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FIRM HOMOMORPHISMS OF RINGS AND SEMIGROUPS

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ABSTRACT. In this paper we define firm homomorphisms between rings without identity in such a way that the category of rings with identity will become a full subcategory of the category of firm rings with firm homomorphisms as morphisms. We prove that firm homomorphisms are in one-to-one correspondence with pairs of compatible concrete functors between certain module categories. This correspondence is given by the restriction of scalars. We also prove the semigroup theoretic analogues of these results and give a list of examples of firm homomorphisms.

1. Introduction

If R and S are rings with identity, the definition of a ring homomorphism $f : R \longrightarrow S$ includes the condition $f(1_R) = 1_S$. Let Ring be the category of rings with identity with their homomorphisms.

If R and S are nonunital rings, then we cannot include the condition $f(1_R) = 1_S$ in the definition of a homomorphism, so we consider ring homomorphisms without that extra condition. This makes Ring a non-full subcategory of the category Rng of all rings. One could ask: is it possible to consider some notion of ring homomorphisms in such a way that the category Ring will become a full subcategory of the category of rings with those homomorphisms as morphisms? In this paper we will define firm homomorphisms of rings and show that Ring is a full subcategory of the category of firm rings with firm homomorphisms as morphisms. A similar problem can be considered for the category of monoids and the category of semigroups. We will also show that the category of monoids with monoid homomorphisms is a full subcategory of the category of firm semigroups with firm homomorphisms as morphisms.

We will study firm homomomorphisms of rings and semigroups in detail. In our main theorem (Theorem 8.1) we will prove that firm homomorphisms between firm rings are in one-to-one correspondence with the pairs of compatible concrete functors between certain module categories. Moreover, those concrete functors will be restriction of scalars functors induced by firm homomorphisms. We also formulate and prove a parallel result

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for semigroups (Theorem 14.2). The proofs for rings and semigroups are often similar, but still at some points different techniques are required.

We will use the terminology of concrete categories and concrete functors given in [1, Definition 5.1] and [1, Definition 5.9].

1.1. Definition. *A* **concrete category** *over a category $\mathcal{X}$ is a pair $(\mathcal{A}, U)$ where $\mathcal{A}$ is a category and $U : \mathcal{A} \longrightarrow \mathcal{X}$ is a faithful functor.*

1.2. Definition. *Let $(\mathcal{A}, U)$ and $(\mathcal{B}, V)$ be concrete categories over a category $\mathcal{X}$. A functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ is a* **concrete functor** *if $U = V \circ F$.*

In this paper we consider different subcategories of the category of all right (or left) modules over a ring R as concrete categories over the category Ab of abelian groups together with a forgetful (or underlying) functor to Ab, which takes each R-module to its underlying abelian group and leaves morphisms unchanged. Thus applying a concrete functor to a module M_R means defining a new multiplication by the elements of a ring S on the abelian group M. The R-multiplication preserving mappings must also preserve the S-multiplication, because concrete functors must map morphisms identically. In the semigroup case we always consider categories of acts as concrete categories over Set. A typical example of a concrete functor is the restriction of scalars functor, which can be considered both in the case of rings and in the case of semigroups.

Concrete functors between categories of modules over unital rings were considered already in [16], where the term *vergessender Funktor* was used. We also point out that concrete functors between the categories of firm modules or acts appear naturally in Morita theory. In the ring case, concrete equivalence functors between different module categories have been used in [5] and [6], also in [4] (to prove Proposition 7). In the semigroup case, concrete functors are used to prove (for example) Theorem 3.2 in [18].

2. Terminology and notation

In this paper, by a ring we mean an associative ring. The construction of tensor product of modules over a ring is well known. We will recall briefly the construction of tensor product of acts over semigroups. If S is a semigroup, M_S is a right S-act and ${}_SN$ is a left S-act, then the tensor product $M \otimes_S N$ is the quotient set of $M \times N$ by the smallest equivalence relation containing the set $\{((ms, n), (m, sn)) \in (M \times N)^2 \mid m \in M, n \in N, s \in S\}$ (cf. [8, Construction 2.5.4]). The equivalence class of a pair (m, n) is denoted by $m \otimes n$.

2.1. Definition. *Let R be a ring (semigroup). A right R-module (resp. a right R-act) M_R is called*

1. **closed** *if the mapping*

$$\lambda_M : M \longrightarrow \mathrm{Hom}_R(R, M), \quad m \mapsto \lambda_M(m) : r \mapsto mr$$

is bijective,

2. **firm** *if the mapping*

$$\mu_M : M \otimes_R R \longrightarrow M, \quad m \otimes r \mapsto mr$$

is bijective (throughout this paper we will write $m \otimes r$ *instead of* $m \otimes_R r$*),*

3. **unitary** *if the mapping* μ_M *is surjective, i.e.* $M = MR$*. (Note that* MR *has a little bit different meaning in the case of modules and acts. In the latter case it consists of products* mr *while in the former case of all finite sums of such products. Also, tensor products of modules and acts are constructed in a different way.)*

Clearly, every firm module or act is unitary, but the converse is not true in general. However, the converse is true if the ring or the semigroup has the identity element.

The terms 'firm' and 'closed' have been introduced by Quillen in [17] for modules over nonunital rings.

2.2. Definition. *A ring (semigroup)* R *is called* **firm** *if the mapping*

$$\mu_R : R \otimes_R R \longrightarrow R, \quad r \otimes r' \mapsto rr'$$

is bijective.

Every unital ring and every monoid is firm. Moreover, every ring or semigroup with local units is firm.

If R is a ring and S is a semigroup, then we will use the following categories (with the usual homomorphisms as morphisms):

MOD_R — the category of all right R-modules,
Mod_R — the category of unitary right R-modules,
DMod_R — the category of firm right R-modules,
CMod_R — the category of closed right R-modules,
Ab — the category of abelian groups,
Ring — the category of unital rings,
Act_S — the category of all right S-acts,
UAct_S — the category of unitary right S-acts,
FAct_S — the category of firm right S-acts,
CAct_S — the category of closed right S-acts,
Set — the category of sets,
Mon — the category of monoids.

For the categories of left modules or acts we will write ${}_R\mathsf{MOD}$, ${}_S\mathsf{Act}$ etc.

The categories DMod_R and CMod_R need not coincide.

2.3. Example. [Firm module which is not closed] Any firm ring R without an identity element (for example the ring of infinite matrices with only a finite number of nonzero entries over a field K) is a firm right module but it is not closed because $\mathrm{Hom}_R(R, R) \ncong R$.

2.4. Example. [Closed module which is not firm] Let R be a firm ring without an identity element. Then the right R-module $M_R := \mathrm{Hom}_R({}_RR_R, R_R)$ is closed by Proposition 5.1(1). To prove that it is not firm, consider the diagram

$$\begin{array}{ccccc} \mathrm{Hom}_R({}_RR_R, R_R) \otimes R & & \xrightarrow{\quad \mu_M \quad} & & \mathrm{Hom}_R({}_RR_R, R_R)\,, \\ & \searrow^{\mathsf{eval}} & & \nearrow_{\lambda_R} & \\ & & R & & \end{array}$$

which can be seen to commute by direct verification. Since R is firm, the map eval is bijective (as we will explain in the beginning of Section 5). Now if M_R was firm, the map μ_M, and therefore λ_R as well, would be bijections. But we saw in Example 2.3 that λ_R cannot be bijective, so M_R cannot be firm.

Similar examples will show that the categories FAct_S and CAct_S need not coincide for a semigroup S. But we can also provide examples that are different from those.

2.5. Example. Let S be a right zero semigroup (i.e. a semigroup satisfying the identity $xy = y$) with at least two elements and let Θ_S be a one-element right S-act. Then $\lambda_\Theta : \Theta \longrightarrow \mathrm{Hom}_S(S, \Theta)$ is clearly bijective and hence Θ_S is closed. It can be shown that Θ_S is not firm. Conversely, [9, Example 2.8] provides a semigroup S such that the left act ${}_SS$ is firm, but not closed. Dually firm right acts exist which are not closed.

Let $f : R \longrightarrow S$ be a ring homomorphism. Any right S-module M can be turned into a right R-module with the multiplication given by $mr := mf(r)$. This construction gives a functor $f^* : \mathsf{MOD}_S \longrightarrow \mathsf{MOD}_R$ such that an S-module M goes to the same abelian group, but endowed with a new multiplication by the elements of R. On homomorphisms the functor f^* acts identically. This functor is called the **restriction of scalars functor** induced by f. We will use the notation ${}^*f$ for the restriction of scalars functor between categories of left modules.

3. Unital rings, homomorphisms and categories

Let R and S be unital rings with identity elements 1_R and 1_S, respectively. In this section, we are going to see that a ring homomorphism $f : R \longrightarrow S$ that preserves the identity (i.e. $f(1_R) = 1_S$) induces some relations between the categories of unitary modules. (Note that a module M_R is unitary if and only if $m1_R = m$ for every $m \in M$.) The generalization of the concept of ring homomorphism for rings without identity will be done using these categorical properties in the following section.

The following result is known (see [16, Satz 3.1]).

3.1. Proposition. *Given rings with identity R and S, there is a bijection between the following sets.*

1. *The concrete functors* $\mathsf{Mod}_S \longrightarrow \mathsf{Mod}_R$.

2. *The concrete functors* ${}_S\mathsf{Mod} \longrightarrow {}_R\mathsf{Mod}$.

3. *The unital ring homomorphisms* $f : R \longrightarrow S$.

The bijection is given by the restriction of the scalars $f \mapsto f^*$.

3.2. Remark. The unital ring homomorphisms $R \longrightarrow S$ clearly form a set. For the class of concrete functors $\mathsf{Mod}_S \longrightarrow \mathsf{Mod}_R$ this is not automatically clear from the beginning, but since this class is in a one-to-one correspondence with a set, it will also be a set.

3.3. Remark. Similarly to Proposition 3.1 one can prove that if R and S are monoids, then concrete functors $\mathsf{UAct}_S \longrightarrow \mathsf{UAct}_R$ are in one-to-one correspondence with monoid homomorphisms $f : R \longrightarrow S$.

Given a unital ring homomorphism $f : R \longrightarrow S$, the restriction of scalars functor $f^* : \mathsf{Mod}_S \longrightarrow \mathsf{Mod}_R$ has two adjoints

$$f_! : \mathsf{Mod}_R \longrightarrow \mathsf{Mod}_S, \qquad f_* : \mathsf{Mod}_R \longrightarrow \mathsf{Mod}_S.$$

The adjoints relations are $f_! \dashv f^* \dashv f_*$. The explicit definitions of the functors for rings with identity are the following:

$$f_!(M_R) = M \otimes_R S, \quad f_*(M_R) = \mathrm{Hom}_R(S_R, M_R).$$

The natural isomorphism induced by the adjunction $f^* \dashv f_*$ is

$$\gamma_{MN} : \mathrm{Hom}_R(f^*(M), N) \longrightarrow \mathrm{Hom}_S(M, \mathrm{Hom}_R(S_R, N_S)), \quad \gamma_{MN}(\alpha)(m)(s) = \alpha(ms),$$

and the adjunction $f_! \dashv f^*$ is given by the natural isomorphism

$$\delta_{NM} : \mathrm{Hom}_S(N \otimes_R S, M) \longrightarrow \mathrm{Hom}_R(N, f^*(M)), \quad \delta_{NM}(\beta)(n) = \beta(n \otimes 1_S).$$

These two adjoints of the functor of restriction of scalars will be the ones that let us give the definition of firm homomorphism of rings.

