural inclusion and projection*

$$\iota : C^{*-1}_{\mathsf{DO}_\lambda}(d_A, d_V) \longrightarrow C^*_{\mathsf{DA}_\lambda}(A, V), \quad \pi : C^*_{\mathsf{DA}_\lambda}(A, V) \longrightarrow C^*_{\mathsf{Alg}}(A, V)$$

of cochain complexes, giving the short exact sequence of cochain complexes

$$0 \to C^{*-1}_{\mathsf{DO}_\lambda}(d_A, d_V) \xrightarrow{\iota} C^*_{\mathsf{DA}_\lambda}(A, V) \xrightarrow{\pi} C^*_{\mathsf{Alg}}(A, V) \to 0.$$

Noting that, by Eq. (5) and the construction of the connecting homomorphism Δ_n, for $[f] \in \mathrm{H}^n_{\mathsf{Alg}}(A, V)$, we have

$$\bar{\iota}(\Delta_n([f])) = [\partial_{\mathsf{DA}_\lambda}(f, 0)] = [(0, (-1)^n \delta f)],$$

which implies that $\Delta_n = (-1)^n \bar{\delta}$. Thus applying the Snake Lemma gives the following relationship among the various cohomology groups.

2.18. Theorem. *We have the following long exact sequence of cohomology groups,*

$$\cdots \to \mathrm{H}^{n-1}_{\mathsf{DO}_\lambda}(d_A, d_V) \xrightarrow{\bar{\iota}} \mathrm{H}^n_{\mathsf{DA}_\lambda}(A, V) \xrightarrow{\bar{\pi}} \mathrm{H}^n_{\mathsf{Alg}}(A, V) \xrightarrow{(-1)^n \bar{\delta}} \mathrm{H}^n_{\mathsf{DO}_\lambda}(d_A, d_V) \to \cdots,$$

where $\bar{\delta} : \mathrm{H}^n_{\mathsf{Alg}}(A, V) \to \mathrm{H}^n_{\mathsf{DO}_\lambda}(d_A, d_V)$ is given by $\bar{\delta}[f] = [\delta f]$. Here $[f]$ and $[\delta f]$ denote the cohomological classes of $f \in C^n_{\mathsf{Alg}}(A, V)$ and $\delta f \in C^n_{\mathsf{DO}_\lambda}(d_A, d_V)$.

Thus the linear maps $\bar{\pi}$ relate the cohomology groups of the differential algebra and those of the underlying algebra, with the error terms controlled by the cohomology groups of the differential operator.

In the next sections, we will need a certain subcomplex of the cochain complex $C^*_{\mathsf{DA}_\lambda}(A, V)$. Let

$$\tilde{C}^n_{\mathsf{DA}_\lambda}(A, V) := \begin{cases} C^n_{\mathsf{Alg}}(A, V) \oplus C^{n-1}_{\mathsf{DO}_\lambda}(d_A, d_V), & n \geq 2, \\ C^1_{\mathsf{Alg}}(A, V), & n = 1, \\ 0, & n = 0. \end{cases} \tag{7}$$

Then it is obvious that $\tilde{C}^*_{\mathsf{DA}_\lambda}(A, V) = \oplus_{n=0}^{\infty} \tilde{C}^n_{\mathsf{DA}_\lambda}(A, V)$ is a subcomplex of the cochain complex $(C^*_{\mathsf{DA}_\lambda}(A, V), \partial_{\mathsf{DA}_\lambda})$, whose differential will be denoted by $\widetilde{\partial}_{\mathsf{DA}_\lambda}$. We denote its cohomology by $\tilde{\mathrm{H}}^*_{\mathsf{DA}_\lambda}(A, V)$, called the reduced cohomology of the differential algebra (A, d_A) with coefficients in the bimodule (V, d_V). Obviously, $\tilde{\mathrm{H}}^n_{\mathsf{DA}_\lambda}(A, V) = \mathrm{H}^n_{\mathsf{DA}_\lambda}(A, V)$ for $n > 2$.

3. Abelian extensions of differential algebras

In this section, we generalize the classification of abelian extensions of associative algebras by the second cohomology group to abelian extensions of differential algebras.

3.1. DEFINITION. Let (V, d_V) be a vector space with a linear endomorphism d_V, which can be considered as a differential algebra of weight λ endowed with the trivial product $uv = 0$ for all $u, v \in V$. An abelian extension of differential algebras of the same weight λ is a short exact sequence of homomorphisms of differential algebras

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \xrightarrow{i} & \hat{A} & \xrightarrow{p} & A & \longrightarrow & 0 \\ & & {\scriptstyle d_V}\big\downarrow & & {\scriptstyle d_{\hat{A}}}\big\downarrow & & {\scriptstyle d_A}\big\downarrow & & \\ 0 & \longrightarrow & V & \xrightarrow{i} & \hat{A} & \xrightarrow{p} & A & \longrightarrow & 0 \end{array}$$

We will call $(\hat{A}, d_{\hat{A}})$ an abelian extension of (A, d_A) by (V, d_V).

3.2. DEFINITION. Abelian extensions $(\hat{A}_1, d_{\hat{A}_1})$ and $(\hat{A}_2, d_{\hat{A}_2})$ of (A, d_A) by (V, d_V) are called isomorphic if there exists an isomorphism of differential algebras $\zeta : (\hat{A}_1, d_{\hat{A}_1}) \to (\hat{A}_2, d_{\hat{A}_2})$ such that the following commutative diagram holds:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (V, d_V) & \xrightarrow{i_1} & (\hat{A}_1, d_{\hat{A}_1}) & \xrightarrow{p_1} & (A, d_A) & \longrightarrow & 0 \\ & & \Big\| & & {\scriptstyle \zeta}\big\downarrow & & \Big\| & & \\ 0 & \longrightarrow & (V, d_V) & \xrightarrow{i_2} & (\hat{A}_2, d_{\hat{A}_2}) & \xrightarrow{p_2} & (A, d_A) & \longrightarrow & 0. \end{array} \tag{8}$$

A section of an abelian extension $(\hat{A}, d_{\hat{A}})$ of (A, d_A) by (V, d_V) is a linear map $s : A \to \hat{A}$ such that $p \circ s = \mathrm{Id}_A$.

