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THE CATEGORY OF L-ALGEBRAS

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ABSTRACT. The category **LAlg** of L-algebras is shown to be complete and cocomplete, regular with a zero object and a projective generator, normal and subtractive, ideal determined, but not Barr-exact. Originating from algebraic logic, L-algebras arise in the theory of Garside groups, measure theory, functional analysis, and operator theory. It is shown that the category **LAlg** is far from protomodular, but it has natural semidirect products which have not been described in category-theoretic terms.

1. Introduction

As a non-additive generalization of abelian categories, Barr-exact [3] and protomodular categories [8] are fundamental. Every topos is Barr-exact; the dual of a topos, and many classical categories (groups, rings, Lie algebras, Heyting algebras, crossed modules, etc.) are Barr-exact and protomodular [10]. Additive categories are Barr-exact if and only if they are abelian, and pointed Barr-exact categories are protomodular if and only if they satisfy the (Split) Short Five Lemma [11]. For a pointed Barr-exact category with pushouts of split monomorphisms, protomodularity is equivalent to the existence of *semidirect products* in the sense of [11].

In the additive context, *exact categories* [47, 15] in the sense of Quillen [63] typically arise as full subcategories of abelian categories. For example, many categories of topological vector spaces, and all *quasi-abelian* categories [75], are exact in a natural way. A non-additive analogue consists in the *regular categories* [3]. Every regular category admits a canonical embedding into a Barr-exact category, its *exact completion* [51, 18].

In this paper, we analyse the category **LAlg** of *L-algebras* [67]. We show that **LAlg** is complete and cocomplete, pointed (i. e. with zero object), and regular (Proposition 4.5), with a natural kind of semidirect product (Section 7) which is not covered by any known categorical construction [11, 56, 57, 58, 13]. L-algebras $(X;\cdot)$ are defined by a single binary operation. There is an element $1 \in X$ (necessarily unique) satisfying $1 \cdot x = x$ and $x \cdot x = x \cdot 1 = 1$, and

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z)$$
$$x \cdot y = y \cdot x = 1 \implies x = y$$

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holds in X. Without the latter implication, X is called a *unital cycloid* [67]. Thus unital cycloids form a variety $\mathbf{Cyc}^*$. Examples of L-algebras are Brouwerian semilattices [49] (e. g., Heyting algebras, locales [54]), MV-algebras [20, 21, 35], measure algebras [55, 32, 71], projection lattices of von Neumann algebras [46, 70], and lattice effect algebras [29, 65, 81]. Many other structures are determined by L-algebras. For example, Artin's braid group [2] is associated with an L-algebra, and projective spaces with an elliptic polarity [16, 39] are L-algebras where the element 1 has been removed [72].

Every L-algebra X is partially ordered ($x \leqslant y :\Leftrightarrow x \cdot y = 1$), with a universal map

$$q_X \colon X \to G(X)$$

into a group, the *structure group* [67] of X. For example, the structure group of a non-degenerate involutive set-theoretic solution to the Yang-Baxter equation [30] comes from an L-algebra [69]. In this and other cases, the structure group is a *right ℓ-group* [69] (a group with a lattice order such that the right multiplications are lattice automorphisms), and q_X embeds X as an L-subalgebra into the negative cone of $G(X)$. (For any right ℓ-group G, the negative cone $G^- := \{g \in G \mid g \leqslant 1\}$ is an L-algebra.)

The class of right ℓ-groups is very wide. Spherical Artin-Tits groups [14, 27] and more generally, all Garside groups [34, 24, 25, 26], are right ℓ-groups. They are structure groups of a finite L-algebra. The structure group of an orthomodular lattice X is a right ℓ-group which determines X up to isomorphism [70]. Two-sided ℓ-groups [6, 23] arise, e. g., as spaces of continuous functions [52]. Mundici's equivalence [61] and its generalization to non-abelian ℓ-groups [28] admit a simple reformulation and proof in terms of the structure group of a commutative L-algebra.

Now let us return to the category **LAlg** of L-algebras. We prove that the variety $\mathbf{Cyc}^*$ is the exact completion of **LAlg** (Theorem 6.1). Unlike general regular categories, **LAlg** can be retrieved from its exact completion by a process similar to the formation of the Lindenbaum algebra in logic. The proof of Theorem 6.1 rests upon the fact that free unital cycloids are L-algebras (Theorem 5.3). Moreover, the partial order of a free L-algebra is trivial in the sense that all non-maximal elements are pairwise incomparable (Theorem 5.3). Such L-algebras have an underlying projective geometry [69, 72]. We show that the regular epimorphisms of an L-algebra are normal and surjective (Proposition 4.3), while monomorphisms in **LAlg** are injective maps (Corollary of Proposition 4.2). There is a reflective full subcategory **ssL** of *self-similar* L-algebras. Its reflector $S\colon \mathbf{LAlg} \to \mathbf{ssL}$ embeds any L-algebra X into its *self-similar closure*. In contrast to **LAlg**, the L-algebras in **ssL** form a variety. Besides its L-algebra operation, a self-similar L-algebra has a monoid structure, and its partial order is a $\wedge$-semilattice. In the above examples, the self-similar closure is the negative cone of the structure group.

Despite the close relationship between L-algebras and groups, the categories **LAlg**, $\mathbf{Cyc}^*$ and **ssL** are not protomodular (Examples 7 and 8), and thus don't have semidirect products in the sense of [11]. On the other hand, it has been known from the beginning that semidirect products of L-algebras exist in a very natural way [68]. In fact, there is a natural concept of action [68] of an L-algebra U on an L-algebra I, which leads to a semidirect product $I \rtimes U$, and there is a corresponding short exact sequence

$$I \hookrightarrow I \rtimes U \overset{p}{\twoheadrightarrow} U$$

with an ideal I of $I \rtimes U$, and a split epimorphism p. The failure of protomodularity cannot be repaired by concepts like "S-protomodularity" [13]. For a semidirect product $I \rtimes U$ of L-algebras, the embedding $U \hookrightarrow I \rtimes U$ is a *strong section* (Definition 7.1), with no relationship to "strong points" [13, 58]. Conversely, we prove that any strongly split short exact sequence $I \hookrightarrow X \twoheadrightarrow U$ extends, up to isomorphism, to a unique short exact sequence $I \hookrightarrow I \rtimes U \twoheadrightarrow U$ (Theorem 7.4).

We prove that the pointed regular category **LAlg** is normal [41] and subtractive [40] (Proposition 8.1), which implies that the upper and lower 3×3 lemma [41] holds in **LAlg**. Furthermore, we show that **LAlg** has a "good theory of ideals" [37], that is, **LAlg** is *ideal determined* [45] in the sense that normal subobjects are mapped to normal subobjects under a regular epimorphism f. As **LAlg** is not Barr-exact, this gives a counter-example to a question in [45]. More importantly, we prove that f respects finite intersections of ideals (Proposition 8.2), which implies that the lattice of ideals of an L-algebra is distributive. Regular epimorphisms of L-algebras are shown to be effective descent morphisms (Proposition 8.3).

If the operation of an L-algebra is interpreted as implication, its axioms provide a logical formalism which specializes to three known types of algebraic logic [73], including quantum logic where the structure group determines the L-algebra. We show that free L-algebras arise from a single axiom and four inference rules which are closely related to the (not so obvious) defining properties of an L-algebra ideal. The logic of L-algebras is shown to be complete (Proposition 5.2). The ideals of an L-algebra are in one-to-one correspondence with the ideals of its self-similar closure (Theorem 3.5). For self-similar L-algebras, the everywhere defined multiplication allows a simple, more customary characterization of ideals (Proposition 3.6).

2. L-algebras as partial monoids

In this section, we recall the concept of an L-algebra [67], a system $(X; \to)$ with a binary operation, which can be interpreted as logical implication. Since applications of L-algebras go far beyond algebraic logic, we use the more convenient notation with a dot instead of an implicational arrow.

Thus, let $(X; \cdot)$ be a set with a binary operation. An element $1 \in X$ is said to be a *logical unit* [67] if the equations

$$1 \cdot x = x \text{ and } x \cdot x = x \cdot 1 = 1 \tag{1}$$

hold for all $x \in X$. Eqs. (1) collect basic properties of a constant 1 which characterizes logical truth. Since $x \cdot x = 1$, a logical unit must be unique. In logical terms, the relation

$$x \leqslant y \ :\Longleftrightarrow \ x \cdot y = 1 \tag{2}$$

interprets logical entailment. The following equation

$$(x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) \tag{3}$$

holds in fundamental systems of algebraic logic [73] like Heyting algebras [59, 54], MV-algebras [20, 21, 35, 71], and orthomodular lattices [46, 70]. If X satisfies Eqs. (1) and (3), it is said to be a *unital cycloid*[1] [67]. Eq. (3) guarantees that the entailment relation (2) is transitive: $x \leqslant y \leqslant z \Rightarrow x \cdot z = (x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) = (y \cdot x) \cdot 1 = 1$. Furthermore, Eqs. (1) and (3) yield

$$y \leqslant z \implies x \cdot y \leqslant x \cdot z. \tag{4}$$

A subset I of a unital cycloid X is said to be an *ideal* [67] if $1 \in I$ and

$$x,\ x \cdot y \in I \implies y \in I \tag{5}$$

$$x \in I \implies (x \cdot y) \cdot y,\ y \cdot x,\ y \cdot (x \cdot y) \in I \tag{6}$$

holds in X. By [67], Proposition 1, every congruence $\equiv$ defines an ideal

$$I := \{x \in X \mid x \equiv 1\},$$

and each ideal I gives rise to a congruence

$$x \equiv y \ :\Longleftrightarrow\ x \cdot y,\ y \cdot x \in I. \tag{7}$$

So the conguence classes form a unital cycloid X/I. The ideal $\{1\}$ leads to a congruence (7) which signifies logical equivalence. It is natural to take it as equality:

$$x \cdot y = y \cdot x = 1 \implies x = y, \tag{8}$$

so that entailment (2) becomes a partial order of X.

2.1. Definition. A set $(X; \cdot)$ with a binary operation is said to be an *L-algebra* [67] if it satisfies Eqs. (1) and (3) together with the implication (8).