4. Firm homomorphisms of rings

In this section R and S will be firm rings. We will consider a ring homomorphsim $R \longrightarrow S$ (a map preserving addition and multiplication). The objective is to find the conditions on this map so that it could be considered a morphism in the category whose objects are the firm rings. We cannot impose the condition that 1_R maps to 1_S because the rings are not unital, but the new definition that we will introduce in this section, applied to unital rings, will extend the usual definition.

In order to simplify the notation, we will sometimes denote the ring homomorphism as $\dot{} : R \longrightarrow S$, so the image of an element $r \in R$ will be $\dot{r}$. In this way, the R-structure induced by this homomorphism on S-modules will be $rm = \dot{r}m$ for any left S-module

${}_SM$, all $m \in M$ and $r \in R$ (and a similar structure for right S-modules). This notation is not the usual one, but it is convenient because we only consider one homomorphism $R \longrightarrow S$ and it simplifies the use of parentheses. We will write module homomorphisms on the opposite side of scalars. In the case of bimorphisms, we will write them on one side or the other, depending on the structure that we are considering.

4.1. Definition. *Let R and S be firm rings and $\dot{} : R \longrightarrow S$ be a ring homomorphism. We will say that*

1. *it is a* **right firm ring homomorphism** *if S considered as a right R-module is firm (i.e. the map $\mu_S : S \otimes_R R \longrightarrow S$ given by $\mu_S(s \otimes r) = s\dot{r}$ is bijective);*
2. *it is a* **left firm ring homomorphism** *if S considered as a left R-module is firm;*
3. *it is a* **firm ring homomorphism** *if it is a right and left firm ring homomorphism.*

There exist right firm homomorphisms which are not left firm (and vice versa). One such homomorphism is constructed in Example 16.8.

4.2. Remark. Unital ring homomorphisms are (left and right) firm ring homomorphisms. To see this, suppose that $\dot{} : R \longrightarrow S$ is a unital ring homomorphism. Then $\mu_S(s \otimes 1_R) = s\dot{1}_R = s1_S = s$ for every $s \in S$, so μ_S is surjective. Suppose that $\mu_S(\sum_i s_i \otimes r_i) = 0$, i.e. $\sum_i s_i\dot{r}_i = 0$. Then

$$\sum_i s_i \otimes r_i = \sum_i s_i \otimes r_i 1_R = \sum_i s_i\dot{r}_i \otimes 1_R = 0 \otimes 1_R$$

in $S \otimes_R R$, showing that the kernel of μ_S is trivial. Hence μ_S is bijective and $\dot{}$ is right firm. Similarly it follows that $\dot{}$ is left firm.

We will give more examples of firm homomorphisms in Section 10.

4.3. Proposition. *If R is a unital ring, S is a firm ring and $\dot{} : R \longrightarrow S$ is a firm ring homomorphism, then S is unital and $\dot{1}_R = 1_S$.*

Proof. Denoting $e = \dot{1}_R$ we have to prove that e is the identity of S. Let $s \in S$. Since the ring homomorphism is right firm, we can find elements $s_i \in S$ and $r_i \in R$ such that $s = \sum_i s_i\dot{r}_i$, so we have

$$se = \sum_i s_i\dot{r}_i e = \sum_i s_i\dot{r}_i\dot{1}_R = \sum_i s_i \dot{\overline{r_i 1_R}} = \sum_i s_i\dot{r}_i = s.$$

Using that the ring homomorphism is left firm, we can prove that $es = s$, so e is the identity of S and the ring homomorphism preserves identity. ∎

If we consider only unital rings, then left or right firm ring homomorphisms preserve identity (i.e. we only have to check on one side).

4.4. PROPOSITION. *If R and S are unital rings and $\dot{}\ : R \longrightarrow S$ is a map preserving addition and multiplication, then the following conditions are equivalent:*

1. *it is a right firm ring homomorphism.*
2. *it is a unital ring homomorphism.*
3. *it is a left firm ring homomorphism.*

PROOF. Unital ring homomorphisms are right and left firm due to Remark 4.2. Suppose now that $\dot{}$ is right firm, then using the surjectivity of μ_S we can find $s_i \in S$ and $r_i \in R$ such that $1_S = \mu_S(\sum_i s_i \otimes r_i) = \sum_i s_i \dot{r}_i$, so we have

$$\dot{1}_R = 1_S \dot{1}_R = (\sum_i s_i \dot{r}_i)\dot{1}_R = \sum_i s_i(\dot{r}_i \dot{1}_R) = \sum_i s_i \dot{\overline{r_i 1_R}} = \sum_i s_i \dot{r}_i = 1_S.$$

Thus (1) and (2) are equivalent. The equivalence between (3) and (2) is symmetric. ∎

5. Functors between module categories

When R is a firm ring (that will be our case), the categories CMod_R and DMod_R are equivalent via the canonical functors

$$- \otimes_R R : \mathsf{CMod}_R \longrightarrow \mathsf{DMod}_R \quad \text{and} \quad \mathrm{Hom}_R(R, -) : \mathsf{DMod}_R \longrightarrow \mathsf{CMod}_R$$

([5, Theorem 15]). These equivalence functors induce the right R-module isomorphisms η_M and ε_M, given by

$$\eta_M(m)(r) = m \otimes r \quad \text{and} \quad \varepsilon_M(\alpha \otimes r) = \alpha(r),$$

such that the following diagrams are commutative:

$M_R \in \mathsf{CMod}_R$

$$\begin{array}{ccc} M \otimes_R R & \xrightarrow{\lambda_{M \otimes_R R}} & \mathrm{Hom}_R(R_R, M \otimes_R R) \\ \downarrow{\scriptstyle \mu_M} & \nearrow{\scriptstyle \eta_M} & \downarrow{\scriptstyle \mathrm{Hom}_R(R, \mu_M)} \\ M_R & \xrightarrow[\lambda_M]{} & \mathrm{Hom}_R(R_R, M_R) \end{array}$$

$M_R \in \mathsf{DMod}_R$

$$\begin{array}{ccc} M \otimes_R R & \xrightarrow{\mu_M} & M_R \\ \downarrow{\scriptstyle \lambda_M \otimes R} & \nearrow{\scriptstyle \varepsilon_M} & \downarrow{\scriptstyle \lambda_M} \\ \mathrm{Hom}_R(R_R, M_R) \otimes_R R & \xrightarrow[\mu_{\mathrm{Hom}_R(R_R, M_R)}]{} & \mathrm{Hom}_R(R_R, M_R). \end{array}$$

These properties are true also on the other side because the condition of being firm is symmetric.

Note that $\eta : 1_{\mathsf{CMod}_R} \Rightarrow \mathrm{Hom}_R(R_R, -) \circ (- \otimes R)$ and $\varepsilon : (- \otimes R) \circ \mathrm{Hom}_R(R_R, -) \Rightarrow 1_{\mathsf{DMod}_R}$ are the unit and counit of the adjunction $- \otimes R \dashv \mathrm{Hom}_R(R_R, -)$, respectively. Also, $\lambda : 1_{\mathsf{MOD}_R} \Rightarrow \mathrm{Hom}_R(R_R, -)$ and $\mu : - \otimes_R R \Rightarrow 1_{\mathsf{MOD}_R}$ are natural transformations.

In some cases, a module M_S will be considered with the induced R-structure, in that case we will use the notation μ_M and λ_M for the R-structure and μ'_M and λ'_M for the S-structure.

Suppose that ${}_SX_R$ is an (S, R)-bimodule and M_R is an R-module. The abelian group $\mathrm{Hom}_R(X_R, M_R)$ is a right S-module with the S-multiplication

$$(gs)(x) = g(sx), \tag{1}$$

$g \in \mathrm{Hom}_R(X_R, M_R)$, $s \in S$, $x \in X$.

5.1. Proposition. *Let R and S be firm rings and $f : R \longrightarrow S, r \mapsto \dot{r}$, a ring homomorphism. Then*

1. $\mathrm{Hom}_R(S_R, -)$ *is a functor from* MOD_R *to* CMod_S*;*
2. $\mathrm{Hom}_R({}_RS, -)$ *is a functor from* ${}_R\mathsf{MOD}$ *to* ${}_S\mathsf{CMod}$*;*
3. $- \otimes_R S$ *is a functor from* MOD_R *to* DMod_S*;*
4. $S \otimes_R -$ *is a functor from* ${}_R\mathsf{MOD}$ *to* ${}_S\mathsf{DMod}$*.*

Proof. The conditions 2 and 4 are the symmetric conditions to 1 and 3, thus we will prove only 1 and 3.

Since S is firm, the map $\mu_S : S \otimes_S S \longrightarrow S$, $s \otimes s' \mapsto ss'$, is an isomorphism of right S-modules. Applying the contravariant functor $\mathrm{Hom}_R(-, M_R) \circ f^* : \mathsf{MOD}_S \longrightarrow \mathsf{MOD}_R$ we see that the map

$$- \circ \mu_S : \mathrm{Hom}_R(S_R, M_R) \longrightarrow \mathrm{Hom}_R(S \otimes_S S_R, M_R)$$

is an isomorphism for every right R-module M_R. Due to the tensor-hom adjunction we have the bijection

$$\sigma : \mathrm{Hom}_R(S \otimes_S S_R, M_R) \longrightarrow \mathrm{Hom}_S(S_S, \mathrm{Hom}_R(S_R, M_R)),\ h \mapsto (s \mapsto (s' \mapsto h(s \otimes s'))).$$

The composite bijection $\sigma \circ (- \circ \mu_S)$ is $\lambda_{\mathrm{Hom}_R(S_R, M_R)}$, because

$$\begin{aligned}
(\sigma \circ (- \circ \mu_S))(g)(s)(s') &= \sigma(g \circ \mu_S)(s)(s') && (\text{def. of } - \circ \mu_S)\\
&= (g \circ \mu_S)(s \otimes s') && (\text{def. of } \sigma)\\
&= g(ss') && (\text{def. of } \mu)\\
&= (gs)(s') && (\text{by } (1))\\
&= \lambda_{\mathrm{Hom}_R(S_R, M_R)}(g)(s)(s') && (\text{def. of } \lambda_{\mathrm{Hom}_R(S_R, M_R)})
\end{aligned}$$

for all $s, s' \in S$ and $g \in \mathrm{Hom}_R(S_R, M_R)$. Hence $\mathrm{Hom}_R(S_R, M_R) \in \mathsf{CMod}_S$. We also have $N \otimes_R S \in \mathsf{DMod}_S$ for every right R-act N_R, because the composite bijection

$$(N \otimes_R S) \otimes_S S \longrightarrow N \otimes_R (S \otimes_S S) \longrightarrow N \otimes_R S,\ (n \otimes s) \otimes s' \mapsto n \otimes (s \otimes s') \mapsto n \otimes ss'$$

is precisely $\mu_{N \otimes_R S}$. The rest of the proof is straightforward. ∎

5.2. Corollary. *If R is a firm ring and $M_R \in \mathsf{MOD}_R$, then* $\mathrm{Hom}_R(R_R, M_R) \in \mathsf{CMod}_R$ *and $M \otimes_R R \in \mathsf{DMod}_R$.*

Proof. We apply Proposition 5.1 for the ring homomorphism $id_R : R \longrightarrow R$. ∎

We will study the firm homomorphisms in more detail, but we will use the definition of firm ring homomorphisms on one side given in Definition 4.1 to prove some results that will be symmetrical on the other side.