Now for an abelian extension

$$0 \to (V, d_V) \xrightarrow{i} (\hat{A}, d_{\hat{A}}) \xrightarrow{p} (A, d_A) \to 0 \tag{9}$$

with a section $s : A \to \hat{A}$, there exists a unique linear map $t : \hat{A} \to V$ such that $t \circ i = \mathrm{Id}_V, t \circ s = 0, i \circ t + s \circ p = \mathrm{Id}_{\hat{A}}$. The left and right multiplications of $s(A)$ on $i(V)$ define linear maps

$$\rho_l : A \to \mathrm{End}_{\mathbf{k}}(V),\ x \mapsto (v \mapsto xv) \text{ and } \rho_r : A \to \mathrm{End}_{\mathbf{k}}(V),\ x \mapsto (v \mapsto vx)$$

respectively by

$$xv := t(s(x)i(v)), \quad vx := t(i(v)s(x)), \quad \forall x \in A, v \in V.$$

Since $p(s(x)i(v)) = p(s(x))p(i(v)) = 0$ and $p(i(v)s(x)) = p(i(v))p(s(x)) = 0$, we have $i(xv) = s(x)i(v)$ and $i(vx) = i(v)s(x)$.

3.3. PROPOSITION. *With the above notations, (V, ρ_l, ρ_r, d_V) is a bimodule over the differential algebra (A, d_A).*

PROOF. For $x, y \in A$ and $v \in V$, since $s(xy) - s(x)s(y) \in i(V)$ implies $s(xy)i(v) = s(x)s(y)i(v)$, we have

$$i((xy)v) = s(xy)i(v) = s(x)s(y)i(v) = s(x)i(yv) = i(x(yv)).$$

Hence, ρ_l is an algebra homomorphism. Similarly, ρ_r is an algebra anti-homomorphism. Moreover, $d_{\hat{A}}(s(x)) - s(d_A(x)) \in i(V)$ means that $d_{\hat{A}}(s(x))i(v) = s(d_A(x))i(v)$. Thus we have

$$\begin{aligned} i(d_V(xv)) &= d_{\hat{A}}(i(xv)) = d_{\hat{A}}(s(x)i(v)) = d_{\hat{A}}(s(x))i(v) + s(x)d_{\hat{A}}(i(v)) + \lambda d_{\hat{A}}(s(x))d_{\hat{A}}(i(v)) \\ &= s(d_A(x))i(v) + s(x)i(d_V(v)) + \lambda s(d_A(x))i(d_V(v)) \\ &= i(d_A(x)v + xd_V(v) + \lambda d_A(x)d_V(v)). \end{aligned}$$

Hence, (V, ρ_l, ρ_r, d_V) is a bimodule over (A, d_A). ■

By the proof of Proposition 3.3, there are unique linear maps $\psi : A \otimes A \to V$ and $\chi : A \to V$ such that

$$\begin{aligned} i(\psi(x, y)) &= s(x)s(y) - s(xy), \quad \forall x, y \in A, \\ i(\chi(x)) &= d_{\hat{A}}(s(x)) - s(d_A(x)), \quad \forall x \in A. \end{aligned}$$

3.4. PROPOSITION. *The pair (ψ, χ) is a 2-cocycle in the reduced cohomology of the diffential algebra (A, d_A) with coefficients in the bimodule (V, d_V) introduced in Proposition 3.3.*

The proof is by direct computations.

The choice of the section s in fact determines an isomorphism of vector spaces

$$\begin{pmatrix} p & t \end{pmatrix} : \hat{A} \cong A \oplus V : \begin{pmatrix} s \\ i \end{pmatrix}.$$

We can transfer the differential algebra structure on $\hat{A}$ to $A \oplus V$ via this isomorphism. It is direct to verify that this endows $A \oplus V$ with a multiplication $\cdot_\psi$ and a differential operator d_χ defined by

$$(x, u) \cdot_\psi (y, v) = (xy, xv + uy + \psi(x, y)), \quad \forall x, y \in A,\ u, v \in V, \tag{10}$$

$$d_\chi(x, v) = (d_A(x), \chi(x) + d_V(v)), \quad \forall x \in A,\ v \in V. \tag{11}$$

Moreover, we get an abelian extension

$$0 \to (V, d_V) \xrightarrow{\begin{pmatrix} 0 & \mathrm{Id} \end{pmatrix}} (A \oplus V, d_\chi) \xrightarrow{\begin{pmatrix} \mathrm{Id} \\ 0 \end{pmatrix}} (A, d_A) \to 0$$

which is easily seen to be isomorphic to the original one (9).

Now we investigate the influence of different choices of sections.

3.5. Proposition.

1. *Different choices of the section s give the same differential bimodule structures on (V, d_V);*

2. *the cohomological class of (ψ, χ) in the reduced cohomology does not depend on the choice of sections.*

Proof. Let s' be another sections of p and let (ψ', χ') be the corresponding 2-cocycle.

Since $p \circ (s' - s) = p \circ s - p \circ s = 0$, introduce $\gamma : A \to V$ by $i(\gamma(x)) = s'(x) - s(x), \forall x \in A$. Since the differential algebra (V, d_V) satisfies $uv = 0$ for all $u, v \in V$, we have

$$s'(x)v = s(x)v + i(\gamma(x)(v)) = s(x)v, \forall x \in A, v \in V.$$

So different choices of the section s give the same differential bimodule structures on (V, d_V);

In order to simplify the presentation, we shall delete i in the following computation.