Now we show that every L-algebra X has a partial multiplication. For each $x \in X$ there is a map $\sigma_x \colon \downarrow x \to X$ from the downset $\downarrow x := \{y \in X \mid y \leqslant x\}$ to X, given by $\sigma_x(y) := x \cdot y$. By Eq. (3), we have

$$\sigma_x(y) \cdot \sigma_x(z) = (x \cdot y) \cdot (x \cdot z) = (y \cdot x) \cdot (y \cdot z) = y \cdot z$$

for $y, z \leqslant x$. Thus (2) implies that each σ_x is an order isomorphism from $\downarrow x$ to a subposet of X. In particular, the σ_x are injective. They give rise to a partial multiplication in X:

2.2. Definition. Let X be an L-algebra, and $x, y \in X$. We say that the product xy is *defined* in X if $x = \sigma_y(z)$ for some $z \leqslant y$. If xy is defined, we set $xy := z$.

Note that the element xy is unique since σ_y is injective.

[1] The terminology comes from *cycle sets* [66] which characterize a class of set-theoretic solutions to the Yang-Baxter equation.

2.3. Proposition. *Let X be an L-algebra, and $x, y, z \in X$. If xy exists, the following equations hold in X:*

$$xy \cdot z = x \cdot (y \cdot z)$$
$$z \cdot xy = \big((y \cdot z) \cdot x\big)(z \cdot y).$$

Proof. By Definition 2.2, $xy \leqslant y$ and $y \cdot xy = x$. Hence Eq. (3) implies that $xy \cdot z = 1 \cdot (xy \cdot z) = (xy \cdot y) \cdot (xy \cdot z) = (y \cdot xy) \cdot (y \cdot z) = x \cdot (y \cdot z)$, which proves the first equation.

Furthermore, $(z \cdot y) \cdot (z \cdot xy) = (y \cdot z) \cdot (y \cdot xy) = (y \cdot z) \cdot x$. Since $xy \cdot y = 1$, (4) implies that $z \cdot xy \leqslant z \cdot y$. So the second equation follows by Definition 2.2. □

As a consequence, we have the following adjointness property:

Corollary 1. *If xy exists, then $xy \leqslant z \Leftrightarrow x \leqslant y \cdot z$.*

Moreover, the partial multiplication is associative:

Corollary 2. *Let X be an L-algebra, and $x, y, z \in X$. Then $(xy)z = x(yz)$ holds if both sides of the equation exist. Furthermore, $1x = x1 = x$ holds in X.*

Proof. We have $(xy)z \cdot x(yz) = xy \cdot \big(z \cdot x(yz)\big) = x \cdot \big(y \cdot (z \cdot x(yz))\big) = x \cdot \big(yz \cdot x(yz)\big) = x \cdot x = 1$ and $x(yz) \cdot (xy)z = x \cdot \big(yz \cdot (xy)z\big) = x \cdot \big(y \cdot (z \cdot (xy)z)\big) = x \cdot (y \cdot xy) = x \cdot x = 1$. Hence $(xy)z = x(yz)$ follows by (8). Furthermore, $x \leqslant 1$ and $1 \cdot x = x$ gives $x1 = x$ by Definition 2.2. Similarly, $x \leqslant x$ and $x \cdot x = 1$ yields $1x = x$. □

Let $\mathbf{Cyc}^*$ be the category of unital cycloids, with maps $f\colon X \to Y$ satisfying $f(x \cdot y) = f(x) \cdot f(y)$ as morphisms. Since $x \cdot x = 1$, a morphism f satisfies $f(1) = 1$. By $\mathbf{LAlg}$ we denote the full subcategory of L-algebras. A subset X of an L-algebra Y is said to be an *L-subalgebra* if it is closed under the operation of Y, that is, X carries the L-algebra structure for which $X \hookrightarrow Y$ is a morphism. An L-subalgebra X is said to be *invariant* if $y \cdot x \in X$ for all $x \in X$ and $y \in Y$. By (6), every ideal is an invariant L-subalgebra. For a morphism $f\colon X \to Y$, the image $\operatorname{Im} f = f(X)$ is an L-subalgebra of Y.

2.4. Proposition. *Every L-algebra morphism $f\colon X \to Y$ is monotone. If $x, y \in X$, and xy exists in X, then $f(x)f(y)$ exists in Y, and $f(xy) = f(x)f(y)$.*

Proof. Assume that $x, y \in X$. If $x \leqslant y$, then $f(x) \cdot f(y) = f(x \cdot y) = f(1) = 1$, which shows that f is monotone. Now assume that xy exists. Then $y \cdot xy = x$ and $xy \leqslant y$. Hence $f(y) \cdot f(xy) = f(y \cdot xy) = f(x)$ and $f(xy) \leqslant f(y)$. By Definition 2.2, this yields $f(xy) = f(x)f(y)$. □

3. Self-similarity

For an L-algebra X, the maps $\sigma_x\colon \downarrow x \to X$ are injective. If they are bijective, the L-algebra is order-isomorphic to each of its downsets, which explains the terminology of the following

3.1. Definition. An L-algebra X is said to be *self-similar* [67] if the maps $\sigma_x\colon \downarrow x \to X$ are bijective for each $x \in X$.

By Definition 2.2, an L-algebra is self-similar if and only if its partial multiplication is everywhere defined. By Proposition 2.3, a self-similar L-algebra satisfies the equations

$$x \cdot yx = y \tag{9}$$
$$xy \cdot z = x \cdot (y \cdot z) \tag{10}$$
$$x \cdot yz = \big((z \cdot x) \cdot y\big)(x \cdot z). \tag{11}$$

Hence Eq. (10) implies that $(x \cdot y)x \leqslant y$, and Eq. (11) yields

$$y \cdot (x \cdot y)x = \big((x \cdot y) \cdot (x \cdot y)\big)(y \cdot x) = y \cdot x.$$

By Definition 2.2, this gives

$$(x \cdot y)x = (y \cdot x)y. \tag{12}$$

3.2. Proposition. *A self-similar L-algebra is a monoid with an operation $\cdot$ which satisfies Eqs. (9), (10), and (12). Conversely, such a monoid is a self-similar L-algebra.*

Proof. It remains to prove the converse. Thus assume that X is a monoid with an operation $\cdot$ which satisfies Eqs. (9), (10), and (12). Then Eq. (10) and (12) give $(x \cdot y) \cdot (x \cdot z) = (x \cdot y)x \cdot z = (y \cdot x)y \cdot z = (y \cdot x) \cdot (y \cdot z)$. Eq. (12) implies (8). Furthermore, Eq. (9) yields $1 \cdot x = 1 \cdot x1 = x$ and $x \cdot x = x \cdot 1x = 1$. By Eqs. (9) and (12), we obtain $x \cdot 1 = x \cdot (x \cdot 1)x = x \cdot (1 \cdot x)1 = x \cdot x = 1$. Thus X is an L-algebra with a globally defined multiplication. Whence X is self-similar. □

By [68], Proposition 1, the full subcategory **ssL** of self-similar L-algebras in **LAlg** is reflective, that is, the inclusion functor $I\colon \mathbf{ssL} \hookrightarrow \mathbf{LAlg}$ has a left adjoint $S\colon \mathbf{LAlg} \to \mathbf{ssL}$. The components of the unit $\eta\colon 1 \to IS$ are inclusions $\eta_X\colon X \hookrightarrow S(X)$, and $S(X)$ is called the *self-similar closure* of X. By [67], Theorem 3, we have the following characterization of $S(X)$:

3.3. Theorem. *Let X be an L-subalgebra of a self-similar L-algebra A. Then A is isomorphic to $S(X)$ if and only if the monoid A is generated by X.*

Remark. Note that by Corollary 1 of Proposition 2.3, the L-algebra structure of a self-similar L-algebra X is determined by the associated monoid structure: $y \cdot z$ is the greatest $x \in X$ with $xy \leqslant z$. By contrast, an arbitrary L-algebra need not be determined by its partial multiplication:

Example 1. There are three isomorphism types of L-algebras $X = \{1, x, y\}$ with incomparable x, y. However, all existing products are trivial: If xy exists, then $y \cdot xy = x$ and $xy \leqslant y$, which is impossible.

By [67], Proposition 4, every self-similar L-algebra A is a $\wedge$-semilattice with $a \wedge b = (a \cdot b)a$ which satisfies the equations

$$a \cdot (b \wedge c) = (a \cdot b) \wedge (a \cdot c) \tag{13}$$
$$(a \wedge b) \cdot c = (a \cdot b) \cdot (a \cdot c). \tag{14}$$

3.4. Proposition. *Let X be an L-algebra. The following are equivalent:*

(a) *For $y, z \in X$, the element $y \cdot z$ is the greatest $x \in X$ so that xy exists and $xy \leqslant z$.*
(b) *X is a $\wedge$-semilattice such that each L-algebra morphism $f\colon X \to Y$ is $\wedge$-preserving.*

Proof. (a) $\Rightarrow$ (b): By assumption, the product $d := (y \cdot z)y$ exists, and $d \leqslant z$. Thus $y \cdot d = y \cdot z$ and $d \leqslant y$. We show that $d = y \wedge z$. If $x \leqslant y, z$, then the second equation in Proposition 2.3 gives $x \cdot d = \big((y \cdot x) \cdot (y \cdot z)\big)(x \cdot y) = (x \cdot y) \cdot (x \cdot z) = 1$. Thus $x \leqslant d$, which proves $d = y \wedge z$. Now let $f\colon X \to Y$ be a morphism in **LAlg**. Then Proposition 2.4 gives $f(d) = \big(f(y) \cdot f(z)\big)f(y)$, and $f(d) \leqslant f(z)$. So the above argument shows that $f(d) = f(y) \wedge f(z)$.

(b) $\Rightarrow$ (a): Let $y, z \in X$ be given. Since $S(X)$ is a $\wedge$-semilattice, (b) implies that X is a sub-semilattice of $S(X)$. Thus Eq. (13) gives $y \cdot (y \wedge z) = (y \cdot y) \wedge (y \cdot z) = y \cdot z$. By Definition 2.2, we obtain $y \wedge z = (y \cdot z)y$. So Corollary 1 of Proposition 2.3 completes the proof. $\square$

Example 2. Every Boolean algebra X is an L-algebra with $x \cdot y := x' \vee y$, where x' denotes the complement of x. Moreover, X has a smallest element 0, and $x' = x \cdot 0$.

Example 3. The free monoid $\langle x \rangle = \{x^n \mid n \in \mathbb{N}\}$ is a self-similar L-algebra with

$$x^n \cdot x^m := \begin{cases} 1 & \text{for } n \geqslant m \\ x^{m-n} & \text{for } n \leqslant m. \end{cases}$$

By Theorem 3.3, the Boolean L-subalgebra $\{x, 1\}$ has $\langle x \rangle$ as its self-similar closure.