5.3. Proposition. *Let R and S be firm rings and $f : R \longrightarrow S, r \mapsto \dot{r}$, a right firm ring homomorphism. Then*

1. *the map $\mu_S : S \otimes_R R \longrightarrow S$ is not only a right R-isomorphism, but also a left S-isomorphism;*
2. $\mathrm{Hom}_S({}_SS_R, -)$ *is a functor from ${}_S\mathsf{MOD}$ to ${}_R\mathsf{CMod}$;*
3. *$- \otimes_S S_R$ is a functor from MOD_S to DMod_R;*
4. *the functors $f^*, - \otimes_S S_R : \mathsf{DMod}_S \longrightarrow \mathsf{DMod}_R$ are naturally isomorphic.*

Proof.

1. The map μ_S is bijective because S_R is in DMod_R, therefore we only need to prove that it is a left S-homomorphism. If $s, t \in S$ and $r \in R$, then we have

$$\begin{aligned}(t(s \otimes r))\mu_S &= (ts \otimes r)\mu_S && \text{(left S-structure of $S \otimes_R R$)}\\ &= (ts)\dot{r} && \text{(def. of μ_S)}\\ &= t(s\dot{r}) && \text{(associativity of multiplication in S)}\\ &= t(s \otimes r)\mu_S. && \text{(def. of μ_S)}\end{aligned}$$

2. Let ${}_SM$ be any left S-module. Then

$$\begin{aligned}\mathsf{Hom}_S({}_SS_R, {}_SM) &\cong \mathsf{Hom}_S({}_SS_R \otimes R_R, {}_SM) && \text{(${}_SS_R$ is right firm)}\\ &\cong \mathsf{Hom}_R({}_RR_R, \mathsf{Hom}_S({}_SS_R, {}_SM)) && \text{(tensor-hom adjunction)}\end{aligned}$$

as left R-modules. The last module is closed by Proposition 5.1(2), hence also $\mathrm{Hom}_S({}_SS_R, {}_SM)$ is in ${}_R\mathsf{CMod}$.

3. Let M_S be in MOD_S. Since $\mu_S : S \otimes_R R \longrightarrow S$ is a left S-isomorphism, we have the composite bijection

$$\begin{aligned}(M \otimes_S S) \otimes_R R &\longrightarrow M \otimes_S (S \otimes_R R) \longrightarrow M \otimes_S S,\\ (m \otimes s) \otimes r &\mapsto m \otimes (s \otimes r) \mapsto m \otimes s\dot{r} = (m \otimes s)r\end{aligned}$$

which takes $(m \otimes s) \otimes r$ to $m \otimes s\dot{r} = (m \otimes s)r$, which is precisely $\mu_{M \otimes_S S_R}$. Thus $M \otimes_S S_R \in \mathsf{DMod}_R$. The rest is again straightforward.

4. If $M_S \in \mathsf{DMod}_S$, then the mapping $\mu'_M : M \otimes_S S \longrightarrow M$, $m \otimes s \mapsto ms$, is bijective. It is an isomorphism of right R-modules, because

$$\mu'_M((m \otimes s)r) = \mu'_M(m \otimes s\dot{r}) = m(s\dot{r}) = (ms)\dot{r} = \mu'_M(m \otimes s)r$$

for every $m \in M$, $s \in S$ and $r \in R$. Since $M \otimes_S S_R \in \mathsf{DMod}_R$, also $f^*(M_S) = M_R \in \mathsf{DMod}_R$, so we can consider f^* as a functor $\mathsf{DMod}_S \longrightarrow \mathsf{DMod}_R$. It is easy to check that μ' is natural in M. Therefore f^* and $- \otimes_S S_R$ are naturally isomorphic.

■

The next proposition will give two necessary and sufficient conditions for a ring homomorphism to be right firm in terms of certain properties of the restriction of scalars functors.

5.4. Proposition. *Let R and S be firm rings and $f : R \longrightarrow S, r \mapsto \dot{r}$, a ring homomorphism. The following conditions are equivalent.*

1. *$f : R \longrightarrow S$ is a right firm ring homomorphism.*

2. *For every ${}_SM \in {}_S\mathsf{CMod}$, ${}^*f({}_SM) \in {}_R\mathsf{CMod}$, so the restriction of scalars is a concrete functor from ${}_S\mathsf{CMod}$ to ${}_R\mathsf{CMod}$.*

3. *For every $M_S \in \mathsf{DMod}_S$, $f^*(M_S) \in \mathsf{DMod}_R$, so the restriction of scalars is a concrete functor from DMod_S to DMod_R.*

Proof. ($1 \Rightarrow 2$). Let ${}_SM$ be in ${}_S\mathsf{CMod}$, that is, $\lambda'_M : {}_SM \longrightarrow \mathrm{Hom}_S({}_SS, {}_SM)$ is a left S-isomorphism. Applying the restriction of scalars functor ${}^*f : {}_S\mathsf{MOD} \longrightarrow {}_R\mathsf{MOD}$ we see that $\lambda'_M : {}_RM \longrightarrow \mathrm{Hom}_S({}_SS_R, {}_SM)$ is a left R-isomorphism, and so is $\mathrm{Hom}_R({}_RR, \lambda'_M)$. Moreover, the diagram

$$\begin{array}{ccc}
{}_RM & \xrightarrow{\quad\lambda_M\quad} & \mathrm{Hom}_R({}_RR, {}_RM) \\
{\scriptstyle \lambda'_M}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_R({}_RR, \lambda'_M)} \\
\mathrm{Hom}_S({}_SS_R, {}_SM) & \xrightarrow{\lambda_{\mathrm{Hom}_S({}_SS_R, {}_SM)}} & \mathrm{Hom}_R({}_RR, \mathrm{Hom}_S({}_SS_R, {}_SM))
\end{array}$$

in ${}_R\mathsf{Mod}$ commutes, because $\lambda : 1_{{}_R\mathsf{Act}} \Rightarrow \mathrm{Hom}_R({}_RR, -)$ is a natural transformation. The lower arrow is bijective because of Proposition 5.3(2) (as $\mathrm{Hom}_S({}_SS_R, {}_SM) \in {}_R\mathsf{CMod}$), so the upper arrow λ_M must also be bijective and therefore ${}_RM = {}^*f({}_SM)$ is in ${}_R\mathsf{CMod}$.

(2 ⇒ 1). Consider the functor $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$. This is the construction of the character module and it is known that it reflects and preserves isomorphisms. For any firm module M, $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is closed. We are going to prove that $\mu_S : S \otimes_R R \longrightarrow S$ is an isomophism, proving that

$$\mathrm{Hom}_{\mathbb{Z}}(\mu_S, \mathbb{Q}/\mathbb{Z}) : \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(S \otimes_R R, \mathbb{Q}/\mathbb{Z})$$

is bijective. But this is true because this map is the composition of

$$\mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Hom}_R(R, \mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z})) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(S \otimes_R R, \mathbb{Q}/\mathbb{Z})$$

and the second one is bijective due to the hom-tensor adjunction and the first one is bijective because $\mathrm{Hom}_{\mathbb{Z}}(S, \mathbb{Q}/\mathbb{Z})$ is a closed module.

(1 ⇒ 3). Let M_S be in DMod_S. Since $\mu : - \otimes_R R \Rightarrow 1_{\mathsf{DMod}_R}$ is a natural transformation, the square

$$\begin{array}{ccc} M \otimes_S S \otimes_R R & \xrightarrow{\quad \mu_{M\otimes S} \quad} & M \otimes_S S \\ {\scriptstyle \mu'_M \otimes R} \downarrow & & \downarrow {\scriptstyle \mu'_M} \\ M \otimes_R R & \xrightarrow{\quad \mu_M \quad} & M \end{array}$$

commutes. The vertical arrows are bijective because M_S is firm. The upper arrow is bijective because of Proposition 5.3(3) (as $M \otimes_S S_R \in \mathsf{DMod}_R$), so the lower arrow μ_M is also a bijection and therefore M_R is in DMod_R.

(3 ⇒ 1). If we apply (3) to the ring S which is a firm right S-module, we get that S_R is in DMod_R and therefore the ring homomophism is right firm. ■

6. Compatibility conditions

In this section we introduce compatibility conditions for pairs of concrete functors between categories of firm and closed modules and prove some connections between these conditions. We will recall the following well-known result about composing natural transformations.

6.1. Proposition. [1, Exercise 6A] *Consider the diagram*

$$\mathcal{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{B} \underset{K}{\overset{H}{\rightrightarrows}} \mathcal{C} \qquad (\Downarrow\alpha,\ \Downarrow\beta)$$

of functors and natural transformations. Then $\beta G \circ H\alpha = K\alpha \circ \beta F$*, i.e. the square*

$$\begin{array}{ccc} HF & \xrightarrow{H\alpha} & HG \\ {\scriptstyle \beta F}\downarrow & \searrow^{\beta * \alpha} & \downarrow{\scriptstyle \beta G} \\ KF & \xrightarrow[K\alpha]{} & KG. \end{array}$$

commutes. The natural transformation $\beta G \circ H\alpha = K\alpha \circ \beta F$ *is denoted by* $\beta * \alpha$ *and called the horisontal composition of* α *and* β*.*

Let R and S be firm rings. We abbreviate

$$H := \mathrm{Hom}_R(R, -), \quad H' := \mathrm{Hom}_S(S, -), \quad T := - \otimes R, \quad T' := - \otimes S.$$

Then $T \dashv H : \mathsf{DMod}_R \longrightarrow \mathsf{CMod}_R$ and $T' \dashv H' : \mathsf{DMod}_S \longrightarrow \mathsf{CMod}_S$. We denote the units and counits of these adjunctions as follows:

$$\eta : 1_{\mathsf{CMod}_R} \Rightarrow HT, \quad \varepsilon : TH \Rightarrow 1_{\mathsf{DMod}_R}, \quad \eta' : 1_{\mathsf{CMod}_S} \Rightarrow H'T', \quad \varepsilon' : T'H' \Rightarrow 1_{\mathsf{DMod}_S}.$$

As observed in the beginning of Section 5, all these natural transformations are actually natural isomorphisms. The triangle identities for the adjunction $T \dashv H$ are

$$1_T = \varepsilon T \circ T\eta,$$
$$1_H = H\varepsilon \circ \eta H,$$

and similar identities hold for the adjunction $T' \dashv H'$.