Now we show that the cohomological class of (ψ, χ) does not depend on the choice of sections. In fact, for all $x, y \in V$, we have

$$\begin{aligned}
\psi'(x, y) &= s'(x)s'(y) - s'(xy) \\
&= (s(x) + \gamma(x))(s(y) + \gamma(y)) - (s(xy) + \gamma(xy)) \\
&= (s(x)s(y) - s(xy)) + s(x)\gamma(y) + \gamma(x)s(y) - \gamma(xy) \\
&= (s(x)s(y) - s(xy)) + x\gamma(y) + \gamma(x)y - \gamma(xy) \\
&= \psi(x, y) + \partial_{\mathrm{Alg}}(\gamma)(x, y)
\end{aligned}$$

and

$$\begin{aligned}
\chi'(x) &= d_{\hat{A}}(s'(x)) - s'(d_A(x)) \\
&= d_{\hat{A}}(s(x) + \gamma(x)) - (s(d_A(x)) + \gamma(d_A(x))) \\
&= (d_{\hat{A}}(s(x)) - s(d_A(x))) + d_V(\gamma(x)) - \gamma(d_A(x)) \\
&= \chi(x) + d_V(\gamma(x)) - \gamma(d_A(x)) \\
&= \chi(x) - \delta(\gamma)(x).
\end{aligned}$$

That is, $(\psi', \chi') = (\psi, \chi) + \widetilde{\partial}_{\mathsf{DA}_\lambda}(\gamma)$. Thus (ψ', χ') and (ψ, χ) form the same cohomological class in $\mathrm{H}^2_{\mathsf{DA}_\lambda}(A, V)$. ∎

We show now the isomorphic abelian extensions give rise to the same cohomology class.

3.6. Proposition. *Let V be a vector space and $d_V \in \mathrm{End}_{\mathbf{k}}(V)$ which is considered as a differential algebra with trivial multiplication. Let (A, d_A) be a differential algebra. Two isomorphic abelian extensions of (A, d) by (V, d_V) give rise to the same cohomology class in $\widetilde{\mathrm{H}}^2_{\mathsf{DA}_\lambda}(A, V)$.*

Proof. Assume that $(\hat{A}_1, \hat{d}_1)$ and $(\hat{A}_2, \hat{d}_2)$ are two isomorphic abelian extensions of (A, d) by (V, d_V) as is given in (8). Let s_1 be a section of $(\hat{A}_1, \hat{d}_1)$. As $p_2 \circ \zeta = p_1$, we have

$$p_2 \circ (\zeta \circ s_1) = p_1 \circ s_1 = \mathrm{Id}_A.$$

Therefore, $\zeta \circ s_1$ is a section of $(\hat{A}_2, \hat{d}_2)$. Denote $s_2 := \zeta \circ s_1$. Since ζ is a homomorphism of differential algebras such that $\zeta|_V = \mathrm{Id}_V$, $\zeta(xv) = \zeta(s_1(x)v) = s_2(x)v = xv$ and similarly $\zeta(vx) = vx, \forall x \in A, v \in V$, so $\zeta|_V : V \to V$ is compatible with the induced differential bimodule structures. For all $x, y \in A$, we have

$$\begin{aligned}\psi_2(x,y) &= s_2(x)s_2(y) - s_2(xy) = \zeta(s_1(x))\zeta(s_1(y)) - \zeta(s_1(xy))\\ &= \zeta(s_1(x)s_1(y) - s_1(xy)) = \zeta(\psi_1(x,y))\\ &= \psi_1(x,y)\end{aligned}$$

and

$$\begin{aligned}\chi_2(x) &= \hat{d}_2(s_2(x)) - s_2(d_A(x)) = \hat{d}_2(\zeta(s_1(x))) - \zeta(s_1(d_A(x)))\\ &= \zeta(\hat{d}_1(s_1(x)) - s_1(d_A(x))) = \zeta(\chi_1(x))\\ &= \chi_1(x).\end{aligned}$$

Consequently, two isomorphic abelian extensions give rise to the same element in $\widetilde{\mathrm{H}}^2_{\mathsf{DA}_\lambda}(A, V)$. ∎

Now we consider the opposite direction.

Let (V, d_V) be a differential bimodule over the differential algebra (A, d_A). Given two linear maps $\psi : A \otimes A \to V$ and $\chi : A \to V$, one can define a binary operation $\cdot_\psi$ and an operator d_χ on $A \oplus V$ by Equations (10)-(11). The following fact is important:

3.7. Proposition. *The triple $(A \oplus V, \cdot_\psi, d_\chi)$ is a differential algebra if and only if (ψ, χ) is a 2-cocycle in the reduced cohomology of the differential algebra (A, d_A) with coefficients in (V, d_V). In this case, we obtain an abelian extension*

$$0 \to (V, d_V) \overset{\begin{pmatrix} 0 & \mathrm{Id}\end{pmatrix}}{\longrightarrow} (A \oplus V, d_\chi) \overset{\begin{pmatrix} \mathrm{Id} \\ 0\end{pmatrix}}{\longrightarrow} (A, d_A) \to 0,$$

and the canonical section $s = \begin{pmatrix} 0 & \mathrm{Id}\end{pmatrix} : (A, d_A) \to (A \oplus V, d_\chi)$ endows V with the original differential bimodule structure.

Proof. If $(A \oplus V, \cdot_\psi, d_\chi)$ is a differential algebra, then the associativity of $\cdot_\psi$ implies

$$x\psi(y \otimes z) - \psi(xy \otimes z) + \psi(x \otimes yz) - \psi(x \otimes y)z = 0, \tag{12}$$

which means $\partial_{\mathrm{Alg}}(\psi) = 0$ in $\mathrm{C}^*_{\mathrm{Alg}}(A, V)$. Since d_χ is an differential operator, for any $x, y \in A, u, v \in V$, we have

$$d_\chi((x,u)) \cdot_\psi d_\chi((y,v)) = d_\chi\Big(d_\chi(x,u) \cdot_\psi (y,v) + (x,u) \cdot_\psi d_\chi(y,v) + \lambda(x,u) \cdot_\psi (y,v)\Big)$$

Then χ, ψ satisfy the following equations:

$$\begin{aligned}
&d_A(x)\chi(y) + \chi(x)d_A(y) + \psi(d(x) \otimes d_A(y)) \\
=&d_V(\chi(x)y) + d_V(\psi(d(x) \otimes y)) + \chi(d_A(x)y) \\
&+ d_V(x\chi(y)) + d_V(\psi(x \otimes d_A(y))) + \chi(xd_A(y)) \\
&+ \lambda d_V(\psi(x \otimes y)) + \lambda\chi(xy),
\end{aligned}$$

which says that $\partial_{\mathsf{DO}_\lambda}(\chi) + \delta(\psi) = 0$. Hence, (ψ, χ) is a 2-cocycle.