Eq. (9)-(10) imply that any self-similar L-algebra A satisfies $ac \cdot bc = a \cdot (c \cdot bc) = a \cdot b$. Hence

$$ac \leqslant bc \iff a \leqslant b. \tag{15}$$

In particular, A is right cancellative. Eq. (12) implies the left Ore condition:

$$\forall\, a, b \;\exists\, c, d\colon ca = db.$$

Hence, for each L-algebra X, the self-similar closure has a group of left fractions $G(X)$, with a natural map

$$q_X\colon X \hookrightarrow S(X) \to G(X). \tag{16}$$

The group $G(X)$ is said to be the *structure group* of X (see [67] for a more detailed description). There are important cases where q_X is injective and the partial order of X extends to a lattice order of $G(X)$ such that (15) holds in $G(X)$. Then the right multiplications in $G(X)$ are order automorphisms, which implies that

$$(a \vee b)c = ac \vee bc, \qquad (a \wedge b)c = ac \wedge bc.$$

Such a group with a right invariant lattice order is said to be a *right ℓ-group* [69].

Now we turn our attention to the ideals in self-similar L-algebras. The following theorem was proved in [68], Corollary 2 of Theorem 1.

3.5. THEOREM. *Let X be an L-algebra. The maps $I \mapsto S(I)$ and $J \mapsto J \cap X$ establish a one-to-one correspondence between the ideals I of X and the ideals J of $S(X)$.*

As a consequence, the functor $S\colon \mathbf{LAlg} \to \mathbf{ssL}$ respects short exact sequences:

COROLLARY. *Every ideal I of an L-algebra X gives rise to a commutative diagram*

$$\begin{array}{ccccc} I & \hookrightarrow & X & \overset{p}{\twoheadrightarrow} & X/I \\ \downarrow & & \downarrow & & \downarrow \\ S(I) & \hookrightarrow & S(X) & \twoheadrightarrow & S(X/I) \end{array}$$

with $S(X/I) \cong S(X)/S(I)$.

Proof. Theorem 3.5 shows that the left-hand square is a pullback, which yields the diagram with $S(X/I)$ replaced by $S(X)/S(I)$. The induced morphism $f\colon X/I \to S(X)/S(I)$ is injective. Indeed, if $p(x)$ and $p(y)$ are mapped to the same element of $S(X)/S(I)$, then $x \cdot y$ and $y \cdot x$ are in $S(I)$, hence in I, which yields $p(x) = p(y)$. By Proposition 3.2, $S(X)/S(I)$ is self-similar. Thus Theorem 3.3 shows that $S(X/I) \cong S(X)/S(I)$. □

3.6. PROPOSITION. *A subset I of a self-similar L-algebra A is an ideal if and only if $1 \in I$ and*

$$x, y \in I \iff xy \in I \tag{17}$$

$$x \in I,\ a \in A \Longrightarrow xa \cdot ax,\ ax \cdot xa \in I. \tag{18}$$

Proof. Assume that I is an ideal of A. If $x, y \in I$, then Eq. (9) gives $y \cdot xy = x \in I$. Hence (5) shows that $xy \in I$. Conversely, assume that $xy \in I$. Then (9) and (6) imply that $x = y \cdot xy \in I$. Furthermore, Eq. (10) gives $xy \cdot y = x \cdot (y \cdot y) = 1 \in I$. Thus (5) yields $y \in I$. This proves (17).

Now assume that $x \in I$ and $a \in A$. Then (11) and (6) give $a \cdot ax = \big((x \cdot a) \cdot a\big)(a \cdot x) \in I$. Hence Eq. (10) yields $xa \cdot ax = x \cdot (a \cdot ax) \in I$. Furthermore, $ax \cdot xa = a \cdot (x \cdot xa) = a \cdot \big((a \cdot x) \cdot x\big)(x \cdot a) = \big(((x \cdot a) \cdot a) \cdot ((a \cdot x) \cdot x)\big)\big(a \cdot (x \cdot a)\big) \in I$, which proves (18).

Conversely, let $1 \in I$ and (17)-(18) be satisfied. Assume that $x, x \cdot y \in I$. Then Eq. (12) implies that $(y \cdot x)y = (x \cdot y)x \in I$. Hence $y \in I$, which yields (5). Now assume that $x \in I$ and $a \in A$. Then $x \cdot (a \cdot ax) = xa \cdot ax \in I$. Hence (5) yields $a \cdot ax \in I$, and thus, Eq. (11) gives $\big((x \cdot a) \cdot a\big)(a \cdot x) = a \cdot ax \in I$. So we obtain $(x \cdot a) \cdot a \in I$ and $a \cdot x \in I$. Finally, the above calculation yields $\big(((x \cdot a) \cdot a) \cdot ((a \cdot x) \cdot x)\big)\big(a \cdot (x \cdot a)\big) = ax \cdot xa \in I$. Whence $a \cdot (x \cdot a) \in I$, which completes the proof of (6). Thus I is an ideal. □

4. The category of L-algebras

In the category of L-algebras, kernels and cokernels behave quite similar to the corresponding notions in more well-known categories. For a morphism $f\colon X \to Y$ of L-algebras, a *kernel* in the categorical sense is given by the subobject $\operatorname{Ker} f := f^{-1}(1)$ of X, which is an ideal of X. Conversely, every ideal I of an L-algebra X gives rise to a congruence (7), hence to a canonical morphism $p\colon X \twoheadrightarrow X/I$ onto an L-algebra X/I (see [67], Section 1).

4.1. Proposition. *Every L-algebra morphism $f\colon X \to Y$ admits a factorization $f\colon X \twoheadrightarrow \operatorname{Im} f \hookrightarrow Y$ with $\operatorname{Im} f \cong X/\operatorname{Ker} f$.*

Proof. For $x, y \in X$, $f(x) = f(y) \Leftrightarrow f(x)\cdot f(y) = f(y)\cdot f(x) = 1 \Leftrightarrow f(x\cdot y) = f(y\cdot x) = 1$. Hence $f(x) = f(y)$ if and only if x and y are congruent modulo $\operatorname{Ker} f$. □

Corollary. *Let X be an L-algebra. There is an one-to-one correspondence between ideals of X and surjective morphisms $X \twoheadrightarrow Y$ in* **LAlg**, *up to an isomorphism of Y.*

Proof. For a surjective morphism $p\colon X \twoheadrightarrow Y$, we have $Y \cong X/\operatorname{Ker} f$. Conversely, every ideal I of X gives rise to a surjective morphism $p\colon X \twoheadrightarrow X/I$. Then $x \in \operatorname{Ker} p \Leftrightarrow x\cdot 1, 1\cdot x \in I \Leftrightarrow x \in I$. □

The category **LAlg** of L-algebras has a zero object $\mathbf{1} = \{1\}$. Accordingly, a morphism which factors through $\mathbf{1}$ is called a *zero morphism.* A sequence

$$X \overset{u}{\rightarrowtail} Y \overset{v}{\twoheadrightarrow} Z \tag{19}$$

in **LAlg** is said to be *short exact* [68] if v is surjective and u is a *kernel* of v in the sense that every morphism f for which vf is a zero morphism factors uniquely through u. In other words, u coincides with $\operatorname{Ker} v \hookrightarrow Y$, up to an isomorphism $X \xrightarrow{\sim} \operatorname{Ker} v$. If v is a split epimorphism, the sequence is said to be *split short exact.*

4.2. Proposition. *For a morphism $f\colon X \to Y$ of L-algebras, the following are equivalent:*

(a) *f is a monomorphism.*
(b) $\operatorname{Ker} f = \mathbf{1}$.
(c) *f is injective.*

Proof. (a) ⇒ (b) holds in any pointed category with kernels, and (c) ⇒ (a) is trivial.

(b) ⇒ (c): Assume that $f(x) = f(y)$. Then $f(x\cdot y) = f(x)\cdot f(y) = 1 = f(y)\cdot f(x) = f(y\cdot x)$, which yields $x\cdot y = y\cdot x = 1$. Hence (8) yields $x = y$. □

Example 4. Let X be a partially ordered set with a greatest element 1. Then

$$x\cdot y := \begin{cases} 1 & \text{for } x \leqslant y \\ y & \text{for } x \nleqslant y \end{cases} \tag{20}$$

makes X into an L-algebra (see [67], Example 1).

Example 5. Epimorphisms of L-algebras need not be surjective. Let $X = \{1, x, y\}$ be the partially ordered set with $y < x < 1$ and the L-algebra structure (20). In the self-similar closure $S(X)$, we have $xy < y$. (Indeed, Eq. (10) gives $xy \cdot y = x \cdot (y \cdot y) = 1$, which yields $xy \leqslant y$. By Eq. (9), $xy = y$ would imply that $x = y \cdot xy = y \cdot y = 1$.) Since $y \leqslant x$ and $x \cdot y = y$, Definition 2.2 gives $y = yx$. By Eq. (9), this yields $x \cdot xy = x \cdot xyx = xy$. Thus $Y := \{1, x, y, xy\}$ is an L-subalgebra of $S(X)$, and Proposition 2.4 implies that $X \hookrightarrow Y$ is an epimorphism in **LAlg**.

The example also shows that L-algebras do not form a variety: The partition $Y = \{1, x\} \sqcup \{y\} \sqcup \{xy\}$ gives a congruence of Y. So there is a surjection $p\colon Y \twoheadrightarrow Z$ onto the cycloid $Z = \{1, t, z\}$ with $1 > t > z$ with $p(x) = 1$, $p(y) = t$, and $p(xy) = z$. Since $t \cdot z = z \cdot t = 1$, the cycloid Z is not an L-algebra.

Recall that coequalizers of parallel pairs of morphisms are also called *regular epimorphisms* [48, 43]. In the category **LAlg**, they coincide with the surjective morphisms:

4.3. Proposition. *For a morphism $f\colon X \to Y$ of L-algebras, the following are equivalent:*

(a) *f is a regular epimorphism.*
(b) *f is a cokernel of a morphism.*
(c) *f is surjective.*

Proof. (a) $\Rightarrow$ (b): By assumption, f is the coequalizer of a morphisms $g, h\colon Z \to X$. As the set of ideals of X is closed with respect to intersection, there is a smallest ideal I of X with $g(z) \cdot h(z) \in I$ and $h(z) \cdot g(z) \in I$ for all $z \in Z$. Hence $X \twoheadrightarrow X/I$ is the coequalizer of g and h, and thus f is the cokernel of $I \hookrightarrow X$. The implications (b) $\Rightarrow$ (c) $\Rightarrow$ (a) follow by Proposition 4.1. $\square$

The Boolean algebra $\mathbb{B} := \{0, 1\}$ with $0 < 1$ is a generator of **LAlg**. Indeed, each element x of an L-algebra X admits a unique morphism $e_x\colon \mathbb{B} \to X$ with $e_x(0) = x$. Note that $\mathbb{B}$ is *projective* with respect to regular epimorphisms (=surjections by Proposition 4.3): For a regular epimorphism $p\colon X \twoheadrightarrow Y$, every morphism $\mathbb{B} \to Y$ factors through p.