Consider a pair

$$F : \mathsf{DMod}_S \longrightarrow \mathsf{DMod}_R \quad \text{and} \quad G : \mathsf{CMod}_S \longrightarrow \mathsf{CMod}_R$$

of concrete functors such that there exist natural transformations

$$\zeta : GH' \Rightarrow HF \quad \text{and} \quad \xi : TG \Rightarrow FT'.$$

$$\begin{array}{ccc} \mathsf{CMod}_S & \xleftarrow{\mathrm{Hom}_S(S,-)} & \mathsf{DMod}_S \\ {\scriptstyle G}\downarrow & \overset{\zeta}{\Longrightarrow} & \downarrow{\scriptstyle F} \\ \mathsf{CMod}_T & \xleftarrow[\mathrm{Hom}_R(R,-)]{} & \mathsf{DMod}_R \end{array} \qquad \begin{array}{ccc} \mathsf{CMod}_S & \xrightarrow{-\otimes S} & \mathsf{DMod}_S \\ {\scriptstyle G}\downarrow & \overset{\xi}{\Longrightarrow} & \downarrow{\scriptstyle F} \\ \mathsf{CMod}_R & \xrightarrow[-\otimes R]{} & \mathsf{DMod}_R \end{array}$$

We introduce the following compatibility conditions.

C1. $\zeta_M \circ \lambda'_M = \lambda_{F(M)}$ for every $M_S \in \mathsf{DMod}_S$,

C2. $\mu'_M \circ \xi_M = \mu_{G(M)}$ for every $M_S \in \mathsf{CMod}_S$,

C3. $\varepsilon F \circ T\zeta = F\varepsilon' \circ \xi H'$,

C4. $H\xi \circ \eta G = \zeta T' \circ G\eta'$.

The following three results can be proved using Proposition 6.1 and the mates correspondence of natural transformations. In particular, Condition C3 (equivalently condition C4) states precisely that the natural transformations ζ and ξ are mates (in the sense of being in the roles of μ and λ in [7, Proposition 2.1]; cf. also [13, page 100]).

6.2. Proposition. *Conditions **C3** and **C4** are equivalent.*

6.3. Proposition. *Assume that **C4** (or, equivalently, **C3**) holds and ζ, ξ are natural isomorphisms. Then **C1** and **C2** are equivalent.*

6.4. Lemma. *Suppose that **C3** holds and ξ is a natural isomorphism. Then also ζ is a natural isomorphism.*

6.5. Definition. *Let $R, S, H, H', T, T', \eta, \eta', \varepsilon, \varepsilon'$ be as above and let $F : \mathsf{DMod}_S \longrightarrow \mathsf{DMod}_R$ and $G : \mathsf{CMod}_S \longrightarrow \mathsf{CMod}_R$ be concrete functors. We say that F, G is a* **pair of compatible concrete functors***, if there exist natural isomorphisms $\zeta : GH' \Rightarrow HF$ and $\xi : TG \Rightarrow FT'$ such that the conditions **C1**–**C4** are satisfied.*

7. Compatibility for restrictions of scalars

If $f : R \longrightarrow S$ is a firm ring homomoprphism between firm rings, then by Proposition 5.4 and its dual the restriction of scalars funtor $f^* : \mathsf{MOD}_S \longrightarrow \mathsf{MOD}_R$ restricts to functors

$$f^+ : \mathsf{DMod}_S \longrightarrow \mathsf{DMod}_R \quad \text{and} \quad f^\times : \mathsf{CMod}_S \longrightarrow \mathsf{CMod}_R.$$

In this section we will show that $f^+, f^\times$ is a pair of compatible concrete functors.

7.1. Proposition. [Compatibility **C1**] *Let R and S be firm rings. Suppose $f : R \longrightarrow S$, $r \mapsto \dot{r}$, is a left firm homomorphism of rings and let M_S be in MOD_S. Then, in the category CMod_R, we have a right R-homomorphism*

$$\zeta_M : \mathrm{Hom}_S({}_R S_S, M_S) \longrightarrow \mathrm{Hom}_R({}_R R_R, M_R),$$

natural in M, such that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Hom}_S({}_R S_S, M_S) & \xrightarrow{\zeta_M} & \mathrm{Hom}_R({}_R R_R, M_R) \\ & \nwarrow^{\lambda'_M} \quad {}^{\lambda_M}\nearrow & \\ & M. & \end{array}$$

Furthermore, if we apply this construction to $M_S = S_S$, we get

$$\zeta_S : \mathrm{Hom}_S(S_S, S_S) \longrightarrow \mathrm{Hom}_R(R_R, S_R)$$

and $\zeta_S(\mathsf{id}_S)$ is precisely the ring homomorphism f.

Proof. Using the dual of Proposition 5.3(2) for the homomorphisms f and id_R we know that the modules $\mathrm{Hom}_S({}_RS_S, M_S)$ and $\mathrm{Hom}_R({}_RR_R, M_R)$ are in CMod_R. Let $\alpha \in \mathrm{Hom}_S({}_RS_S, M_S)$. We define a map $\zeta_M(\alpha) : R \longrightarrow M$ by

$$\zeta_M(\alpha)(r) = \alpha(\dot{r}). \tag{2}$$

With this definition, the last claim of the proposition is trivial because $\zeta_S(\mathsf{id}_S)(r) = \mathsf{id}_S(\dot{r}) = f(r)$ for all $r \in R$. Also, the triangle is commutative because, for every $m \in M$ and $r \in R$,

$$(\zeta_M \circ \lambda'_M)(m)(r) = \zeta_M(\lambda'_M(m))(r) = \lambda'_M(m)(\dot{r}) = m\dot{r} = mr = \lambda_M(m)(r).$$

We have to check several properties of ζ_M.

1. $\zeta_M(\alpha)$ is a right R-homomorphism. For every $u, r \in R$,

$$\zeta_M(\alpha)(ru) = \alpha(\dot{\overline{ru}}) = \alpha(\dot{r}\dot{u}) = \alpha(\dot{r})\dot{u} = \zeta_M(\alpha)(r)u.$$

2. ζ_M is a right R-homomorphism. For every $\alpha \in \mathrm{Hom}_S({}_RS_S, M_S)$, $r, u \in R$ we have

$$\begin{aligned}
\zeta_M(\alpha r)(u) &= (\alpha r)(\dot{u}) && \text{(def. of } \zeta_M) \\
&= \alpha(\dot{r}\dot{u}) && (R\text{-structure of } \mathrm{Hom}_S({}_RS_S, M_S)) \\
&= \alpha(\dot{\overline{ru}}) && (\dot{\ }\text{ preserves multiplication}) \\
&= \zeta_M(\alpha)(ru) && \text{(def. of } \zeta_M) \\
&= (\zeta_M(\alpha)r)(u). && (\text{right } R\text{-structure of } \mathrm{Hom}_R({}_RR_R, M_R))
\end{aligned}$$

This proves that $\zeta_M(\alpha r) = \zeta_M(\alpha)r$ for all $r \in R$ and all $\alpha \in \mathrm{Hom}_S({}_RS_S, M_S)$.

3. ζ_M is natural in M. Let $k : M_S \longrightarrow N_S$ be a morphism in MOD_S. The square

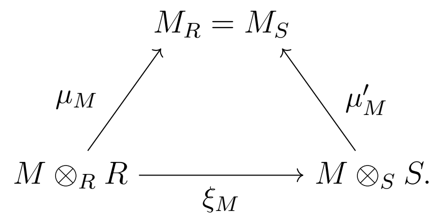
$$\begin{array}{ccc}
\mathrm{Hom}_S({}_RS_S, M_S) & \xrightarrow{\ \zeta_M\ } & \mathrm{Hom}_R({}_RR_R, M_R) \\
{\scriptstyle \mathrm{Hom}_S(S,k)}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_R(R,k)} \\
\mathrm{Hom}_S({}_RS_S, N_S) & \xrightarrow{\ \zeta_N\ } & \mathrm{Hom}_R({}_RR_R, N_R)
\end{array}$$

commutes, because, for every $\alpha \in \mathrm{Hom}_S({}_RS_S, M_S)$ and $r \in R$, we have

$$\begin{aligned}
\mathrm{Hom}_R(R,k)(\zeta_M(\alpha))(r) &= (k \circ \zeta_M(\alpha))(r) && (\text{def. of } \mathrm{Hom}_R({}_RR_R, -)) \\
&= k(\zeta_M(\alpha)(r)) && \text{(composition of maps)} \\
&= k(\alpha(\dot{r})) && (\text{def. of } \zeta_M(\alpha)) \\
&= \zeta_N(k\alpha)(r) && (\text{def. of } \zeta_N(k\alpha)) \\
&= \zeta_N(\mathrm{Hom}_S(S,k)(\alpha))(r). && (\text{def. of } \mathrm{Hom}_S(S, -))
\end{aligned}$$

■

7.2. Proposition. [Compatibility **C2**] *Let* $\dot{}\ : R \longrightarrow S$ *be a right firm ring homomorphism between firm rings and let* $M_S \in \mathsf{MOD}_S$*. Then we have a right* R*-homomorphism in the category* DMod_R, $\xi_M : M \otimes_R R \longrightarrow M \otimes_S S$*, natural in* M *such that the following diagram is commutative:*



Furthermore, if we apply this construction to $M = \mathrm{Hom}_S(S_S, S_S)$ *and we use the canonical isomorphism* $\varepsilon'_S : \mathrm{Hom}_S(S_S, S_S) \otimes_S S \longrightarrow S_S$*, then we have* $\varepsilon'_S(\xi_{\mathrm{Hom}_S(S_S,S_S)}(\mathsf{id}_S \otimes r)) = \dot{r}$ *for all* $r \in R$*.*

Proof. By Proposition 5.3, $M \otimes_S S \in \mathsf{DMod}_R$, and due to Corollary 5.2 we have $M \otimes_R R \in \mathsf{DMod}_R$. The definition of $\xi_M : M \otimes_R R \longrightarrow M \otimes_S S$ is

$$\xi_M(m \otimes r) = m \otimes \dot{r}.$$

To see that ξ_M is well defined we note that the mapping

$$\overline{\xi_M} : M \times R \longrightarrow M \otimes_S S, \quad (m, r) \mapsto m \otimes \dot{r}$$

is R-balanced, because

$$\overline{\xi_M}(m, ur) = m \otimes \dot{\overline{ur}} = m \otimes \dot{u}\dot{r} = m\dot{u} \otimes \dot{r} = mu \otimes \dot{r} = \overline{\xi_M}(mu, r)$$

for all $m \in M$ and $u, r \in R$. For any $r \in R$ we have $\dot{r} \in S = SR$, so we can find $s_i \in S$ and $r_i \in R$ such that $\dot{r} = \sum_i s_i \dot{r}_i$. We have

$$\begin{aligned}
\varepsilon'_S(\xi_{\mathrm{Hom}_S(S_S,S_S)}(\mathsf{id}_S \otimes r)) &= \varepsilon'_S(\mathsf{id}_S \otimes \dot{r}) && (\text{def. of } \xi_{\mathrm{Hom}_S(S_S,S_S)})\\
&= \varepsilon'_S(\mathsf{id}_S \otimes \Sigma_i s_i \dot{r}_i) && (\text{expansion of } \dot{r})\\
&= \varepsilon'_S(\Sigma_i \mathsf{id}_S s_i \otimes \dot{r}_i) && (\mathrm{Hom}_S(S_S, S_S) \otimes_S S \text{ is } S\text{-balanced})\\
&= \Sigma_i(\mathsf{id}_S s_i)(\dot{r}_i) && (\text{def. of } \varepsilon'_S)\\
&= \mathsf{id}_S(\Sigma_i s_i \dot{r}_i) && (S\text{-structure of } \mathrm{Hom}_S(S_S, S_S))\\
&= \mathsf{id}_S(\dot{r}) = \dot{r}. && (\text{expansion of } \dot{r})
\end{aligned}$$

This proves the last claim. We still need to check several properties of ξ.