Conversely, if (ψ, χ) is a 2-cocycle, one can easily check that $(A \oplus V, \cdot_\psi, d_\chi)$ is a differential algebra.

The last statement is clear by Proposition 3.3. ∎

Finally, we show the following result:

3.8. Proposition. *Two cohomologous 2-cocyles give rise to isomorphic abelian extensions.*

Proof. Given two 2-cocycles (ψ_1, χ_1) and (ψ_2, χ_2), we can construct two abelian extensions $(A \oplus V, \cdot_{\psi_1}, d_{\chi_1})$ and $(A \oplus V, \cdot_{\psi_2}, d_{\chi_2})$ via Equations (10) and (11). If they represent the same cohomology class in $\widetilde{\mathrm{H}}^2_{\mathsf{DA}_\lambda}(A, V)$, then there exists a linear map $\gamma : A \to V$ such that

$$(\psi_1, \chi_1) = (\psi_2, \chi_2) + (\partial_{\mathrm{Alg}}(\gamma_1), -\delta^1(\gamma)).$$

Define $\zeta : A \oplus M \to A \oplus M$ by

$$\zeta(x, v) := (x, -\gamma(x) + v).$$

Then ζ is an isomorphism of these two abelian extensions $(A \oplus M, \cdot_{\psi_1}, d_{\chi_1})$ and $(A \oplus M, \cdot_{\psi_2}, d_{\chi_2})$. ∎

Combining the results obtained in this section, we have shown the following theorem.

3.9. Theorem. *The second reduced cohomology group of a differential algebra classifies abelian extensions.*

3.10. Remark. In particular, a vector space V with linear endomorphism d_V can serve as a trivial bimodule over the differential algebra (A, d_A). In this situation, central extensions of (A, d_A) by (V, d_V), i.e. abelian extensions satisfying $xv = vx = 0, \forall x \in A, v \in V$, are classified by the second cohomology group $\mathrm{H}^2_{\mathsf{DA}_\lambda}(A, V)$ of (A, d_A) with coefficients in the trivial bimodule (V, d_V). Note that for a trivial bimodule (V, d_V), since $\partial_{\mathsf{DO}_\lambda} v = 0$ for all $v \in V$, we have

$$\mathrm{H}^2_{\mathsf{DA}_\lambda}(A, V) = \tilde{\mathrm{H}}^2_{\mathsf{DA}_\lambda}(A, V).$$

4. Formal deformations of differential algebras

In this section, we study formal deformations of a differential algebra. In particular, we show that if the second cohomology group $\tilde{\mathrm{H}}^2_{\mathsf{DA}_\lambda}(A,A)$ vanishes, then the differential algebra (A,d_A) is rigid. Although the approach follows the traditional paradigm of the classical theory of deformation theory of associative algebras, the proofs are technically more difficult.

4.1. Definition. Let (A,d_A) be a differential algebra with multiplication μ_A. Consider a 1-parameter-ized family

$$\mu_t=\sum_{i=0}^{\infty}\mu_i t^i,\ \ \mu_i\in C^2_{\mathsf{Alg}}(A,A),\quad d_t=\sum_{i=0}^{\infty}d_i t^i,\ \ d_i\in C^1_{\mathsf{DO}_\lambda}(d_A,d_A).$$

The family is called a 1-parameter formal deformation of the differential algebra (A,d_A) if the pair (μ_t,d_t) endows the $\mathbf{k}[[t]]$-module $(A[[t]],\mu_t,d_t)$ with a differential algebra structure over $\mathbf{k}[[t]]$ such that $(\mu_0,d_0)=(\mu_A,d_A)$.

The pair (μ_t,d_t) generates a 1-parameter formal deformation of the differential algebra (A,d_A) if and only if for all $x,y,z\in A$, the following equalities hold:

$$\begin{aligned}\mu_t(\mu_t(x,y),z)&=\mu_t(x,\mu_t(y,z)),&(13)\\ d_t(\mu_t(x,y))&=\mu_t(d_t(x),y)+\mu_t(x,d_t(y))+\lambda\mu_t(d_t(x),d_t(y)).&(14)\end{aligned}$$

Expanding these equations and collecting coefficients of t^n, we see that Eqs. (13) and (14) are equivalent to the systems of equations:

$$\sum_{i=0}^{n}\mu_i(\mu_{n-i}(x,y),z)=\sum_{i=0}^{n}\mu_i(x,\mu_{n-i}(y,z)),\tag{15}$$

$$\sum_{\substack{k,l\geq 0\\ k+l=n}}d_l\mu_k(x,y)=\sum_{\substack{k,l\geq 0\\ k+l=n}}(\mu_k(d_l(x),y)+\mu_k(x,d_l(y)))+\lambda\sum_{\substack{k,l,m\geq 0\\ k+l+m=n}}\mu_k(d_l(x),d_m(y))\tag{16}$$

4.2. Remark. For $n=0$, Eq. (15) is equivalent to the associativity of μ_A, and Eq. (16) is equivalent to the fact that d_A is a differential operator of weight λ.

4.3. Proposition. *Let $(A[[t]],\mu_t,d_t)$ be a 1-parameter formal deformation of a differential algebra (A,d_A). Then (μ_1,d_1) is a 2-cocycle of the differential algebra (A,d_A) with coefficients in the regular bimodule (A,d_A).*

Proof. For $n=1$, Eq. (15) is equivalent to $\partial\mu_1=0$, and Eq. (16) is equivalent to

$$\partial_{\mathsf{DO}_\lambda}d_1+\delta\mu_1=0,$$

implying that (μ_1,d_1) is a 2-cocycle. ■

If $\mu_t = \mu_A$ (that is, $\mu_0 = \mu_A$ and $\mu_n = 0$, $\forall n \geq 1$) in the above 1-parameter formal deformation of the differential algebra (A, d_A), we obtain a 1-parameter formal deformation of the differential operator d_A. Consequently, we have

4.4. Corollary. *Let d_t be a 1-parameter formal deformation of the differential operator d_A. Then d_1 is a 1-cocycle of the differential operator d_A with coefficients in the regular bimodule (A, d_A).*

Proof. In the special case when $n = 1$, Eq. (16) is equivalent to $\partial_{\mathsf{DO}_\lambda} d_1 = 0$, which implies that d_1 is a 1-cocycle of the differential operator d_A with coefficients in the regular bimodule (A, d_A). ■

4.5. Definition. The 2-cocycle (μ_1, d_1) is called the infinitesimal of the 1-parameter formal deformation $(A[[t]], \mu_t, d_t)$ of (A, d_A).