4.4. Proposition. *There is a free L-algebra $L\langle S\rangle$ over any set S, and the canonical map $e_S\colon S \to L\langle S\rangle$ is injective. Moreover, $L\langle S\rangle$ is isomorphic to the copower $\mathbb{B}^{(S)} := \coprod_{s \in S} \mathbb{B}$.*

Proof. Since L-algebras form a quasivariety [22], there is a free L-algebras $L\langle S\rangle$ by [22], Proposition 4.5. To show that the map $e_S\colon S \to L\langle S\rangle$ is injective, consider the partially ordered set $\widetilde{S} := S \sqcup \{1\}$ (disjoint union) with an antichain S and $x < 1$ for all $x \in S$. We endow $\widetilde{S}$ with the L-algebra structure (20). So the injection $i\colon S \hookrightarrow \widetilde{S}$ extends to a morphism $f\colon L\langle S\rangle \to \widetilde{S}$ with $fe_S = i$. Thus e_S is injective. Since $L\langle S\rangle \cong \mathbb{B}$ for a singleton $S = \{s\}$, the universal property yields $L\langle S\rangle \cong \mathbb{B}^{(S)}$ for arbitrary S. $\square$

A category is said to be *regular* [3] if it has finite limits and coequalizers of kernel pairs, and regular epimorphisms are *stable under pullback*, that is, in a pullback diagram

$$\begin{array}{ccc} X & \xrightarrow{\ p\ } & Y \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle g} \\ Z & \xrightarrow{\ q\ } & T \end{array}$$

where q is a regular epimorphism, p is a regular epimorphism. Every morphism f of a regular category admits a factorization $f = ip$ into a regular epimorphism p followed by a monomorphism i.

4.5. Proposition. *The category of L-algebras is complete and cocomplete, and regular.*

Proof. This follows since **LAlg** is a quasi-variety. By [1], Theorems 3.22 and 3.24, a quasi-variety is complete and cocomplete. It is regular by [62], Corollary 4.6. □

5. The logic of L-algebras

An L-algebra X is said to be *discrete* [69] if $x < y$ implies that $y = 1$. In other words, the elements of $S^1(X) := X \smallsetminus \{1\}$ are pairwise incomparable. By [69], Proposition 18, $S^1(X)$ consists of the atoms of a geometric lattice: For distinct $x, y \in S^1(X)$, the connecting line is $\{z \in S^1(X) \mid x \cdot y \leqslant x \cdot z\}$. In this section, we show that free L-algebras are discrete, a fact that is closely related to the logic of L-algebras.

Let S be a set of variables. The *logic* $\mathscr{L}(S)$ of L-algebras $(X; \to)$ generated by S consists of a single axiom

$$\vdash \big((x \to y) \to (x \to z)\big) \to \big((y \to x) \to (y \to z)\big) \tag{21}$$

and the following inference rules, reflecting the properties of ideals:

$$x,\ x \to y \vdash y \tag{22}$$
$$x \vdash y \to x \tag{23}$$
$$x \vdash (x \to y) \to y \tag{24}$$
$$x \vdash y \to (x \to y). \tag{25}$$

Thus $\mathscr{L}(S)$ can be regarded as a Hilbert style deductive system, but we use it in a similar fashion like a sequent calculus. For brevity, we use expressions like $A, B \vdash C, D \vdash E$, which means that using the inference rules, C and D can be derived from A and B, and E follows by C and D.

Let $(T(S); \to)$ be the free magma over S, that is, the set of all implicational terms in S. To apply the inference rules, the variables can be substituted with any terms in $T(S)$. The *theory* $\mathscr{T}(S)$ of L-algebras consists of the axiom (21), with arbitrary terms in $T(S)$

inserted for the variables, together with its consequences by the inference rules. Thus $\mathscr{T}(S) \subset T(S)$. The axiom (21) can be interpreted as an inference rule with no terms on the left-hand side. The relation

$$x \equiv y \;:\Longleftrightarrow \vdash x \to y \text{ and } \vdash y \to x \tag{26}$$

is a congruence on $T(S)$:

5.1. Proposition. *The relation* (26) *is an equivalence relation, and* $x \equiv y$ *implies that* $z \to x \equiv z \to y$ *and* $x \to z \equiv y \to z$.

Proof. Note first that *modus ponens* implies the deduction theorem: $\vdash x \to y$ implies $x \vdash y$. Indeed, $\vdash x \to y$ gives $x \vdash x,\, x \to y \vdash y$. Assume that $x \equiv y$. By (23) and (21), we have $x \to y \vdash (x \to z) \to (x \to y) \vdash (z \to x) \to (z \to y)$. Thus $z \to x \equiv z \to y$ follows by symmetry.

Now assume that $x \equiv y$ and $y \equiv z$. By (23) and (21), we have $y \to z \vdash (y \to x) \to (y \to z) \vdash (x \to y) \to (x \to z)$. So (22) yields $x \to y,\, y \to z \vdash x \to z$. By symmetry, it follows that $\equiv$ is transitive. The symmetry of the relation (26) is trivial. Now (24) and (25) give $1 \vdash (1 \to y) \to y$ and $1 \vdash y \to (1 \to y)$ for any term $1 \in \mathscr{T}$. Hence $1 \to y \equiv y$. So (25) yields $1 \vdash y \to (1 \to y) \equiv y \to y$, which proves the reflexivity.

By (24) and (25), we have $x \to y \vdash \big((x \to y) \to (x \to z)\big) \to (x \to z)$ and $x \to y \vdash (x \to z) \to \big((x \to y) \to (x \to z)\big)$. Assume that $x \equiv y$. Then (21) implies that $x \to z \equiv (x \to y) \to (x \to z) \equiv (y \to x) \to (y \to z)$. By (25), we obtain $y \to x \vdash (y \to z) \to \big((y \to x) \to (y \to z)\big) \equiv (y \to z) \to (x \to z)$. Thus $x \to z \equiv y \to z$ follows by symmetry. $\square$

Recall that the *Lindenbaum algebra* [77, 7] of a logical theory is obtained by factoring out the equivalence relation of provable equivalent sentences. For L-algebras, this equivalence relation is the congruence (26). By Definition 2.1, we obtain:

Corollary. *The Lindenbaum algebra* $L(S)$ *of* $\mathscr{L}(S)$ *is an* L*-algebra.*

Proof. We have already shown that $1 \to x \equiv x$ holds for $1 \in \mathscr{T}(S)$. Furthermore, (23) implies that $1 \vdash x \to 1$. Hence $x \to 1 \equiv 1$. Furthermore, $x \equiv x$ gives $x \to x \equiv 1$. Thus 1 represents a logical unit in the Lindenbaum algebra. By (21), $L(S)$ is an L-algebra. $\square$

Let $F(S)$ be the free unital cycloid over S. Then $F(S)/\{1\}$ is isomorphic to the free L-algebra $L\langle S\rangle$ over S. The following result shows that the logic of L-algebras is complete.

5.2. Proposition. *Let* $p\colon T(S) \twoheadrightarrow F(S)$ *be the natural extension of* $S \hookrightarrow F(S)$ *to the free magma* $T(S)$ *over* S*. Then* $p^{-1}(1) = \mathscr{T}(S)$.

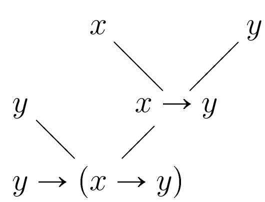
Proof. By (21)-(25), a simple induction shows that $p(\mathscr{T}(S)) = 1$. Conversely, assume that $p(a) = 1$ for some $a \in T(S)$. To show that $a \in \mathscr{T}(S)$, we have to verify that the equations

$$1 \to a = a, \qquad a \to a = a \to 1 = 1 \tag{27}$$
$$(a \to b) \to (a \to c) = (b \to a) \to (b \to c) \tag{28}$$

correspond to equivalences (26) in $T(S)$. By (22), this will imply that any $a \in \mathscr{T}(S)$ remains in $\mathscr{T}(S)$ if a subterm of a is changed by one of the equations (27) and (28). Now (21) shows that $(x \to y) \to (x \to z) \equiv (y \to x) \to (y \to z)$. By Proposition 5.1, $x \equiv x$ holds for all $x \in T(S)$. Hence $x \to x \equiv 1$. Furthermore, (24) and (25) give $1 \to x \equiv x$, and (23) yields $x \to 1 \equiv 1$. □

5.3. Theorem. *The free unital cycloid $F(S)$ over a set S is a discrete L-algebra.*

Proof. Since $S \to L\langle S\rangle$ is a composed map $S \to F(S) \twoheadrightarrow L\langle S\rangle$, Proposition 4.4 implies that the canonical map $S \to F(S)$ is injective. As an intermediate step toward $F(S)$, let $F_0(S)$ be the free magma generated by $S \sqcup \{1\}$ modulo the equations (27), that is, the free system $(X; \to)$ with a logical unit 1. Note first that by successive application of the rules $1 \to x \vdash x$ and $x \to x \vdash 1$, and $x \to 1 \vdash 1$, any term $a \in F_0(S)$ can be transformed into a term $\nu(a)$ of shortest length. To see this, it is convenient to represent the terms of $F_0(S)$ by labelled binary trees, e. g.,



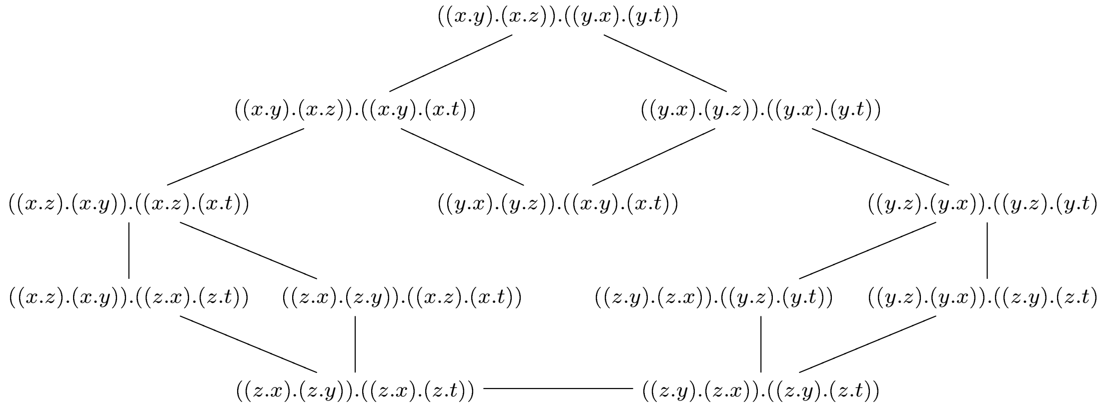
Thus any $a \in F_0(S)$ corresponds to a binary tree where the leaves are labelled with elements of $S \sqcup \{1\}$. Using Eqs. (27), the labels 1 of any $a \in F_0(S) \smallsetminus \{1\}$ can be removed, and it is easily seen that this process leads to a unique normal form $\nu(a)$ of a which does not contain 1 or $b \to b$ as a subterm. We call $a \in F_0(S)$ *reduced* if $\nu(a) = a$.