1. The diagram is commutative because, for every $m \in M$ and $r \in R$,

$$(\mu'_M \circ \xi_M)(m \otimes r) = \mu'_M(\xi_M(m \otimes r)) = \mu'_M(m \otimes \dot{r}) = m\dot{r} = \mu_M(m \otimes r).$$

2. ξ_M is a right R-homomorphism. For any $m \in M$, $r, u \in R$ we have

$$\begin{aligned}
\xi_M((m \otimes r)u) &= \xi_M(m \otimes ru) && \text{(right } R\text{-module structure of } M \otimes_R R) \\
&= m \otimes \overset{\cdot}{\overline{ru}} && \text{(def. of } \xi_M) \\
&= m \otimes \dot{r}\dot{u} && (\dot{}\ \text{preserves multiplication}) \\
&= (m \otimes \dot{r})\dot{u} && \text{(right } S\text{-module structure of } M \otimes_S S) \\
&= (m \otimes \dot{r})u && \text{(right } R\text{-module structure of } M \otimes_S S) \\
&= \xi_M(m \otimes r)u. && \text{(def. of } \xi_M)
\end{aligned}$$

The rest of the proof is straightforward.

3. ξ is natural in M. Let $k : M_S \longrightarrow N_S$ be a right S-homomorphism. This is also a right R-homomorphism and we can consider the following diagram:

$$\begin{array}{ccc}
M \otimes_R R & \xrightarrow{\quad \xi_M \quad} & M \otimes_S S \\
{\scriptstyle k \otimes R}\downarrow & & \downarrow{\scriptstyle k \otimes S} \\
N \otimes_R R & \xrightarrow{\quad \xi_N \quad} & N \otimes_S S
\end{array}.$$

To see that it commutes, we observe that, for all $m \in M$ and all $r \in R$,

$$\begin{aligned}
((k \otimes S) \circ \xi_M)(m \otimes r) &= (k \otimes S)(m \otimes \dot{r}) && \text{(def. of } \xi_M) \\
&= k(m) \otimes \dot{r} && \text{(def. of } - \otimes S \text{ over morphisms)} \\
&= \xi_N(k(m) \otimes r) && \text{(def. of } \xi_N) \\
&= (\xi_N \circ (k \otimes R))(m \otimes r). && \text{(def. of } - \otimes R \text{ over morphisms)}
\end{aligned}$$

■

All the results that we have proved for right firm ring homomorphisms are true for left firm ring homomorphisms changing the sides on each one of the categories considered. If we have both conditions, i.e. we have a firm ring homomorphism, then restriction of scalars provides concrete functors $\mathsf{CMod}_S \longrightarrow \mathsf{CMod}_R$ and $\mathsf{DMod}_S \longrightarrow \mathsf{DMod}_R$, but we also have the category equivalences $\mathsf{CMod}_R \longrightarrow \mathsf{DMod}_R$ and $\mathsf{CMod}_S \longrightarrow \mathsf{DMod}_S$. The next proposition will show that in the case of a (left and right) firm ring homomorphism, the concrete functors and the category equivalences are compatible.

7.3. Proposition. [Compatibility **C3**] *Let* $\dot{}\ : R \longrightarrow S$ *be a firm homomorphism between firm rings. Then, for any* $M_S \in \mathsf{DMod}_S$*, the following diagram of* R*-homomorphisms is commutative:*

$$\begin{array}{ccc} \mathrm{Hom}_S(S_S, M_S) \otimes_R R & \xrightarrow{\xi_{\mathrm{Hom}_S(S_S,M_S)}} & \mathrm{Hom}_S(S_S, M_S) \otimes_S S \\ \zeta_M \otimes R \Big\downarrow & & \Big\downarrow \varepsilon'_M \\ \mathrm{Hom}_R(R_R, M_R) \otimes_R R & \xrightarrow{\quad \varepsilon_M \quad} & M \end{array} .$$

Hence $\varepsilon f^+ \circ T\zeta = f^+\varepsilon' \circ \xi H'$.

PROOF. If $\alpha \in \mathrm{Hom}_S(S_S, M_S)$ and $r \in R$, then

$$\begin{aligned} \varepsilon'_M(\xi_{\mathrm{Hom}_S(S_S,M_S)}(\alpha \otimes r)) &= \varepsilon'_M(\alpha \otimes \dot{r}) && \text{(def. of } \xi_{\mathrm{Hom}_S(S_S,M_S)}) \\ &= \alpha(\dot{r}) && \text{(def. of } \varepsilon'_M) \\ &= \zeta_M(\alpha)(r) && \text{(def. of } \zeta_M) \\ &= \varepsilon_M(\zeta_M(\alpha) \otimes r) && \text{(def. of } \varepsilon_M) \\ &= \varepsilon_M((\zeta_M \otimes R)(\alpha \otimes r)). && \text{(def. of } - \otimes R \text{ over morphisms)} \end{aligned}$$

■

7.4. PROPOSITION. *The natural transformations* $\xi : Tf^\times \Rightarrow f^+T'$ *and* $\zeta : f^\times H' \Rightarrow Hf^+$ *defined in the previous propositions are natural isomorphisms.*

PROOF. We will prove that ξ_M is bijective for every M_S. Then ζ is a natural isomorphism because of Lemma 6.4.

First we verify that ξ_M is surjective. For all $m \in M$ and $s \in S$ we can write $s = \sum_i s_i\dot{r}_i$ because S_R is firm and in particular $SR = S$. So we have

$$m \otimes s = m \otimes \sum_i s_i\dot{r}_i = \sum_i m \otimes s_i\dot{r}_i = \sum_i ms_i \otimes \dot{r}_i = \xi_M\left(\sum_i ms_i \otimes r_i\right)$$

in $M \otimes_S S$. This proves that all $m \otimes s$ are in $\mathsf{Im}(\xi_M)$. Since they are generators of $M \otimes_S S$, we conclude that ξ_M is surjective.

Now we show that ξ_M is injective. Using that $\mu'_M\xi_M = \mu_M$, we know that $\mathsf{Ker}(\xi_M) \subseteq \mathsf{Ker}(\mu_M)$. But $\mathsf{Ker}(\mu_M)R = 0$, so we have $\mathsf{Ker}(\xi_M)R = 0$. The kernel of a surjective homomorphism between firm modules is unitary (by [15, Proposition 10]), so $0 = \mathsf{Ker}(\xi_M)R = \mathsf{Ker}(\xi_M)$. ■

8. Main theorem for rings

Now we can prove our main theorem.

8.1. Theorem. *Given firm rings R and S, there is a bijection between the following sets:*

1. *The pairs of compatible concrete functors*

$$F : \mathsf{DMod}_S \longrightarrow \mathsf{DMod}_R \quad \textit{and} \quad G : \mathsf{CMod}_S \longrightarrow \mathsf{CMod}_R.$$

2. *The pairs of compatible concrete functors*

$$F : {}_S\mathsf{DMod} \longrightarrow {}_R\mathsf{DMod} \quad \textit{and} \quad G : {}_S\mathsf{CMod} \longrightarrow {}_R\mathsf{CMod}.$$

3. *The (left and right) firm ring homomorphisms $R \longrightarrow S$.*

The bijection is given by the restriction of scalars.

Proof. Condition (3) is symmetric, therefore we only need to see the bijection between (1) and (3). Given a firm ring homorphism $f : R \longrightarrow S$, the restriction of scalars provides the functors $f^+ : \mathsf{DMod}_S \longrightarrow \mathsf{DMod}_R$ and $f^\times : \mathsf{CMod}_S \longrightarrow \mathsf{CMod}_R$ such that they form a compatible pair. The required natural transformations ζ and ξ for this pair are constructed in Proposition 7.1 and Proposition 7.2, respectively. Conditions **C1**–**C3** are satisfied by Proposition 7.1, Proposition 7.2 and Proposition 7.3, respectively. Condition **C4** follows from Proposition 6.2. Thus each firm homomorphism induces a pair of concrete functors.

Suppose now that we have a pair of compatible concrete functors $F : \mathsf{DMod}_S \longrightarrow \mathsf{DMod}_R$ and $G : \mathsf{CMod}_S \longrightarrow \mathsf{CMod}_R$. We are going to make the proof in several steps.

1. If we apply condition **C3** to the firm module S_S, $\mathsf{id}_S \in \mathrm{Hom}_S(S_S, S_S)$ and $r \in R$, then we obtain that

$$\varepsilon'_S(\xi_{\mathrm{Hom}_S(S_S,S_S)}(\mathsf{id}_S \otimes r)) = \varepsilon_S((\zeta_S \otimes R)(\mathsf{id}_S \otimes r)) = \zeta_S(\mathsf{id}_S)(r).$$

This common value will be called $f(r)$ and the mapping $f := \zeta_S(\mathsf{id}_S) : R \longrightarrow S$, which is actually a right R-homomorphism, will be the candidate to be the ring homomorphism.

2. For every $M_S \in \mathsf{DMod}_S$, $m \in M$ and $r \in R$, we are going to prove that $mr = mf(r)$. (Here mr is the product of m and r in the right R-module $F(M_S)$.)