4.6. Definition. Let $(A[[t]], \mu_t, d_t)$ and $(A[[t]], \bar{\mu}_t, \bar{d}_t)$ be 1-parameter formal deformations of (A, d_A). A formal isomorphism from $(A[[t]], \bar{\mu}_t, \bar{d}_t)$ to $(A[[t]], \mu_t, d_t)$ is a power series

$$\Phi_t = \sum_{i \geq 0} \phi_i t^i : A[[t]] \to A[[t]],$$

where $\phi_i : A \to A$ are linear maps with $\phi_0 = \mathrm{Id}_A$, such that

$$\begin{aligned} \Phi_t \circ \bar{\mu}_t &= \mu_t \circ (\Phi_t \times \Phi_t), && (17)\\ \Phi_t \circ \bar{d}_t &= d_t \circ \Phi_t. && (18) \end{aligned}$$

Two 1-parameter formal deformations $(A[[t]], \mu_t, d_t)$ and $(A[[t]], \bar{\mu}_t, \bar{d}_t)$ are said to be equivalent if there exists a formal isomorphism $\Phi_t : (A[[t]], \bar{\mu}_t, \bar{d}_t) \to (A[[t]], \mu_t, d_t)$.

4.7. Theorem. *The infinitesimals of two equivalent 1-parameter formal deformations of (A, d_A) are in the same cohomology class in $\tilde{\mathrm{H}}^2_{\mathsf{DA}_\lambda}(A, A)$.*

Proof. Let $\Phi_t : (A[[t]], \bar{\mu}_t, \bar{d}_t) \to (A[[t]], \mu_t, d_t)$ be a formal isomorphism. For all $x, y \in A$, we have

$$\begin{aligned} \Phi_t \circ \bar{\mu}_t(x, y) &= \mu_t \circ (\Phi_t \times \Phi_t)(x, y),\\ \Phi_t \circ \bar{d}_t(x) &= d_t \circ \Phi_t(x). \end{aligned}$$

Expanding the above identities and comparing the coefficients of t, we obtain

$$\begin{aligned} \bar{\mu}_1(x, y) &= \mu_1(x, y) + \phi_1(x)y + x\phi_1(y) - \phi_1(xy),\\ \bar{d}_1(x) &= d_1(x) + d_A(\phi_1(x)) - \phi_1(d_A(x)). \end{aligned}$$

Thus, we have

$$(\bar{\mu}_1, \bar{d}_1) = (\mu_1, d_1) + \partial_{\mathsf{DA}_\lambda}(\phi_1),$$

which implies that $[(\bar{\mu}_1, \bar{d}_1)] = [(\mu_1, d_1)]$ in $\tilde{\mathrm{H}}^2_{\mathsf{DA}_\lambda}(A, A)$. ■

4.8. DEFINITION. A 1-parameter formal deformation $(A[[t]], \mu_t, d_t)$ of (A, d_A) is called trivial if it is equivalent to the deformation $(A[[t]], \mu_A, d_A)$, that is, there exists $\Phi_t = \sum_{i\geq 0} \phi_i t^i : A[[t]] \to A[[t]]$, where $\phi_i : A \to A$ are linear maps with $\phi_0 = \mathrm{Id}_A$, such that

$$\begin{aligned}\Phi_t \circ \mu_t &= \mu_A \circ (\Phi_t \times \Phi_t), &(19)\\ \Phi_t \circ d_t &= d_A \circ \Phi_t. &(20)\end{aligned}$$

4.9. DEFINITION. A differential algebra (A, d_A) is said to be rigid if every 1-parameter formal deformation is trivial.

4.10. THEOREM. *Regarding (A, d_A) as the regular bimodule over itself, if $\tilde{\mathrm{H}}^2_{\mathsf{DA}_\lambda}(A, A) = 0$, then the differential algebra (A, d_A) is rigid.*

PROOF. Let $(A[[t]], \mu_t, d_t)$ be a 1-parameter formal deformation of (A, d_A). By Proposition 4.3, (μ_1, d_1) is a 2-cocycle. By $\tilde{\mathrm{H}}^2_{\mathsf{DA}_\lambda}(A, A) = 0$, there exists a 1-cochain $\phi_1 \in C^1_{\mathsf{Alg}}(A, A)$ such that

$$(\mu_1, d_1) = -\widetilde{\partial}_{\mathsf{DA}_\lambda}(\phi_1). \tag{21}$$

Then setting $\Phi_t = \mathrm{Id}_A + \phi_1 t$, it is easily verified that we get a new deformation $(A[[t]], \bar{\mu}_t, \bar{d}_t)$, where

$$\begin{aligned}\bar{\mu}_t(x, y) &= \big(\Phi_t^{-1} \circ \mu_t \circ (\Phi_t \times \Phi_t)\big)(x, y),\\ \bar{d}_t(x) &= \big(\Phi_t^{-1} \circ d_t \circ \Phi_t\big)(x).\end{aligned}$$

Thus, $(A[[t]], \bar{\mu}_t, \bar{d}_t)$ is equivalent to $(A[[t]], \mu_t, d_t)$. Moreover, we have

$$\begin{aligned}\bar{\mu}_t(x, y) &= (\mathrm{Id}_A - \phi_1 t + \phi_1^2 t^2 + \cdots + (-1)^i \phi_1^i t^i + \cdots)(\mu_t(x + \phi_1(x)t, y + \phi_1(y)t)),\\ \bar{d}_t(x) &= (\mathrm{Id}_A - \phi_1 t + \phi_1^2 t^2 + \cdots + (-1)^i \phi_1^i t^i + \cdots)(d_t(x + \phi_1(x)t)).\end{aligned}$$

Therefore,

$$\begin{aligned}\bar{\mu}_t(x, y) &= xy + (\mu_1(x, y) + x\phi_1(y) + \phi_1(x)y - \phi_1(xy))t + \bar{\mu}_2(x, y)t^2 + \cdots,\\ \bar{d}_t(x) &= d_A(x) + (d_A(\phi_1(x)) + d_1(x) - \phi_1(d_A(x)))t + \bar{d}_2(x)t^2 + \cdots.\end{aligned}$$