Next we show that the left-hand side of Eq. (28), with reduced $a, b, c \in F_0(S)$, is reduced if and only if the right-hand side is reduced. Assume that $(a \to b) \to (a \to c)$ is reduced. Then $1 \notin \{a, b, c\}$ and $b \neq a$, and $b \neq c$ since $(a \to b) \neq (a \to c)$. Thus $b \to a$ and $b \to c$ are reduced. In particular, $b \to c \neq 1$. If $b \to a = b \to c$, then $a = c$, which yields $a \to c = 1$, a contradiction. So the right-hand side of Eq. (28) is reduced.

Let $q\colon F_0(S) \twoheadrightarrow F(S)$ be the natural morphism which extends the embedding $S \hookrightarrow F(S)$ to $F_0(S)$. We define a *displacement* of a reduced term $a \in F_0(S)$ to be a modification of a that results from a finite sequence of replacements of subterms $(x \to y) \to (x \to z)$ by $(y \to x) \to (y \to z)$. For example, the term

$$\big((x \to y) \to (x \to z)\big) \to \big((y \to x) \to (y \to t)\big)$$

admits the following displacements (for reasons of space, we represent arrows by dots):



We call a term $a \in F_0(S)$ *fully reduced* if all displacements of a are reduced. So there is a fully reduced term in the inverse image $q^{-1}(a)$ of any $a \in F(S)$.

For $1 \in \{a, b, c\}$, Eq. (28) follows by Eqs. (27). If $a = b$, both sides of Eq. (28) are equal, and for $a = c$, both sides are 1, which again follows by Eqs. (27). If $a \to b$ and $a \to c$ are reduced, $a \to b = a \to c$ implies that $b = c$. Thus if for given $a, b, c \in F_0(S)$, Eq. (28) does not follow by Eqs. (27), both sides of the equation must be reduced.

Now we show that $a \to b = 1$ in $F(S)$ implies that $a = b$ or $b = 1$. Suppose that $a \to b = 1$ with $a \neq b$ and $b \neq 1$. Then $a \neq 1$, and there are fully reduced $a_0, b_0 \in F_0(S)$ with $q(a_0) = a$ and $q(b_0) = b$. Since $a \to b = 1$, there is a sequence of displacements via Eq. (28) transforming $a_0 \to b_0$ into 1. As $a \neq b$, the terms a_0 and b_0 can be chosen as $a_0 = x \to y$ and $b_0 = x \to z$, such that the first step of this transformation changes $a_0 \to b_0$ into $(y \to x) \to (y \to z)$. By induction, $q(y \to x) = q(y \to z)$ or $q(y \to z) = 1$. But $q(y \to z) = 1$ would give $q(y) = q(z)$ or $q(z) = 1$, which is impossible. Now the above diagram shows that $q(y \to x) = q(y \to z)$ is not possible unless $q(x) = q(z)$. Thus $b = q(x \to z) = 1$, a contradiction. So the implication $a \to b = 1 \implies a = b$ or $b = 1$ holds in $F(S)$, which proves that $F(S)$ is a discrete L-algebra. □

Corollary. *The free L-algebra $F\langle S\rangle$ over a set S is discrete.*

6. The exact completion of **LAlg**

Regular categories $\mathscr{C}$ admit a calculus of relations [36, 51, 76, 17]. A *relation* in $\mathscr{C}$ is a monomorphism $R \to A \times B$ in $\mathscr{C}$. It can be viewed as a morphism $R\colon A \to B$ in the category $\mathrm{Rel}(\mathscr{C})$ of relations in $\mathscr{C}$. Its identity morphisms are $\binom{1_A}{1_A}\colon A \to A \times A$ for each object A of $\mathscr{C}$. A morphism in $\mathscr{C}$ is a relation given by its graph. Thus $\mathscr{C}$ is a subcategory of $\mathrm{Rel}(\mathscr{C})$. By definition, a relation $R\colon A \to B$ is given by a pair of jointly monic morphisms $A \overset{p}{\leftarrow} R \overset{q}{\to} B$. If p and q are interchanged, $R\colon A \to B$ turns into its opposite relation R°. In $\mathrm{Rel}(\mathscr{C})$ the relation $A \overset{p}{\leftarrow} R \overset{q}{\to} B$ is equal to qp°. The morphism sets $\mathrm{Hom}(A, B)$ of $\mathrm{Rel}(\mathscr{C})$ are partially ordered such that composition is functorial. In other words, $\mathrm{Rel}(\mathscr{C})$ is a locally posetal bicategory [4]. A relation $R\colon A \to B$ in $\mathrm{Rel}(\mathscr{C})$

belongs to $\mathscr{C}$ if and only if $1_A \leqslant R^\circ R$ and $RR^\circ \leqslant 1_B$. A morphism $f\colon A \to B$ in $\mathscr{C}$ is monic if and only if $f^\circ f = 1_A$, and a regular epimorphism if and only if $ff^\circ = 1_B$. For calculations in $\mathrm{Rel}(\mathscr{C})$ it is useful to note that the equation $ab^\circ = c^\circ d$ holds for any pullback

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle c} \\ C & \xrightarrow{d} & D \end{array}$$

in $\mathscr{C}$. For example, this shows that the difference kernel of a morphism $f\colon A \to B$ in $\mathscr{C}$ is the relation $E := f^\circ f$, which is an *equivalence relation* in $\mathrm{Rel}(\mathscr{C})$, a self-adjoint idempotent $E \geqslant 1_A$. Let $\mathscr{E}(\mathscr{C})$ be the set of these idempotents. If $E \in \mathscr{E}(\mathscr{C})$ *splits*, that is, $E = QP$ with $PQ = 1$ for some $P, Q \in \mathrm{Rel}(\mathscr{C})$, then $Q = P^\circ$ by [33], 2.162. Hence $P \in \mathscr{C}$, and E is its difference kernel. Conversely, the difference kernel $f^\circ f$ of a morphism $f \in \mathscr{C}$ is a splitting idempotent. Indeed, $f = ip$ with a monomorphism i and a regular epimorphism p. Hence $pp^\circ = 1$ and $i^\circ i = 1$, which yields $f^\circ f = p^\circ i^\circ ip = p^\circ p$. Thus $E \in \mathscr{E}(\mathscr{C})$ splits if and only if E is a kernel pair. For $\mathscr{C} = \mathbf{LAlg}$ it is easily checked that an equivalence relation $R \hookrightarrow X \times X$ is the same as a congruence relation of X, that is, a set-theoretic equivalence relation $\sim$ such that $x \sim x'$ and $y \sim y'$ implies that $x \cdot y \sim x' \cdot y'$.

EXAMPLE 6. A category $\mathscr{C}$ is said to be *Barr-exact* [3] if it is regular and every equivalence relation is a kernel pair, that is, every idempotent $E \in \mathscr{E}(\mathscr{C})$ splits. Since unital cycloids form a variety, $\mathbf{Cyc}^*$ is Barr-exact by Lawvere's theorem [50, 62]. Moreover, $\mathbf{Cyc}^*$ is monadic over the category $\mathbf{Set}$ of sets by [80], Proposition 3.2. The L-algebra $Y = \{1, x, y, xy\}$ in Example 5 has a congruence relation $\sim$ given by the partition $Y = \{1, x\} \sqcup \{y\} \sqcup \{xy\}$, but the unital cycloid $Y/\sim$ is not an L-algebra. So the idempotent in $\mathscr{E}(\mathbf{LAlg})$ associated with $\sim$ does not split. Hence $\mathbf{LAlg}$ is not Barr-exact.

Any regular category $\mathscr{C}$ embeds into a Barr-exact category $\mathscr{C}_{\mathrm{ex}}$, the *exact completion* [51, 33, 19, 18] of $\mathscr{C}$, which can be obtained as follows. By splitting the idempotents in $\mathscr{E}(\mathscr{C})$, we get a full subcategory $\mathscr{K}$ of of $\mathrm{Rel}(\mathscr{C})$ with object class $\mathscr{E}(\mathscr{C})$ and morphisms $R\colon E \to F$ given by $R \in \mathrm{Rel}(\mathscr{C})$ with $R = RE = FR$. Then $\mathscr{C}_{\mathrm{ex}}$ is the subcategory of maps in $\mathscr{K}$, that is, morphisms $R\colon E \to F$ with $E \leqslant R^\circ R$ and $RR^\circ \leqslant F$.

6.1. THEOREM. *The exact completion of* $\mathbf{LAlg}$ *is* $\mathbf{Cyc}^*$.

Proof. Let $E \subset X \times X$ be an equivalence relation on an L-algebra X. Thus E determines a congruence relation on X, and the congruence classes form a unital cycloid $\overline{X}$. So we have an exact diagram

$$E \rightrightarrows X \overset{p}{\twoheadrightarrow} \overline{X}.$$

Now let F be an equivalence relation on an L-algebra Y with exact sequence

$$F \rightrightarrows Y \overset{q}{\twoheadrightarrow} \overline{Y}.$$

Then a morphism $R\colon E \to F$ in $\mathbf{LAlg}_{\mathrm{ex}}$ gives rise to a relation $f\colon \overline{X} \to \overline{Y}$ with $f := qRp^\circ$. Hence $1 = pp^\circ = pEp^\circ \leqslant pR^\circ Rp^\circ \leqslant pR^\circ q^\circ qRp^\circ = f^\circ f$ and $ff^\circ = qRp^\circ pR^\circ q^\circ =$

$qRER^\circ q^\circ = qRR^\circ q^\circ \leqslant qFq^\circ = qq^\circ \leqslant 1$. Thus f is a morphism of unital cycloids. Furthermore, R is determined by f since $R = FRE = q^\circ qRp^\circ p = q^\circ fp$.