Consider the S-homomorphism, $\lambda'_M(m) : S_S \longrightarrow M_S$ given by $\lambda'_M(m)(s) = ms$ and the commutative diagram induced by the naturality of ε' and ξ:

$$\begin{array}{ccc}
G(\mathrm{Hom}_S(S_S,S_S)) \otimes_R R & \xrightarrow{\mathrm{Hom}_S(S,\lambda'_M(m)) \otimes R} & G(\mathrm{Hom}_S(S_S,M_S)) \otimes_R R \\
\Big\downarrow {\scriptstyle \xi_{\mathrm{Hom}_S(S_S,S_S)}} & & \Big\downarrow {\scriptstyle \xi_{\mathrm{Hom}_S(S_S,M_S)}} \\
\mathrm{Hom}_S(S_S,S_S) \otimes_S S & \xrightarrow{\mathrm{Hom}_S(S,\lambda'_M(m)) \otimes S} & \mathrm{Hom}_S(S_S,M_S) \otimes_S S \\
\Big\downarrow {\scriptstyle \varepsilon'_S} & & \Big\downarrow {\scriptstyle \varepsilon'_M} \\
S_S & \xrightarrow{\lambda'_M(m)} & M_S.
\end{array}$$

We calculate:

$$\begin{aligned}
mf(r) &= \lambda'_M(m)(f(r)) && \text{(def. of } \lambda'_M(m))\\
&= \lambda'_M(m)(\varepsilon'_S(\xi_{\mathrm{Hom}_S(S_S,S_S)}(\mathsf{id}_S \otimes r))) && \text{(def. of } f(r))\\
&= \varepsilon'_M(\xi_{\mathrm{Hom}_S(S_S,M_S)}(\lambda'_M(m) \otimes r)) && \text{(comm. of the diagram above)}\\
&= \varepsilon_M((\zeta_M \otimes R)(\lambda'_M(m) \otimes r)) && \text{(condition } \mathbf{C3})\\
&= \varepsilon_M(\zeta_M(\lambda'_M(m)) \otimes r) && \text{(def. of } \zeta_M \otimes R)\\
&= \zeta_M(\lambda'_M(m))(r) && \text{(def. of } \tau_M)\\
&= \lambda_{F(M)}(m)(r) && \text{(condition } \mathbf{C1})\\
&= mr. && \text{(def. of } \lambda_{F(M)}(m))
\end{aligned}$$

3. For every $M_S \in \mathsf{CMod}_S$, $m \in M$ and $r \in R$, we are going to prove that $mr = mf(r)$. (Here mr is the product of m and r in the right R-module $G(M_S)$.)

   Consider the canonical S-isomorphism $\eta'_M : M \longrightarrow \mathrm{Hom}_S(S, M \otimes_S S)$ given by $\eta'_M(m)(s) = m \otimes s$. Applying the naturality of ζ to $\eta'_M(m) : S_S \longrightarrow M \otimes_S S$ we have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}_S(S,S) & \xrightarrow{\mathrm{Hom}_S(S,\eta'_M(m))} & \mathrm{Hom}_S(S, M \otimes_S S)\\
\downarrow{\scriptstyle \zeta_S} & & \downarrow{\scriptstyle \zeta_{M\otimes_S S}}\\
\mathrm{Hom}_R(R, F(S)) & \xrightarrow{\mathrm{Hom}_R(R,\eta'_M(m))} & \mathrm{Hom}_R(R, F(M \otimes_S S)).
\end{array}$$

   The commuativity of this diagram over the element $\mathsf{id}_S \in \mathrm{Hom}_S(S,S)$ gives

$$\zeta_{M\otimes_S S}(\eta'_M(m)) = \eta'_M(m) \circ \zeta_S(\mathsf{id}_S) = \eta'_M(m) \circ f \tag{3}$$

   because of the definition of f made in step 1. From condition **C4** we conclude that $\zeta_{M\otimes_S S} \circ G(\eta'_M) = \mathrm{Hom}_R(R, \xi_M) \circ \eta_{G(M)}$. Then we have

$$\begin{aligned}
mf(r) &= \mu'_M(m \otimes f(r)) && \text{(def. of } \mu'_M)\\
&= \mu'_M(\eta'_M(m)(f(r))) && \text{(def. of } \eta'_M(m))\\
&= \mu'_M(\zeta_{M\otimes_S S}(\eta'_M(m))(r)) && \text{(by (3))}\\
&= \mu'_M(\mathrm{Hom}_R(R,\xi_M)(\eta_{G(M)}(m))(r)) && \text{(condition } \mathbf{C4})\\
&= \mu'_M(\xi_M(\eta_{G(M)}(m)(r))) && \text{(def. of } \mathrm{Hom}_R(R,\xi_M))\\
&= \mu'_M(\xi_M(m \otimes r)) && \text{(def. of } \eta_M)\\
&= \mu_{G(M)(m\otimes r)} && \text{(condition } \mathbf{C2})\\
&= mr. && \text{(def. of } \mu_{G(M)})
\end{aligned}$$

4. f is a ring homomorphism. For all $r, r' \in R$ we have

$$\begin{aligned} f(r+r') &= f(r) + f(r'), && (f = \zeta_S(\mathsf{id}_S) \text{ is a right } R\text{-homomorphism}) \\ f(rr') &= f(r)r' && (f = \zeta_S(\mathsf{id}_S) \text{ is a right } R\text{-homomorphism}) \\ &= f(r)f(r'). && (\text{step 2 applied to } S_S \in \mathsf{DMod}_S) \end{aligned}$$

5. f is a left and right firm ring homomorphism. We have seen in steps 2 and 3 that the functors F and G are the restriction of scalars functors of the ring homomorphism f, hence Proposition 5.4 (and its symmetric) say that f is a left and right firm ring homomorphism.

Finally we will prove the one-to-one correspondence. Denote the set in (1) by X and the set in (3) by Y. Let (F, G) be a pair of compatible concrete functors. By $X \longrightarrow Y$ it is mapped to the ring homomorphism $f = \zeta_S(\mathsf{id}_S)$. Since F and G are the restriction of scalars funtors induced by f, we receive back the pair (F, G) with the map $Y \longrightarrow X$. Thus the composite $X \longrightarrow Y \longrightarrow X$ is the identity map.

We prove that also the composite $Y \longrightarrow X \longrightarrow Y$ is the identity map. Let $f : R \longrightarrow S$ be a firm ring homomorphism in Y. It is mapped to a pair $(f^+, f^\times)$ of compatible restrictions of $f^* : \mathsf{MOD}_S \longrightarrow \mathsf{MOD}_R$. This pair is mapped to a firm semigroup homomorphism $g := \zeta_S(\mathsf{id}_S)$. We need to show that $f = g$. Take $r \in R$. Since R is a firm ring, there exist $r_i, r'_i \in R$ such that $r = \sum_i r_i r'_i$. Then

$$\begin{aligned} f(r) &= f(\sum_i r_i r'_i) && (r = \textstyle\sum_i r_i r'_i) \\ &= \sum_i \mathsf{id}_S(f(r_i)f(r'_i)) && (f \text{ is a ring homomorphism}) \\ &= \sum_i (\mathsf{id}_S \cdot f(r_i))(f(r'_i)) && (\text{right } S\text{-structure of } \mathrm{Hom}_S(S_S, S_S)) \\ &= \sum_i (\mathsf{id}_S \cdot r_i)(f(r'_i)) && (\text{right } R\text{-structure of } \mathrm{Hom}_S(S_S, S_S)) \\ &= \sum_i \zeta_S(\mathsf{id}_S \cdot r_i)(r'_i) && (\text{def of } \zeta_S) \\ &= \sum_i (\zeta_S(\mathsf{id}_S) r_i)(r'_i) && (\zeta_S \text{ is an } R\text{-homomorphism}) \\ &= \sum_i \zeta_S(\mathsf{id}_S)(r_i r'_i) && (R\text{-structure of } \mathrm{Hom}_R(R_R, S_R)) \\ &= \zeta_S(\mathsf{id}_S)(\sum_i r_i r'_i) && (\zeta_S(\mathsf{id}_S) \text{ is a ring homomorphism}) \\ &= g(r). && (g = \zeta_S(\mathsf{id}_S),\ r = \textstyle\sum_i r_i r'_i) \end{aligned}$$

■

9. The category of firm rings

It turns out that firm homomorphisms compose.

9.1. LEMMA. *The composition of right (left) firm ring homomorphisms is a right (left) firm ring homomorphism.*

PROOF. Let $f : R \longrightarrow S$ and $g : S \longrightarrow T$ be right firm ring homomorphisms. The composition $g \circ f$ is a ring homomorphism. By Proposition 5.3,

$$M_T \in \mathsf{DMod}_T \implies g^*(M_T) \in \mathsf{DMod}_S \implies (f^* \circ g^*)(M_T) = f^*(g^*(M_T)) \in \mathsf{DMod}_R.$$

But $f^* \circ g^* = (g \circ f)^*$, so the implication $3 \Rightarrow 1$ in Proposition 5.3 yields that $g \circ f$ is a right firm homomorphism. Using a symmetric argument, we get the result on the left. ■

Clearly, if R is a firm ring, then id_R is a firm homomorphism. This allows us to give the following definition.

9.2. DEFINITION. *The category* FRng *will be the category of firm rings with firm ring homomorphisms as morphisms.*

9.3. THEOREM. *The category* Ring *of unital rings and unital ring homomorphisms is a full subcategory of the category* FRng *of firm rings and firm ring homomorphisms. Furthermore, if R is a unital ring and S is a firm ring, then* $\mathsf{FRng}(R, S) \neq \emptyset$ *implies* $S \in \mathsf{Ring}$ *and* $\mathsf{FRng}(R, S) = \mathsf{Ring}(R, S)$.

PROOF. The category is a full subcategory due to Remark 4.2 and Proposition 4.4. The last claim is Proposition 4.3. ■

9.4. PROPOSITION. *Let S be a firm ring. Then* $\mathsf{FRng}(\mathbb{Z}, S) \neq 0$ *if and only if S is a unital ring.*

PROOF. If S is unital, the map $n \mapsto n1_S$ is an unital ring homomorphism. Hence it is firm by Remark 4.2. The converse is true because of Theorem 9.3. ■

This proposition allows us clarify the role of the compatibility conditions in Theorem 8.1. For any firm ring S, taking $R = \mathbb{Z}$ we have the forgetful functors

$$U : \mathsf{DMod}_S \longrightarrow \mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}} = \mathsf{DMod}_{\mathbb{Z}} \quad \text{and} \quad V : \mathsf{CMod}_S \longrightarrow \mathsf{Ab} = \mathsf{Mod}_{\mathbb{Z}} = \mathsf{CMod}_{\mathbb{Z}}.$$

If these functors were always compatible, this would be a contradiction between Proposition 9.4 and Theorem 8.1. To see this, suppose U and V are compatible, then for any firm module $M_S \in \mathsf{DMod}_S$ and any closed module $N_S \in \mathsf{CMod}_S$, the compatibility conditions **C1** and **C2** provide bijections ζ_M and ξ_N such that $\zeta_M \circ \lambda'_M = \lambda_M$ and $\mu'_N \circ \xi_N = \mu_N$, but in this case $\lambda_M : M \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M)$ and $\mu_N : N \otimes_{\mathbb{Z}} \mathbb{Z} \longrightarrow N$ are bijections, therefore λ'_M and μ'_M should be bijections. This proves that in this case firm and closed S-modules are the same, therefore $\mathsf{End}(S) = \mathsf{End}(S) \otimes_S S = S$, so S is unital.

9.5. PROPOSITION. 0 *is a terminal object in the category* FRng.

PROOF. For any firm ring R, the constant map $0 : R \longrightarrow 0$ is firm because $0 \otimes_R R = 0 = R \otimes_R 0$, so $\mathsf{FRng}(R, 0) = \{0\}$. ■

10. Examples of firm homomorphisms of rings

We have already proved that the category Ring is a full subcategory of FRng, but we are going to see some extra properties and examples of firm homomorphisms.