By Eq. (21), we have

$$\begin{aligned}\bar{\mu}_t(x, y) &= xy + \bar{\mu}_2(x, y)t^2 + \cdots,\\ \bar{d}_t(x) &= d_A(x) + \bar{d}_2(x)t^2 + \cdots.\end{aligned}$$

Then by repeating the argument, we can show that $(A[[t]], \mu_t, d_t)$ is equivalent to $(A[[t]], \mu_A, d_A)$. Thus, (A, d_A) is rigid. ∎

4.11. Remark. Expanding Eq. (15) and Eq. (16) for $n=2$, we obtain that

$$\begin{array}{rcl}\partial_{\mathrm{Alg}}(\mu_2) & = & \mu_1\circ(\mu_1\otimes \mathrm{Id})-\mu_1\circ(\mathrm{Id}\otimes\mu_1),\\ \partial_{\mathsf{DA}_\lambda}(d_2)+\delta(\mu_2) & = & -d_1\circ\mu_1+\mu_1\circ(d_1\otimes\mathrm{Id}+\mathrm{Id}\otimes d_1)+\lambda\mu_1\circ(d_1\otimes d_A+d_A\otimes d_1)\\ & & +\lambda\mu\circ(d_1\otimes d_1).\end{array}$$

It is not difficult to verify that the right hand side is a 3-cocycle in $(C^*_{\mathsf{DA}_\lambda}(A,V),\partial_{\mathsf{DA}_\lambda})$. So once the third cohomology group vanishes, each infinitesimal deformation (μ_1,d_1) can be extended to a second order deformation. More generally, one verifies that when $\mathrm{H}^3_{\mathsf{DA}_\lambda}(A,A)=0$, each infinitesimal deformation (μ_1,d_1) can be extended to a formal deformation.

Appendix: Proof of Lemma 2.11

Now we prove Lemma 2.11. For this purpose, we first prove another lemma.

4.12. Lemma. *For $f\in C^n_{\mathsf{Alg}}(A,V)$ and $x_1,\dots,x_{n+1}\in A$ with $n\geq 1$, we have*

$$\begin{aligned}&\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\leq i_1<\cdots<i_k\leq n}\sum_{j=1}^{n}(-1)^j f^{(i_1,\dots,i_k)}(x_{1,j-1},x_jx_{j+1},x_{j+2,n+1})\\
=&\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\leq i_1<\cdots<i_k\leq n+1}\sum_{\substack{1\leq j\leq n\\ j\neq i_r-1,i_r,\forall r}}(-1)^j f(x_1,\dots,d_A(x_{i_1}),\dots,x_jx_{j+1},\dots,d_A(x_{i_k}),\dots,x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\leq i_1<\cdots<i_k\leq n+1}\sum_{\substack{1\leq r\leq k\\ i_r-1\neq i_{r-1}}}(-1)^{i_r-1}f(x_1,\dots,d_A(x_{i_1}),\dots,x_{i_r-1}d_A(x_{i_r}),\dots,d_A(x_{i_k}),\dots,x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\leq i_1<\cdots<i_k\leq n+1}\sum_{\substack{1\leq r\leq k-1\\ i_r+1\neq i_{r+1}}}(-1)^{i_r}f(x_1,\dots,d_A(x_{i_1}),\dots,d_A(x_{i_r})x_{i_r+1},\dots,d_A(x_{i_k}),\dots,x_{n+1})\\
&+\sum_{k=2}^{n+1}\lambda^{k-1}\sum_{1\leq i_1<\cdots<i_k\leq n+1}\sum_{\substack{1\leq r\leq k-1\\ i_r+1=i_{r+1}}}(-1)^{i_r}f(x_1,\dots,d_A(x_{i_1}),\dots,d_A(x_{i_r})d_A(x_{i_{r+1}}),\dots,d_A(x_{i_k}),\dots,x_{n+1})\end{aligned}$$

Proof. Applying the conventions $i_0=0$ and $i_{k+1}=n+2$ in the first equality below, with k as the index in the following formula, by Eq. (1), we have