Conversely, let $f\colon \overline{X} \to \overline{Y}$ be a morphism in $\mathbf{Cyc}^*$. Define $R := q^\circ fp$. Then $RE = R$ and $FR = F^\circ q^\circ fp = (qF)^\circ fp = q^\circ fp = R$. Furthermore, $E = p^\circ p \leqslant p^\circ f^\circ fp = p^\circ f^\circ qq^\circ fp = R^\circ R$, and $RR^\circ = q^\circ fpp^\circ f^\circ q = q^\circ ff^\circ q \leqslant q^\circ q = F$. Thus R is a morphism $R\colon E \to F$ in $\mathbf{LAlg}_{\mathrm{ex}}$. Furthermore, $f = qq^\circ fpp^\circ = qRp^\circ$. So we have a bijection $R \mapsto f$ between morphisms $R\colon E \to F$ in $\mathbf{LAlg}_{\mathrm{ex}}$ and morphisms $f\colon \overline{X} \to \overline{Y}$ in $\mathbf{Cyc}^*$. Hence $\mathbf{LAlg}_{\mathrm{ex}}$ is a full subcategory of $\mathbf{Cyc}^*$. Each unital cycloid Y admits a regular epimorphism $p\colon X \twoheadrightarrow Y$ from a free unital cycloid X onto Y. By Theorem 5.3, X is an L-algebra. The difference kernel $E \rightrightarrows X$ of p is an equivalence relation E on X, and p is its coequalizer. Whence $\mathbf{LAlg}_{\mathrm{ex}}$ is equivalent to $\mathbf{Cyc}^*$.

Remark. Note that the regular category $\mathbf{LAlg}$ can be retrieved from its exact completion $\mathbf{Cyc}^*$: For a unital cycloid X, the ideal $\{1\}$ determines a congruence (7) on X, which leads to an L-algebra $\overline{X} = X/\equiv$. If the operation of X is interpreted as logical implication, $\overline{X}$ is the Lindenbaum algebra [7] of X.

7. Semidirect products beyond protomodularity

Based on the notion of protomodularity [8], a strengthening of the concept of Barr-exact category was introduced by Janelidze, Márki, and Tholen [43]. They call a category *semi-abelian* if it is Barr-exact and protomodular, with finite coproducts and a zero object. For an object B of a category $\mathscr{C}$ with pullbacks, the objects of the category $\mathrm{Pt}_{\mathscr{C}}(B)$ of *points* are triples (E, p, s) with $p\colon E \to B$ and $s\colon B \to E$ satisfying $ps = 1_B$. A morphism $(E, p, s) \to (F, q, t)$ in $\mathrm{Pt}_{\mathscr{C}}(B)$ is given by a morphisms $f\colon E \to F$ in $\mathscr{C}$ with $fs = t$ and $qf = p$. Then $\mathscr{C}$ is said to be *protomodular* if for each morphism $v\colon C \to B$ in $\mathscr{C}$, the pullback functor $v^*\colon \mathrm{Pt}_{\mathscr{C}}(B) \to \mathrm{Pt}_{\mathscr{C}}(C)$ reflects isomorphisms. If $\mathscr{C}$ has pullbacks and a zero object, protomodularity is equivalent to the Split Short Five Lemma [53, 43], which states that in a commutative diagram

$$\begin{array}{ccccc} X & \rightarrowtail & Y & \twoheadrightarrow & Z \\ \| & & \downarrow f & & \| \\ X & \rightarrowtail & T & \twoheadrightarrow & Z \end{array}$$

with split short exact rows the morphism f is invertible. For varieties, a slightly simpler criterion is available [12]. The following example shows that neither $\mathbf{LAlg}$ nor $\mathbf{Cyc}^*$ is protomodular.

Example 7. Let $Y = \{1, x, y, xy\}$ be the L-algebra of Example 5. Thus $1 > x > y > xy$, and $X := \{1, x, y\}$ is an L-subalgebra. Furthermore, $I := \{1, x\}$ is an ideal of Y. So we have a commutative diagram

$$\begin{array}{ccccc} I & \hookrightarrow & X & \overset{p}{\twoheadrightarrow} & \mathbb{B} \\ \| & & \downarrow & & \| \\ I & \hookrightarrow & Y & \overset{q}{\twoheadrightarrow} & \mathbb{B} \end{array}$$

with split short exact rows, which shows that **LAlg** and **Cyc*** are not protomodular.

Since $S(X) = S(Y)$, Example 7 does not provide a counterexample to protomodularity of **ssL** just by applying the functor $S\colon \mathbf{LAlg} \to \mathbf{ssL}$ and Theorem 3.5 (Corollary).

Example 8. Let $X = \{1, x, y\}$ be the L-algebra of Example 7. In Example 5, we have shown that $yx = y$ holds in $S(X)$. Thus, each element of $S(X)$ is of the form $x^i y^j$ with $i, j \in \mathbb{N}$. To show that these elements are all distinct, assume that $x^i y^j = x^k y^\ell$. If $j = \ell$, then $x^i = x^k$, since $S(X)$ is right cancellative. Hence $i = k$. Otherwise, assume that $j < \ell$. Then Eq. (11) gives $x^i \leqslant y^j \cdot x^k y^\ell = \big((y^\ell \cdot y^j) \cdot x^k\big)(y^j \cdot y^\ell) = x^k y^{\ell - j}$. Hence $1 = x^i \cdot x^k y^{\ell-j} = \big((y^{\ell-j} \cdot x^i) \cdot x^k\big)(x^i \cdot y^{\ell-j}) \leqslant x^i \cdot y^{\ell-j}$, and thus $x^i \leqslant y^{\ell-j} \leqslant y$. On the other hand, using Eq. (11), $y \leqslant x^i$ follows by induction. So we obtain $x^i = y$ for some $i > 0$. Hence $y^2 = yx^i = y$, and thus $y = 1$, a contradiction. So the $x^i y^j$ are all distinct. Using Eqs. (10)-(11), we obtain

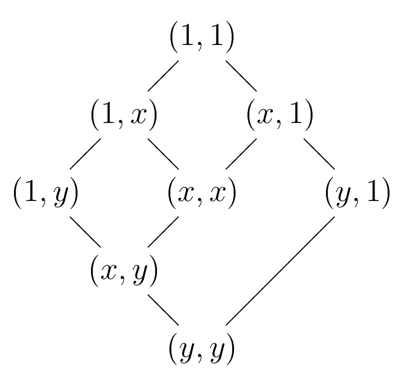
$$x^i y^j \cdot x^k y^\ell := \begin{cases} 1 & \text{for } j > \ell \text{ or } (j = \ell \text{ and } i \geqslant k) \\ x^{k-i} & \text{for } j = \ell \text{ and } i < k \\ x^k y^{\ell - j} & \text{for } j < \ell. \end{cases}$$

Thus (2) gives a linear (lexicographic) order of $S(X)$.

Now it is easily checked that $x \mapsto x$ and $y \mapsto xy$ defines an L-algebra isomorphism of $S(X)$ onto the L-subalgebra $A := S(X) \smallsetminus \{y^n \mid n > 0\}$. By Proposition 3.6, $\langle x \rangle = \{x^n \mid n \in \mathbb{N}\}$ is an ideal of A and of $S(X)$, with $A/\langle x \rangle \cong \langle xy \rangle = \{xy^n \mid n > 0\} \cup \{1\} \cong \langle y \rangle \cong S(X)/\langle x \rangle$. So the variety **ssL** of self-similar L-algebras is not protomodular.

A slight weakening of protomodularity is the *Mal'cev property* (see [9], Proposition 17). A regular category $\mathscr{C}$ is said to be a *Mal'cev category* [17] if every reflexive relation $R \to X \times X$ in $\mathscr{C}$ is effective. The following example shows that **LAlg** is not a Mal'cev category.

Example 9. Let $X = \{1, x, y\}$ be the L-algebra of Example 7. Consider the L-subalgebra $R := (X \times X) \smallsetminus \{(y, x)\}$ of $X \times X$. The partial order of R is a lattice:



Thus R is a reflexive relation in **LAlg** which is not symmetric. Hence **LAlg** is not a Mal'cev category.

Another concept related to protomodularity is the existence of *semidirect products* [11], which means that for morphisms $v\colon C \to B$, the pullback functor $v^*\colon \mathrm{Pt}_{\mathscr{C}}(B) \to \mathrm{Pt}_{\mathscr{C}}(C)$ is monadic. A Barr-exact category $\mathscr{C}$ has semidirect products if and only if $\mathscr{C}$ has pushouts of split monomorphisms and is protomodular. Thus **LAlg** has no semidirect products in the sense of [11]. Nevertheless, semidirect products of L-algebras have been constructed in a very natural way [68], while the categorical concept of semidirect product [11] does not apply here. The reason is that the notion of split short exact sequence is too weak in the category of L-algebras.

7.1. Definition. We say that a short exact sequence $X \overset{u}{\rightarrowtail} Y \overset{v}{\twoheadrightarrow} Z$ *strongly splits* if it admits a *strong section*, that is, a morphism $s\colon Z \to Y$ with $vs = 1_Z$ such that $y \cdot s(z) = s(v(y) \cdot z)$ holds for $y \in Y$ and $z \in Z$.

The condition $y \cdot s(z) = s(v(y) \cdot z)$ says that $s(Z)$ is an invariant L-subalgebra of Y. Indeed, $y \cdot s(z) = s(z_0)$ implies that $z_0 = vs(z_0) = v(y \cdot s(z)) = v(y) \cdot z$. In particular, it yields $y \cdot sv(y) = 1$, that is,

$$y \leqslant sv(y) \tag{29}$$

for all $y \in Y$. So the short exact sequence (19) strongly splits in the sense of Definition 4 of [68]. The converse holds for *KL-algebras* [67], that is, L-algebras satisfying the inequality

$$y \leqslant x \cdot y, \tag{30}$$

which have been studied in [68]. Indeed, (29) and (30) give $s(v(y) \cdot z) = sv(y) \cdot s(z) \leqslant \big(sv(y) \cdot y\big) \cdot \big(sv(y) \cdot s(z)\big) = (y \cdot sv(y)) \cdot (y \cdot s(z)) = y \cdot s(z) \leqslant sv(y \cdot s(z)) = s(v(y) \cdot z)$.