10.1. EXAMPLE. Let $I \neq R$ be a two-sided ideal of R. Then the inclusion $j : I \longrightarrow R$ is not a firm ring homomorphism.

PROOF. If it were the case, $R = RI \subseteq I$, so $R = I$. ■

10.2. PROPOSITION. *Let R be a firm ring, I a two-sided ideal of R such that R/I is firm, then the projection $p : R \longrightarrow R/I$ is a firm homomorphism.*

A ring R is said to have **local units** ([2]) if for every finite subset $\{r_1, \dots, r_n\} \subseteq R$ there exists an idempotent $e \in R$ such that $er_i = r_i e = r_i$ for every $i \in \{1, \dots, n\}$. Every ring with local units is firm.

10.3. PROPOSITION. *Let R be a ring with local units and S a firm ring with a firm ring homomorphism $f : R \longrightarrow S$. Then S is a ring with local units and its local units are the images of the local units of R.*

PROOF. Let $s_1, \dots, s_n \in S$. Since f is firm, for every $i \in \{1, \dots, n\}$, there exist $s'_{j_i}, s''_{k_i} \in S, r'_{j_i}, r''_{k_i} \in R$ such that $s_i = \sum_{j_i} s'_{j_i} f(r'_{j_i}) = \sum_{k_i} f(r''_{k_i}) s''_{k_i}$. Also, there exists an idempotent $e \in R$ such that $r'_{j_i} e = r'_{j_i}$ and $er''_{k_i} = r''_{k_i}$ for every j_i, k_i. Hence

$$s_i = \sum_{j_i} s'_{j_i} f(r'_{j_i}) f(e) = s_i f(e), \quad s_i = \sum_{k_i} f(e) f(r''_{k_i}) s''_{k_i} = f(e) s_i.$$

■

10.4. PROPOSITION. *Let R be a ring with enough idempotents and $f : R \longrightarrow S$ be a firm ring homomorphism. Then S is a ring with enough idempotents.*

PROOF. As R is a ring with enough idempotents, we can find a set $\{e_i : i \in I\}$ of orthogonal idempotents of R such that $\alpha : \coprod_{i \in I} Re_i \longrightarrow R$ given by $(r_i e_i)_{i \in I} \mapsto \sum_{i \in I} r_i e_i$ and $\beta : \coprod_{i \in I} e_i R \longrightarrow R$ given by $(e_i r_i)_{i \in I} \mapsto \sum_{i \in I} e_i r_i$ are isomorphisms.

Using the fact that the tensor functor preserves coproducts and that f is a firm ring homomorphism, we get the isomorphisms

$$\coprod_{i \in I} S \otimes_R Re_i \longrightarrow S \otimes_R \coprod_{i \in I} Re_i \longrightarrow S \otimes_R R \longrightarrow S,$$

$$\coprod_{i \in I} e_i R \otimes_R S \longrightarrow \coprod_{i \in I} e_i R \otimes_R S \longrightarrow R \otimes_R S \longrightarrow S,$$

and the images of each one of the blocks $S \otimes_R Re_i$ and $e_iR \otimes_R S$ in S are $Sf(e_i)$ and $f(e_i)S$. This proves that the set $\{f(e_i) : i \in I\}$ is a set of orthogonal idempotents for S. ■

For a nonempty set I and a ring R, let $M_I^F(R)$ be the ring of matrices with indices in I and entries in R such that the number of nonzero entries is finite. If I is an infinite set, then $M_I^F(R)$ is a nonunital ring, but it has local units.

10.5. Proposition. *Let $f : R \longrightarrow S$ be a homomorphism of unital rings and I be a nonempty set. The map $\overline{f} : M_I^F(R) \longrightarrow M_I^F(S)$ which applies f to all entries of a matrix is a firm ring homomorphism.*

Proof. For every finite set of matrices $X_1, \ldots, X_n \in M_I^F(S)$ there exists a matrix $E \in M_I^F(S)$ such that $X_iE = X_i$ for all $i \in \{1, \ldots, n\}$. It has finitely many diagonal entries 1_S and zeroes elsewhere. Since $f(1_R) = 1_S$, $E = \overline{f}(E')$ for a similar matrix $E' \in M_I^F(R)$. Now it is clear that μ is surjective.

Suppose that $\sum_{k=1}^n Y_k\overline{f}(X_k) = 0$, where $X_k \in M_I^F(R)$ and $Y_k \in M_I^F(S)$. Let E be as above. Then

$$\sum_k Y_k \otimes X_k = \sum_k Y_k \otimes X_kE = \sum_k Y_k\overline{f}(X_k) \otimes E = 0 \otimes E = 0.$$

We have shown that $\ker \mu = 0$, so μ is bijective and $\overline{f}$ is a firm homomorphism. ■

11. Firm homomorphisms of semigroups

Now we start considering the case of semigroups. From now on, R and S will stand for semigroups and $f : R \longrightarrow S$ will be a semigroup homomorphism, unless otherwise stated. To shorten the notation, we often write $\dot{r}$ instead of $f(r)$. Sometimes we will write $\cdot : R \longrightarrow S$ instead of $f : R \longrightarrow S$.

Similarly to the case of rings, we have the (concrete) restriction of scalars functor

$$f^* : \mathsf{Act}_S \longrightarrow \mathsf{Act}_R, \ \ A_S \mapsto A_R,$$

where the R-action on the set A is defined by $ar := af(s)$. Dually we also have a restriction of scalars functor ${}^*f : {}_S\mathsf{Act} \longrightarrow {}_R\mathsf{Act}$.

11.1. Definition. *We call a semigroup homomorphism $f : R \longrightarrow S$* **right firm** *if the act $S_R = f^*(S_S)$ is firm. Left firm homomorphisms are defined dually. A semigroup homomorphism is called* **firm** *if it is both right firm and left firm.*

We will give a list of examples of (right) firm semigroup homomorphisms in our final section. Immediately from the definition we have the following result.

11.2. LEMMA. *A semigroup homomorphism $f : R \longrightarrow S$, $r \mapsto \dot{r}$, is right firm if and only if $S = Sf(R)$ and, for all $s, t \in S$ and $r, p \in R$,*

$$s\dot{r} = t\dot{p} \Longrightarrow s \otimes r = t \otimes p \text{ in } S \otimes_R R.$$

The equality $S = Sf(R)$ means precisely that S_R is a unitary act.

Our aim is to prove the semigroup theoretic analogue of Theorem 8.1. For that we need the compatibility conditions **C1**–**C4**. We define them precisely as for rings in Section 6, just replacing DMod_R by FAct_R and CMod_R by CAct_R. The proofs of Section 6 will go through for semigroups without any change.

To prove Proposition 12.4 (the analogue of Proposition 5.4), we will need the notion of a character act and some lemmas about character acts. Let $\mathbf{2} = \{0, 1\}$ be a two element set. For a right S-act A_S we can consider the set $\mathbf{2}^A = \{g \mid g : A \longrightarrow \mathbf{2} \text{ is a map}\}$ as a left S-act with the action

$$(sg)(a) := g(as), \tag{4}$$

$s \in S$, $a \in A$. This act is called a **character act** of A (cf. [8, Definition 3.12.3]). This construction gives a contravariant functor $\mathsf{Act}_S \longrightarrow {}_S\mathsf{Act}$, which takes a morphism $h : A_S \longrightarrow B_S$ to a morphism $- \circ h : \mathbf{2}^B \longrightarrow \mathbf{2}^A$. We call this functor a **character functor**.

11.3. LEMMA. *For every semigroup S, the character functor $\mathsf{Act}_S \longrightarrow {}_S\mathsf{Act}$ reflects isomorphisms.*

PROOF. Let $h : A_S \longrightarrow B_S$ be a morphism in Act_S. First we prove that if $- \circ h$ is surjective, then h is injective. (In fact, this part of the proof is essentially the same as the proof of [8, Proposition 3.12.4(2)].) Suppose that $h(a) = h(a')$, where $a, a' \in A$, but $a \neq a'$. Choose a map $g : A \longrightarrow \mathbf{2}$ such that $g(a) \neq g(a')$. Then there exists a map $k : B \longrightarrow \mathbf{2}$ such that $k \circ h = g$. Therefore $g(a) = k(h(a)) = k(h(a')) = g(a')$, a contradiction.

We will also prove that if $- \circ h$ is injective, then h is surjective. Suppose that h is not surjective, i.e. there exists an element $b_0 \in B \setminus h(A)$. Define a map $k_1 : B \longrightarrow \mathbf{2}$ by $k_1(b) = 1$ for every $b \in B$, and a map $k_2 : B \longrightarrow \mathbf{2}$ by

$$k_2(b) = \begin{cases} 1, & \text{if } b \in h(A), \\ 0, & \text{if } b \notin h(A). \end{cases}$$

Then $k_1 \neq k_2$, but $k_1 \circ h = k_2 \circ h$. Thus $- \circ h$ is not injective, a contradiction. ∎

11.4. LEMMA. *For every firm semigroup S, $\mathbf{2}^S \in {}_S\mathsf{CAct}$.*

PROOF. To avoid confusion, in this proof we will write all maps to the left of their arguments. We need to prove that the map $\lambda_{\mathbf{2}^S} : \mathbf{2}^S \longrightarrow \mathrm{Hom}_S({}_SS, \mathbf{2}^S)$ defined by

$$\lambda_{\mathbf{2}^S}(g)(s)(s') := (sg)(s') = g(s's),$$

$g \in \mathbf{2}^S$, $s, s' \in S$, is bijective. Consider the composite

$$\mathbf{2}^S \xrightarrow{-\circ\mu_S} \mathbf{2}^{S\otimes_S S} = \mathsf{Set}(S \otimes_S S, \mathbf{2}) \xrightarrow{\sigma} \mathrm{Hom}_S({}_SS, \mathsf{Set}(S_S, \mathbf{2})) = \mathrm{Hom}_S({}_SS, \mathbf{2}^S),$$

where $\mathsf{Set}(A, B)$ denotes the set of all mappings $A \longrightarrow B$ and σ is defined by

$$\sigma(h)(s)(s') := h(s' \otimes s),$$

$h \in \mathsf{Set}(S \otimes_S S, \mathbf{2})$, $s, s' \in S$. Since

$$(\sigma \circ (- \circ \mu_S))(g)(s)(s') = \sigma(g \circ \mu_S)(s)(s') = (g \circ \mu_S)(s' \otimes s) = g(s's) = \lambda_{\mathbf{2}^S}(g)(s)(s'),$$

we see that $\sigma \circ (- \circ \mu_S) = \lambda_{\mathbf{2}^S}$. Now $- \circ \mu_S$ is an isomorphism, because $\mu_S : S \otimes_S S \longrightarrow S$ is an isomorphism, and σ is bijective because of the tensor-hom adjunction

$$S \otimes_S - \dashv \mathsf{Set}(S_S, -) : \mathsf{Set} \longrightarrow {}_S\mathsf{Act}.$$

Consequently, $\lambda_{\mathbf{2}^S}$ is bijective. ∎

Recall that a category is called **balanced** if all bimorphisms are isomorphisms (see [1, Definition 7.49]). A semigroup is called **factorisable** if $S = S^2$.