$$\begin{aligned}&\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\leq i_1<\cdots<i_k\leq n}\sum_{j=1}^{n}(-1)^j f^{(i_1,\dots,i_k)}(x_{1,j-1},x_jx_{j+1},x_{j+2,n+1})\\
=&\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\leq i_1<\cdots<i_k\leq n}\sum_{\substack{1\leq j\leq n\\ i_s<j<i_{s+1}\\ 0\leq s\leq k}}(-1)^j f(\dots,d_A(x_{i_1}),\dots,d_A(x_{i_s}),\dots,x_jx_{j+1},\dots,d_A(x_{i_{s+1}+1}),\dots,d_A(x_{i_k+1}),\dots)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n}\sum_{r=1}^{k}(-1)^{i_r}f(\ldots,d_A(x_{i_1}),\ldots,d_A(x_{i_r}x_{i_r+1}),\ldots,d_A(x_{i_k+1}),\ldots)\\
=&\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le j\le n\\ j\neq i_r,i_r-1,\,\forall r}}(-1)^{j}f(\cdots,d_A(x_{i_1}),\ldots,x_jx_{j+1},\ldots,d_A(x_{i_k}),\ldots)\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n}\sum_{r=1}^{k}(-1)^{i_r}f(x_1,\ldots,d_A(x_{i_1}),\ldots,d_A(x_{i_r})x_{i_r+1},\ldots,d_A(x_{i_k+1}),\ldots,x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n}\sum_{r=1}^{k}(-1)^{i_r}f(x_1,\ldots,d_A(x_{i_1}),\ldots,x_{i_r}d_A(x_{i_r+1}),\ldots,d_A(x_{i_k+1}),\ldots,x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k}\sum_{1\le i_1<\cdots<i_k\le n}\sum_{r=1}^{k}(-1)^{i_r}f(x_1,\ldots,d_A(x_{i_1}),\ldots,d_A(x_{i_r})d_A(x_{i_r+1}),\ldots,d_A(x_{i_k+1}),\ldots,x_{n+1})\\
=&\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le j\le n\\ j\neq i_r,i_r-1,\,\forall r}}(-1)^{j}f(\ldots,d_A(x_{i_1}),\ldots,x_jx_{j+1},\ldots,d_A(x_{i_k}),\ldots)\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le r\le k\\ i_r-1\neq i_{r-1}}}(-1)^{i_r-1}f(x_1,\ldots,d_A(x_{i_1}),\ldots,x_{i_r-1}d_A(x_{i_r}),\ldots,d_A(x_{i_k}),\ldots,x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le r\le k-1\\ i_r+1\neq i_{r+1}}}(-1)^{i_r}f(x_1,\ldots,d_A(x_{i_1}),\ldots,d_A(x_{i_r})x_{i_r+1},\ldots,d_A(x_{i_k}),\ldots,x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_{k+1}\le n+1}\sum_{\substack{1\le r\le k\\ i_r+1=i_{r+1}}}(-1)^{i_r}f(x_1,\ldots,d_A(x_{i_1}),\ldots,d_A(x_{i_r})d_A(x_{i_{r+1}}),\ldots,d_A(x_{i_{k+1}}),\ldots,x_{n+1})\\
=&\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le j\le n\\ j\neq i_r,i_r-1,\,\forall r}}(-1)^{j}f(\ldots,d_A(x_{i_1}),\ldots,x_jx_{j+1},\ldots,d_A(x_{i_k}),\ldots)\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le r\le k\\ i_r-1\neq i_{r-1}}}(-1)^{i_r-1}f(x_1,\ldots,d_A(x_{i_1}),\ldots,x_{i_r-1}d_A(x_{i_r}),\ldots,d_A(x_{i_k}),\ldots,x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le r\le k-1\\ i_r+1\neq i_{r+1}}}(-1)^{i_r}f(x_1,\ldots,d_A(x_{i_1}),\ldots,d_A(x_{i_r})x_{i_r+1},\ldots,d_A(x_{i_k}),\ldots,x_{n+1})\\
&+\sum_{k=2}^{n+1}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le r\le k-1\\ i_r+1=i_{r+1}}}(-1)^{i_r}f(x_1,\ldots,d_A(x_{i_1}),\ldots,d_A(x_{i_r})d_A(x_{i_{r+1}}),\ldots,d_A(x_{i_k}),\ldots,x_{n+1}).
\end{aligned}
$$

■

Now we are ready to prove Lemma 2.11.

PROOF. (of Lemma 2.11) As $d_V(xv) = d_A(x)v+x \vdash_\lambda d_V(v)$, $d_V(vx) = vd_A(x)+d_V(v) \dashv_\lambda x$ for $v \in V, x \in A$, we have

$$\begin{aligned}
d_V(\partial_{\mathsf{Alg}} f(x_{1,n+1})) = & \, d_A(x_1)f(x_{2,n+1}) + x_1 \vdash_\lambda d_V(f(x_{2,n+1})) \\
& + \sum_{j=1}^{n}(-1)^j d_V(f(x_{1,j-1}, x_j x_{j+1}, x_{j+2,n+1})) \\
& + (-1)^{n+1} f(x_{1,n}) d_A(x_{n+1}) + (-1)^{n+1} d_V(f(x_{1,n})) \dashv_\lambda x_{n+1}.
\end{aligned}$$