Using the partial multiplication of L-algebras, we define semidirect products as follows:

7.2. Definition. Let X be an L-algebra with an ideal I and an invariant L-subalgebra U. We say that X is a *semidirect product* of I and U if $I \cap U = \{1\}$ and $X = \{xu | x \in I, u \in U\}$.

Every semidirect product X of I and U gives rise to a split short exact sequence

$$I \hookrightarrow X \overset{p}{\twoheadrightarrow} U \tag{31}$$

with $p(xu) = u$. Indeed, we have

7.3. Proposition. *Let X be an L-algebra with an ideal I and an L-subalgebra U. If X is a semidirect product of I and U, each element of X has a unique expression xu with $x \in I$ and $u \in U$. The short exact sequence* (31) *strongly splits.*

Proof. Assume that $xu = yv$ with $x, y \in I$ and $u, v \in U$. By Eq. (10), this implies that $xu \leqslant v$ and $x \leqslant u \cdot v$. Hence $u \cdot v \in I \cap U = \{1\}$, and thus $u \leqslant v$. By symmetry, we obtain $u = v$. By Definition 2.2, this implies that $x = u \cdot xu = v \cdot yv = y$. In particular, the map p in (31) is well defined.

For $x \in I$ and $u \in U$, we have $x \cdot u = v$ for some $v \in U$. As above, this yields $u \leqslant v$. On the other hand, (6) implies that $v \cdot u = (x \cdot u) \cdot u \in I \cap U = \{1\}$. Thus $v = u$, that is,

$$x \cdot u = u. \tag{32}$$

Now Eq. (11) yields $xu\cdot yv = \big((v\cdot xu)\cdot y\big)(xu\cdot v)$. Hence $p(xu\cdot yv) = xu\cdot v = x\cdot(u\cdot v) = u\cdot v$, which shows that p is an L-algebra morphism with kernel I.

To show that the short exact sequence (31) strongly splits, we have to verify that $xu \cdot v = u \cdot v$ for $x \in I$ and $u, v \in U$. This follows by Eqs. (10) and (32). □

By Proposition 7.3, the natural map $I \times U \to X$ into a semidirect product X is bijective. Therefore, we write $I \rtimes U$ for a semidirect product X of I and U. The following result exhibits a connection between strongly split short exact sequences, semidirect products, and the Short Five lemma:

7.4. Theorem. *Let $I \hookrightarrow X \overset{p}{\twoheadrightarrow} U$ be a short exact sequence in* **LAlg** *with a strong section $s\colon U \to X$. There exists a semidirect product $\widetilde{X} = I \rtimes s(U)$ with an L-subalgebra X such that the diagram*

$$\begin{array}{ccccc} I & \hookrightarrow & \widetilde{X} & \overset{q}{\twoheadrightarrow} & U \\ \| & & \uparrow & & \| \\ I & \hookrightarrow & X & \overset{p}{\twoheadrightarrow} & U \end{array} \tag{33}$$

commutes, and sq is the projection of $\widetilde{X}$ onto $s(U)$. The L-algebra $\widetilde{X}$ is uniquely determined, up to isomorphism.

Proof. Assume that there exists a commutative diagram (33) with a semidirect product $\widetilde{X} = I \rtimes s(U)$. By Proposition 7.3, the map $(x,u) \mapsto xs(u)$ gives a bijection $I \times U \xrightarrow{\sim} \widetilde{X}$. For $x, y \in I$ and $u, v \in U$, Eqs. (10) and (11) give $xs(u) \cdot ys(v) = x \cdot \big(s(u) \cdot ys(v)\big) = x \cdot (s(v \cdot u) \cdot y)s(u \cdot v) = \big((s(u \cdot v) \cdot x) \cdot (s(v \cdot u) \cdot y)\big)\big(x \cdot s(u \cdot v)\big)$. Since $x \cdot s(u \cdot v) = s(u \cdot v)$ by Eq. (32), we obtain

$$xs(u) \cdot ys(v) = \big((s(u \cdot v) \cdot x) \cdot (s(v \cdot u) \cdot y)\big)s(u \cdot v).$$

So the L-algebra $\widetilde{X}$ is unique, up to isomorphism. Therefore, we define $\widetilde{X} := I \times U$ with

$$(x,u) \cdot (y,v) := \big((s(u \cdot v) \cdot x) \cdot (s(v \cdot u) \cdot y), u \cdot v\big). \tag{34}$$

Since $x \cdot s(u) = s(u)$ holds for $x \in I$ and $u \in U$, we have $s(u) \cdot (x \cdot y) = (x \cdot s(u)) \cdot (x \cdot y) = (s(u) \cdot x) \cdot (s(u) \cdot y)$ for $x, y \in I$. Thus

$$s(u) \cdot (x \cdot y) = \big(s(u) \cdot x\big) \cdot \big(s(u) \cdot y\big)$$

holds for $x, y \in I$ and $u \in U$. So we obtain

$$\big((x,u) \cdot (y,v)\big) \cdot \big((x,u) \cdot (z,w)\big) = \big(A \cdot B, (u \cdot v) \cdot (u \cdot w)\big)$$

with

$$\begin{aligned} A &= s((u\cdot v)\cdot(u\cdot w))\cdot\big((s(u\cdot v)\cdot x)\cdot(s(v\cdot u)\cdot y)\big) \\ &= \big(s((u\cdot v)\cdot(u\cdot w))\cdot(s(u\cdot v)\cdot x)\big)\cdot\big(s((u\cdot v)\cdot(u\cdot w))\cdot(s(v\cdot u)\cdot y)\big) \\ &= \big(s((u\cdot v)\cdot(u\cdot w))\cdot(s(u\cdot v)\cdot x)\big)\cdot\big(s((v\cdot u)\cdot(v\cdot w))\cdot(s(v\cdot u)\cdot y)\big) \\ B &= s((u\cdot w)\cdot(u\cdot v))\cdot\big((s(u\cdot w)\cdot x)\cdot(s(w\cdot u)\cdot z)\big) \\ &= \big(s((u\cdot w)\cdot(u\cdot v))\cdot(s(u\cdot w)\cdot x)\big)\cdot\big(s((u\cdot w)\cdot(u\cdot v))\cdot(s(w\cdot u)\cdot z)\big) \\ &= \big(s((u\cdot v)\cdot(u\cdot w))\cdot(s(u\cdot v)\cdot x)\big)\cdot\big(s((w\cdot u)\cdot(w\cdot v))\cdot(s(w\cdot u)\cdot z)\big). \end{aligned}$$

By Eq. (3), $A\cdot B$ is symmetric in (x,u) and (y,v), which shows that $\widetilde{X}$ satisfies Eq. (3). By Eq. (34), $(1,1)$ is a logical unit. Furthermore, $(x,u)\cdot(y,v)=(1,1)$ is equivalent to $u\leqslant v$ and $x\leqslant s(v\cdot u)\cdot y$. So the implication (8) holds in $\widetilde{X}$, which proves that $\widetilde{X}$ is an L-algebra.

Now we define a map $f\colon X\to\widetilde{X}$ with $f(x):=\big(sp(x)\cdot x, p(x)\big)$. Assume that $x,y\in X$. Then $p(sp(x)\cdot x)=1$ implies that $sp(x)\cdot x\in I$. Since s is a strong section, we have

$$x\cdot s(u)=s(u)$$

for $x\in I$ and $u\in U$. With (29) and Eq. (3), this yields

$$\begin{aligned} f(x)\cdot f(y) &= \big((sp(x\cdot y)\cdot(sp(x)\cdot x))\cdot(sp(y\cdot x)\cdot(sp(y)\cdot y)), p(x)\cdot p(y)\big) \\ &= \big((sp(x\cdot y)\cdot(sp(x)\cdot x))\cdot(sp(x\cdot y)\cdot(sp(x)\cdot y)), p(x\cdot y)\big) \\ &= \big(((sp(x)\cdot x)\cdot sp(x\cdot y))\cdot((sp(x)\cdot x)\cdot(sp(x)\cdot y)), p(x\cdot y)\big) \\ &= \big(sp(x\cdot y)\cdot((x\cdot sp(x))\cdot(x\cdot y)), p(x\cdot y)\big) = \big(sp(x\cdot y)\cdot(1\cdot(x\cdot y)), p(x\cdot y)\big) \\ &= \big(sp(x\cdot y)\cdot(x\cdot y), p(x\cdot y)\big) = f(x\cdot y). \end{aligned}$$

Thus, f is an L-algebra morphism. If $f(x)\leqslant f(y)$, then $sp(x\cdot y)\cdot(x\cdot y)=1$ and $p(x\cdot y)=1$. Together with (29), this yields $x\cdot y=sp(x\cdot y)=1$. Thus f is injective, and $f|_I\colon I\hookrightarrow\widetilde{X}$ gives the embedding $x\mapsto(x,1)$. By Eq. (34), we have $(1,u)\cdot(x,u)=(x,1)$ and $(x,u)\cdot(1,u)=(1,1)$. So Definition 2.2 gives $(x,u)=(x,1)(1,u)$ for all $(x,u)\in\widetilde{X}$. Furthermore, $u\mapsto(1,u)$ makes U into an L-subalgebra of $\widetilde{X}$, with $(x,u)\cdot(1,v)=(1,u\cdot v)$, which shows that U is invariant. Thus $\widetilde{X}$ is a semidirect product of I and $s(U)$, which fits into a commutative diagram (33). □

Remarks. 1. Besides the quasi-variety of L-algebras, there are important varieties with semidirect products which are not covered by the categorical approach of [11]. For example, the category of monoids or monoids with operations [57] is not protomodular, and thus has no semidirect products in the sense of [11]. To remedy, the categorical concept of semidirect product was generalized [58] by considering *regular points* (also called *strong points* [13])

$$K \overset{i}{\rightarrowtail} E \underset{s}{\overset{p}{\rightleftarrows}} B$$

for which $\binom{i}{s}\colon K\amalg B\to E$ is a regular epimorphism. This led to the concept of S-*protomodular* category [13], where S is a pullback-stable class of regular points. Note that semidirect products $I \rtimes U$ of L-algebras do not fit into this pattern since $I \cup U$ is an L-subalgebra of $I \rtimes U$. So the corresponding point is not regular, unless I or U is trivial.

2. By [68], Definition 5, a semidirect product $I \rtimes U$ in **LAlg** is given by an *action* of U on I, that is, a map $\varrho\colon U \to \mathrm{End}(I)$ which satisfies $\varrho(1) = 1$ and $\varrho(u\cdot v)\varrho(u) = \varrho(v\cdot u)\varrho(v)$. For a semidirect product $I \rtimes U$, the corresponding action is given by $\varrho(u)(x) := u \cdot x$. Eq. (34) shows that the action determines the structure of $I \rtimes U$. By [74], Corollary 1 of Proposition 3, the products $a = ux$ with $u \in U$ and $x \in I$ exist in $I \rtimes U$ and are meets $a = u \wedge x$; and by [74], Corollary 2 of Proposition 3, each $a \in I \rtimes U$ has a unique representation $a = u \wedge x$ with $u \in U$ and $x \in I$ if and only if $\varrho(u) \in \mathrm{Aut}(I)$ for all $u \in U$.

3. An anonymous referee pointed out that semidirect products of L-algebras satisfy the Schreier condition for monoids, introduced by Rédei [64], with respect to the partial multiplication in an L-algebra. Indeed, Proposition 7.3 shows that in a semidirect product of L-algebras related to a short exact sequence (31), every fibre $p^{-1}(u)$ contains a greatest element u (the "generator" with respect to the Schreier condition) such that each element of $p^{-1}(u)$ is of the form xu with a unique $x \in I$.

8. Ideals of L-algebras

Let $\mathscr{C}$ be a *pointed* category, i. e. with a zero object 0. A monomorphism $f \in \mathscr{C}$ is said to be *normal* [60] if f is a *kernel* of some $g \in \mathscr{C}$, that is, the equalizer of g and a zero morphism. Similarly, an epimorphism is said to be *normal* if it is a normal monomorphism in $\mathscr{C}^{\mathrm{op}}$. The concept of *normal* subobject or quotient object is defined analogously. For $\mathscr{C} = \mathbf{LAlg}$, the normal subobjects of an L-algebra X coincide with the ideals of X, and the normal quotient objects of X are of the form X/I for some ideal I of X.

A category $\mathscr{C}$ is said to be *normal* [41] if $\mathscr{C}$ is pointed and regular such that every regular epimorphism is normal. By Proposition 4.3, **LAlg** is a normal category. Note that the dual statement is false: An equalizer in **LAlg** need not be a kernel. For example, the equalizer of the two projections $\mathbb{B}^2 \twoheadrightarrow \mathbb{B}$ is $\{0, 1\} \subset \mathbb{B}^2$, which is not an ideal.

In the context of universal algebra, Ursini [79] introduced the concept of subtractive variety. More generally, a pointed category $\mathscr{C}$ with finite limits is said to be *subtractive* [40] if every reflexive relation $r\colon R \to X \times X$ in $\mathscr{C}$ for which $\binom{1}{0}\colon X \to X \times X$ factors through r, the morphism $\binom{0}{1}\colon X \to X \times X$ also factors through r. Janelidze [41] characterized subtractive resp. protomodular categories by three versions of the 3×3 lemma. Let

$$
\begin{array}{ccccc}
A_1 & \rightarrowtail & B_1 & \twoheadrightarrow & C_1 \\
\downarrow & & \downarrow & & \downarrow \\
A_2 & \rightarrowtail & B_2 & \twoheadrightarrow & C_2 \\
\downarrow & & \downarrow & & \downarrow \\
A_3 & \rightarrowtail & B_3 & \twoheadrightarrow & C_3
\end{array}
$$

be a commutative diagram in a pointed regular category $\mathscr{C}$ with short exact columns and two short exact rows. For the remaining row (let us call it the *target row*) it is only assumed that the composed morphism is zero. The 3×3 lemma then states that the target row $A \overset{a}{\rightarrowtail} B \overset{b}{\twoheadrightarrow} C$ is *short exact*, that is, a is a kernel of b, and b is cokernel of a. If the target row is the first (second, third) one, we speak of the *lower (middle, upper)* 3×3 lemma. By [41], Theorem 5.3, $\mathscr{C}$ is protomodular if and only if it satisfies the middle 3×3 lemma. By [41], Theorem 5.4, a normal category $\mathscr{C}$ is subtractive if and only if it satisfies the upper, or equivalently, the lower 3×3 lemma.

8.1. Proposition. *The category* **LAlg** *is subtractive and normal.*

Proof. The normality of **LAlg** follows by Proposition 4.3. Let $r\colon R \hookrightarrow X \times X$ be a reflexive relation of L-algebras such that $\binom{1}{0}\colon X \to X \times X$ factors through r. Then $(x,x) \in R$ and $(x,1) \in R$ for all $x \in X$. Hence $(1,x) = (x,1) \cdot (x,x) \in R$, and thus $\binom{0}{1}\colon X \to X \times X$ factors through r. $\square$

Corollary. *The category* **ssL** *of self-similar L-algebras is a subtractive normal variety.*

Proof. Let $p\colon X \twoheadrightarrow Y$ be a regular epimorphism in **LAlg** with X self-similar. By the corollary of Theorem 3.5, Y is self-similar. Hence p is a cokernel in **ssL**, and thus **ssL** is a normal variety. Since **LAlg** is subtractive, the full subcategory **ssL** is subtractive. $\square$

Remark. The variety **Cyc*** is subtractive, but not normal. Let $Y = \{1, x, y, xy\}$ be the L-algebra of Example 5. The ideal $I = \{1, x\}$ gives rise to a congruence relation with $1 \equiv x$ and $y \equiv xy$. Its coequalizer is not a cokernel.

Gumm and Ursini ([38], Corollary 1.9) characterized subtractive normal varieties as pointed varieties with "una buona teoria degli ideali" [78]. In [38], these varieties have been called *ideal determined*. More generally, a normal category $\mathscr{C}$ is said to be *ideal determined* [45] if each regular epimorphism maps normal subobjects to normal subobjects. Note that a monomorphism $j\colon J \to Y$ in a pointed regular category $\mathscr{C}$ is said to be an *ideal* [44, 37] if there is a commutative diagram

$$\begin{array}{ccc} I & \overset{q}{\twoheadrightarrow} & J \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \underset{p}{\twoheadrightarrow} & Y \end{array}$$

with regular epimorphisms p, q and a normal monomorphism i. Thus, a normal category is ideal determined if and only if its ideals are normal monomorphisms. The following result shows that **LAlg** is ideal determined.

8.2. Proposition. *Let $f\colon X \to Y$ be a morphism of L-algebras. The inverse image $f^{-1}(J)$ of an ideal J of Y is an ideal of X. If f is surjective, ideals of X are mapped to ideals of Y, and $f(I \cap J) = f(I) \cap f(J)$ for ideals I, J of X.*

Proof. The proof of the first statement is straightforward. Thus, let f be surjective. We have to verify (5)-(6) for $f(I)$. The implication (6) being obvious, assume that $x, y \in X$ with $f(x) \in f(I)$ and $f(x) \cdot f(y) \in f(I)$. So we can assume that $x \in I$, and there is an element $z \in I$ with $f(x \cdot y) = f(z)$. Since $t := z \cdot (x \cdot y) \equiv z \cdot y$ modulo I, we have $t \cdot y \in I$. Hence $f(t) = f(z) \cdot f(x \cdot y) = 1$ and $f(y) = f(t) \cdot f(y) = f(t \cdot y) \in f(I)$. Thus $f(I)$ is an ideal of Y.

Now let I and J be ideals of X. Then $f(I \cap J) \subset f(I) \cap f(J)$. Conversely, every element of $f(I) \cap f(J)$ is of the form $f(x) = f(y)$ with $x \in I$ and $y \in J$. Hence $z := (x \cdot y) \cdot y \in I \cap J$ and $f(x \cdot y) = f(x) \cdot f(y) = 1$. Thus $f(z) = f(x \cdot y) \cdot f(y) = 1 \cdot f(y) = f(y)$, which shows that $f(I) \cap f(J) \subset f(I \cap J)$. □

Corollary 1. *The normal category* **LAlg** *is ideal determined. Up to isomorphism, the categorical ideals $I \hookrightarrow X$ in* **LAlg** *[44] coincide with the normal monomorphisms $I \to X$.*

Corollary 1 provides a negative answer to Question 4.1 of [45] which askes whether ideal determined categories are Barr-exact.

Corollary 2. *The normal variety* **ssL** *is ideal determined.*

Proof. This follows by the corollary of Theorem 3.5. □

Corollary 3. *The lattice of ideals of an L-algebra X is distributive.*

Proof. Let I, J, K be ideals of X. Consider the canonical morphism $f\colon X \twoheadrightarrow X/K$. Then $f^{-1}f(I)$ is an ideal of X with $I \cup K \subset f^{-1}f(I)$. So the ideal $I \vee K$ generated by $I \cup K$ is contained in $f^{-1}f(I)$. The canonical morphism $p\colon X \twoheadrightarrow X/(I \vee K)$ factors through f. Hence $p = gf$ for some morphism $g\colon X/K \to X/(I \vee K)$. Thus $p(f^{-1}f(I)) \subset gf(I) = p(I) = \{1\}$, which proves that $f^{-1}f(I) = I \vee K$. Similarly, $f^{-1}f(J) = J \vee K$. Hence $(I \vee K) \cap (J \vee K) = f^{-1}f(I) \cap f^{-1}f(J) = f^{-1}\big(f(I) \cap f(J)\big) = f^{-1}f(I \cap J) = (I \cap J) \vee K$, and thus $(I \cap J) \vee K = (I \vee K) \cap (J \vee K)$. □

Recall that a morphism $p\colon E \to B$ in a regular category $\mathscr{C}$ is said to be an *effective descent morphism* [5, 42, 31] if the pullback functor $p^*\colon \mathscr{C}/B \to \mathscr{C}/E$ is monadic. Let $\mathrm{Reg}(\mathscr{C})$ be the category of regular epimorphisms, with commutative squares as morphisms. If $\mathrm{Reg}(\mathscr{C})$ is regular, every regular epimorphism in $\mathscr{C}$ is an effective descent morphism ([31], Theorem 2.3). If, in addition, $\mathscr{C}$ is finitely cocomplete, every regular epimorphism in $\mathrm{Reg}(\mathscr{C})$ is an effective descent morphism ([31], Corollary 2.4).

8.3. Proposition. *In the category* **LAlg** *of L-algebras, every regular epimorphism is an effective descent morphism.*

Proof. Since $\mathscr{C} :=$ **LAlg** is pointed and ideal-determined, $\mathrm{Reg}(\mathscr{C})$ is a regular category (see [31], Section 3.2). Hence every regular epimorphism in $\mathscr{C}$ or $\mathrm{Reg}(\mathscr{C})$ is an effective descent morphism. □

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