Hence, we only need to check Eq. (3) as follows. By Lemma 4.12, we have

$$\begin{aligned}
& \sum_{k=1}^{n+1} \lambda^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq n+1} (\partial_{\mathsf{Alg}} f)^{(i_1,\ldots,i_k)}(x_{1,n+1}) \\
= & \lambda^n \partial_{\mathsf{Alg}} f(d_A(x_1),\ldots,d_A(x_{n+1})) + \sum_{k=1}^{n} \lambda^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq n+1} (\partial_{\mathsf{Alg}} f)^{(i_1,\ldots,i_k)}(x_{1,n+1}) \\
= & \lambda^n d_A(x_1) f(d_A(x_2),\ldots,d_A(x_{n+1})) + \lambda^n \sum_{i=1}^{n} (-1)^i f(d_A(x_1),\ldots,d_A(x_i)d_A(x_{i+1}),\ldots,d_A(x_{n+1})) \\
& + (-1)^{n+1}\lambda^n f(d_A(x_1),\ldots,d_A(x_n)) d_A(x_{n+1}) \\
& + d_A(x_1) f(x_{2,n+1}) + \sum_{k=2}^{n} \lambda^{k-1} \sum_{2\leq i_2<\cdots<i_k\leq n+1} d_A(x_1) f^{(i_2-1,\ldots,i_k-1)}(x_{2,n+1}) \\
& + \sum_{k=1}^{n} \lambda^{k-1} \sum_{2\leq i_1<\cdots<i_k\leq n+1} x_1 f^{(i_1-1,\ldots,i_k-1)}(x_{2,n+1}) \\
& + \sum_{k=1}^{n} \lambda^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq n+1} \sum_{\substack{1\leq j\leq n \\ j\neq i_r-1, i_r,\, \forall r}} (-1)^j f(x_1,\ldots,d_A(x_{i_1}),\ldots,x_j x_{j+1},\ldots,d_A(x_{i_k}),\ldots,x_{n+1}) \\
& + \sum_{k=1}^{n} \lambda^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq n+1} \sum_{\substack{1\leq r\leq k \\ i_r-1\neq i_{r-1}}} (-1)^{i_r-1} f(x_1,\ldots,d_A(x_{i_1}),\ldots,x_{i_r-1} d_A(x_{i_r}),\ldots,d_A(x_{i_k}),\ldots,x_{n+1}) \\
& + \sum_{k=1}^{n} \lambda^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq n+1} \sum_{\substack{1\leq r\leq k-1 \\ i_r+1\neq i_{r+1}}} (-1)^{i_r} f(x_1,\ldots,d_A(x_{i_1}),\ldots,d_A(x_{i_r}) x_{i_r+1},\ldots,d_A(x_{i_k}),\ldots,x_{n+1}) \\
& + \sum_{k=1}^{n} \lambda^{k-1} \sum_{1\leq i_1<\cdots<i_k\leq n+1} \sum_{\substack{1\leq r\leq k-1 \\ i_r+1= i_{r+1}}} (-1)^{i_r} f(x_1,\ldots,d_A(x_{i_1}),\ldots,d_A(x_{i_r}) d_A(x_{i_{r+1}}),\ldots,d_A(x_{i_k}),\ldots,x_{n+1}) \\
& + (-1)^{n+1} f(x_{1,n}) d_A(x_{n+1}) + \sum_{k=2}^{n} \lambda^{k-1} (-1)^{n+1} \sum_{1\leq i_1<\cdots<i_{k-1}\leq n} f^{(i_1,\ldots,i_{k-1})}(x_{1,n}) d_A(x_{n+1})
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\lambda^{k-1}(-1)^{n+1}\sum_{1\le i_1<\cdots<i_k\le n} f^{(i_1,\dots,i_k)}(x_{1,n})x_{n+1}\\
=&\lambda^n d_A(x_1)f(d_A(x_2),\dots,d_A(x_{n+1}))+(-1)^{n+1}\lambda^n f(d_A(x_1),\dots,d_A(x_n))d_A(x_{n+1})\\
&+d_A(x_1)f(x_{2,n+1})+\sum_{k=1}^{n-1}\lambda^k\sum_{1\le i_1<\cdots<i_k\le n} d_A(x_1)f^{(i_1,\dots,i_k)}(x_{2,n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n} x_1f^{(i_1,\dots,i_k)}(x_{2,n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le j\le n\\ j\neq i_r-1,i_r,\,\forall r}}(-1)^j f(x_1,\dots,d_A(x_{i_1}),\dots,x_jx_{j+1},\dots,d_A(x_{i_k}),\dots,x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le r\le k\\ i_r-1\neq i_{r-1}}}(-1)^{i_r-1} f(x_1,\dots,d_A(x_{i_1}),\dots,x_{i_r-1}d_A(x_{i_r}),\dots,d_A(x_{i_k}),\dots,x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le r\le k-1\\ i_r+1\neq i_{r+1}}}(-1)^{i_r} f(x_1,\dots,d_A(x_{i_1}),\dots,d_A(x_{i_r})x_{i_r+1},\dots,d_A(x_{i_k}),\dots,x_{n+1})\\
&+\sum_{k=1}^{n+1}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le r\le k-1\\ i_r+1= i_{r+1}}}(-1)^{i_r} f(x_1,\dots,d_A(x_{i_1}),\dots,d_A(x_{i_r})d_A(x_{i_{r+1}}),\dots,d_A(x_{i_k}),\dots,x_{n+1})\\
&+(-1)^{n+1}f(x_{1,n})d_A(x_{n+1})+\sum_{k=1}^{n-1}\lambda^k(-1)^{n+1}\sum_{1\le i_1<\cdots<i_k\le n} f^{(i_1,\dots,i_k)}(x_{1,n})d_A(x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}(-1)^{n+1}\sum_{1\le i_1<\cdots<i_k\le n} f^{(i_1,\dots,i_k)}(x_{1,n})x_{n+1}\\
=&d_A(x_1)f(x_{2,n+1})+(-1)^{n+1}f(x_{1,n})d_A(x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n} x_1f^{(i_1,\dots,i_k)}(x_{2,n+1})\\
&+\sum_{k=1}^{n}\lambda^{k}\sum_{1\le i_1<\cdots<i_k\le n} d_A(x_1)f^{(i_1,\dots,i_k)}(x_{2,n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le j\le n\\ j\neq i_r-1,i_r,\,\forall r}}(-1)^j f(x_1,\dots,d_A(x_{i_1}),\dots,x_jx_{j+1},\dots,d_A(x_{i_k}),\dots,x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le r\le k\\ i_r-1\neq i_{r-1}}}(-1)^{i_r-1} f(x_1,\dots,d_A(x_{i_1}),\dots,x_{i_r-1}d_A(x_{i_r}),\dots,d_A(x_{i_k}),\dots,x_{n+1})
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le r\le k-1\\ i_r+1\ne i_{r+1}}}(-1)^{i_r}f(x_1,\ldots,d_A(x_{i_1}),\ldots,d_A(x_{i_r})x_{i_r+1},\ldots,d_A(x_{i_k}),\ldots,x_{n+1})\\
&+\sum_{k=2}^{n+1}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n+1}\sum_{\substack{1\le r\le k-1\\ i_r+1= i_{r+1}}}(-1)^{i_r}f(x_1,\ldots,d_A(x_{i_1}),\ldots,d_A(x_{i_r})d_A(x_{i_{r+1}}),\ldots,d_A(x_{i_k}),\ldots,x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k}(-1)^{n+1}\sum_{1\le i_1<\cdots<i_k\le n}f^{(i_1,\ldots,i_k)}(x_{1,n})d_A(x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}(-1)^{n+1}\sum_{1\le i_1<\cdots<i_k\le n}f^{(i_1,\ldots,i_k)}(x_{1,n})x_{n+1}\\
=&d_A(x_1)f(x_{2,n+1})+(-1)^{n+1}f(x_{1,n})d_A(x_{n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n}x_1\vdash_\lambda f^{(i_1,\ldots,i_k)}(x_{2,n+1})\\
&+\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n}\sum_{j=1}^{n}(-1)^j f^{(i_1,\ldots,i_k)}(x_{1,j-1},x_jx_{j+1},x_{j+2,n+1})\\
&+(-1)^{n+1}\sum_{k=1}^{n}\lambda^{k-1}\sum_{1\le i_1<\cdots<i_k\le n}f^{(i_1,\ldots,i_k)}(x_{1,n})\dashv_\lambda x_{n+1}.
\end{aligned}
$$

■

References

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Department of Mathematics and Computer Science, Rutgers University
Newark, NJ 07102, United States

School of Mathematics and Information Science, Guangzhou University
Guangzhou 510006, China

Department of Mathematics, Jilin University
Changchun 130012, Jilin, China

School of Mathematical Sciences, Shanghai Key laboratory of PMMP, East China Normal University
Shanghai 200241, China

Email: liguo@rutgers.edu
ynli@gzhu.edu.cn
shengyh@jlu.edu.cn
gdzhou@math.ecnu.edu.cn

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