11.5. Lemma. *If S is a factorisable semigroup, then the category FAct_S is balanced.*

Proof. Let S be a factorisable semigroup. We know that $S \otimes S$ is a firm semigroup by [10, Theorem 2.6]. According to [10, Proposition 4.9], the categories FAct_S and $\mathsf{FAct}_{S \otimes S}$ are equivalent. Thus it suffices to prove that $\mathsf{FAct}_{S \otimes S}$ is balanced.

Take a bimorphism h in $\mathsf{FAct}_{S \otimes S}$. By [11, Theorem 2.10], h is an extremal monomorphism. Since it is also an epimorphism and $h = \mathsf{id} \circ h$, it must be an isomorphism, proving that $\mathsf{FAct}_{S \otimes S}$ is balanced.

Recalling that epimorphisms in the category FAct_S are precisely surjective morphisms ([11, Corollary 1.4]), we can say that surjective monomorphisms must be bijective. ∎

11.6. Lemma. [9, Proposition 3.9] *Let S be a firm semigroup and M_S a right S-act. Then*

1. *$M_S \in \mathsf{FAct}_S$ if and only if the right S-act homomorphism*

$$\varepsilon'_M : \mathrm{Hom}_S(S_S, M_S) \otimes_S S \longrightarrow M, \quad \alpha \otimes s \mapsto \alpha(s)$$

is bijective;

2. *$M_S \in \mathsf{CAct}_S$ if and only if the right S-act homomorphism*

$$\eta'_M : M \longrightarrow \mathrm{Hom}_S(S_S, M \otimes_S S), \quad m \mapsto (s \mapsto m \otimes s)$$

is bijective.

12. Functors between act categories

In this section we study functors between different categories of acts. These results will be parallel to those in Section 5.

If ${}_SX_R$ is an (S,R)-biact and M_R is a right R-act, then the hom-set $\mathrm{Hom}_R(X_R, M_R)$ is a right S-act with the action

$$(gs)(x) = g(sx), \tag{5}$$

$g \in \mathrm{Hom}_R(X_R, M_R)$, $s \in S$, $x \in X$. If $f : R \longrightarrow S$ is a semigroup homomorphism, then we have a natural (S,R)-biact ${}_SS_R$.

12.1. Proposition. *Let R and S be firm semigroups and $\dot{}: R \longrightarrow S$ a semigroup homomorphism. Then*

1. $\mathrm{Hom}_R(S_R, -)$ *is a functor from* Act_R *to* CAct_S*;*
2. $\mathrm{Hom}_R({}_RS, -)$ *is a functor from* ${}_R\mathsf{Act}$ *to* ${}_S\mathsf{CAct}$*;*
3. $- \otimes_R S$ *is a functor from* Act_R *to* FAct_S*;*
4. $S \otimes_R -$ *is a functor from* ${}_R\mathsf{Act}$ *to* ${}_S\mathsf{FAct}$*.*

Proof. The conditions 2 and 4 are the symmetric conditions to 1 and 3, thus we will prove only 1 and 3.

Since S is firm, the map $\mu_S : S \otimes_S S \longrightarrow S$, $s \otimes s' \mapsto ss'$, is an isomorphism of right S-acts. Applying the hom-functor $\mathrm{Hom}_R(-, M_R)$ we see that the map

$$- \circ \mu_S : \mathrm{Hom}_R(S_R, M_R) \longrightarrow \mathrm{Hom}_R(S \otimes_S S_R, M_R)$$

is an isomorphism for every right R-act M_R. Due to the tensor-hom adjunction we have the bijection

$$\sigma : \mathrm{Hom}_R(S \otimes_S S_R, M_R) \longrightarrow \mathrm{Hom}_S(S_S, \mathrm{Hom}_R(S_R, M_R)),\ h \mapsto (s \mapsto (s' \mapsto h(s \otimes s'))).$$

The composite bijection $\sigma \circ (- \circ \mu_S)$ is $\lambda_{\mathrm{Hom}_R(S_R,M_R)}$, because

$$\begin{aligned}
(\sigma \circ (- \circ \mu_S))(g)(s)(s') &= \sigma(g \circ \mu_S)(s)(s') && \text{(def. of } - \circ \mu_S) \\
&= (g \circ \mu_S)(s \otimes s') && \text{(def. of } \sigma) \\
&= g(ss') && \text{(def. of } \mu) \\
&= (gs)(s') && \text{(by (5))} \\
&= \lambda_{\mathrm{Hom}_R(S_R,M_R)}(g)(s)(s') && \text{(def. of } \lambda_{\mathrm{Hom}_R(S_R,M_R)})
\end{aligned}$$

for all $s, s' \in S$ and $g \in \mathrm{Hom}_R(S_R, M_R)$. Hence $\mathrm{Hom}_R(S_R, M_R) \in \mathsf{CAct}_S$. We also have $N \otimes_R S \in \mathsf{FAct}_S$ for every right R-act N_R, because the composite bijection

$$(N \otimes_R S) \otimes_S S \longrightarrow N \otimes_R (S \otimes_S S) \longrightarrow N \otimes_R S,\ (n \otimes s) \otimes s' \mapsto n \otimes (s \otimes s') \mapsto n \otimes ss'$$

is precisely $\mu_{N \otimes_R S}$. The rest of the proof is straightforward. ∎

12.2. Corollary. *If R is a firm semigroup and $M_R \in \mathsf{Act}_R$, then* $\mathrm{Hom}_R(R_R, M_R) \in \mathsf{CAct}_R$ *and* $M \otimes_R R \in \mathsf{FAct}_R$.

Proof. We apply Proposition 12.1 for the homomorphism $\mathsf{id}_R : R \longrightarrow R$. ∎

Let ${}_SA_R$ be an (S, R)-biact and ${}_SM$ be a left S-act. We will write the elements of the hom-set $\mathrm{Hom}_S({}_SA_R, {}_SM)$ to the right of their arguments and equip this set with the left R-action

$$(a)(r\alpha) = (ar)\alpha, \tag{6}$$

$\alpha \in \mathrm{Hom}_S({}_SA_R, {}_SM)$, $a \in A$, $r \in R$. The following result is an analogue of Proposition 5.3.

12.3. Proposition. *Let R and S be firm semigroups and $f : R \longrightarrow S, r \mapsto \dot{r}$, a right firm semigroup homomorphism. Then*

1. *the map $\mu_S : S \otimes_R R \longrightarrow S$ is not only a right R-isomorphism, but also a left S-isomorphism;*
2. $\mathrm{Hom}_S({}_SS_R, -)$ *is a functor from* ${}_S\mathsf{Act}$ *to* ${}_R\mathsf{CAct}$*;*
3. $- \otimes_S S_R$ *is a functor from* Act_S *to* FAct_R*;*
4. *the functors f^*, $- \otimes_S S_R : \mathsf{FAct}_S \longrightarrow \mathsf{FAct}_R$ are naturally isomorphic.*

Proof.

1. The map μ_S is bijective because S_R is in FAct_R, therefore we only need to prove that it is a left S-homomorphism. If $s, t \in S$ and $r \in R$, then we have

$$\begin{aligned}
(t(s \otimes r))\mu_S &= (ts \otimes r)\mu_S && \text{(left } S\text{-action of } S \otimes_R R) \\
&= (ts)\dot{r} && \text{(def. of } \mu_S) \\
&= t(s\dot{r}) && \text{(associativity of multiplication in } S) \\
&= t(s \otimes r)\mu_S. && \text{(def. of } \mu_S)
\end{aligned}$$

2. Let ${}_SM$ be any left S-act. Then

$$\begin{aligned}
\mathsf{Hom}_S({}_SS_R, {}_SM) &\cong \mathsf{Hom}_S({}_SS_R \otimes R_R, {}_SM) && ({}_SS_R \text{ is right firm}) \\
&\cong \mathsf{Hom}_R({}_RR_R, \mathsf{Hom}_S({}_SS_R, {}_SM)) && \text{(tensor-hom adjunction)}
\end{aligned}$$

as left R-acts. The last act is closed by Proposition 12.1(2), hence also the left R-act $\mathrm{Hom}_S({}_SS_R, {}_SM)$ is in ${}_R\mathsf{CAct}$.

3. Let M_S be in Act_S. Since $\mu_S : S \otimes_R R \longrightarrow S$ is a left S-isomorphism, we have the composite bijection

$$(M \otimes_S S) \otimes_R R \longrightarrow M \otimes_S (S \otimes_R R) \longrightarrow M \otimes_S S,$$
$$(m \otimes s) \otimes r \mapsto m \otimes (s \otimes r) \mapsto m \otimes sr = (m \otimes s)r$$

mapping $(m \otimes s) \otimes r$ to $m \otimes sr = (m \otimes s)r$, which is precisely $\mu_{M \otimes_S S}$. Thus $M \otimes_S S_R \in \mathsf{FAct}_R$.

4. If $M_S \in \mathsf{FAct}_S$, then the mapping $\mu'_M : M \otimes_S S \longrightarrow M$, $m \otimes s \mapsto ms$, is bijective. It is an isomorphism of right R-acts, because
$$\mu'_M((m \otimes s)r) = \mu'_M(m \otimes s\dot{r}) = m(s\dot{r}) = (ms)\dot{r} = \mu'_M(m \otimes s)r$$
for every $m \in M$, $s \in S$ and $r \in R$. It is easy to check that μ' is natural in M.

■

In the following proposition, using the restriction of action we can consider an S-act A_S also as an R-act A_R. We will make a convention to write μ_A for the mapping $\mu_{A_R} : A \otimes_R R \longrightarrow A$ and μ'_A for the mapping $\mu_{A_S} : A \otimes_S S \longrightarrow A_S$. In particular, $\mu_S : S \otimes_R R \longrightarrow S$ and $\mu'_S : S \otimes_S S \longrightarrow S$. A similar convention is used for the mappings λ_A and λ'_A. The next proposition will give two necessary and sufficient conditions for a semigroup homomorphism to be right firm in terms of restricting certain restriction of scalars functors.

12.4. PROPOSITION. *Let R and S be firm semigroups and $f : R \longrightarrow S$, $r \mapsto \dot{r}$, a semigroup homomorphism. The following conditions are equivalent.*

1. *$f : R \longrightarrow S$ is a right firm semigroup homomorphism.*
2. *For every ${}_SM \in {}_S\mathsf{CAct}$, ${}^*f({}_SM) \in {}_R\mathsf{CAct}$, so the restriction of scalars is a concrete functor from ${}_S\mathsf{CAct}$ to ${}_R\mathsf{CAct}$.*
3. *For every $M_S \in \mathsf{FAct}_S$, $f^*(M_S) \in \mathsf{FAct}_R$, so the restriction of scalars is a concrete functor from FAct_S to FAct_R.*

PROOF. (1 ⇒ 2). Let ${}_SM$ be in ${}_S\mathsf{CAct}$, that is, $\lambda'_M : {}_SM \longrightarrow \mathrm{Hom}_S({}_SS, {}_SM)$ is a left S-isomorphism. Applying the restriction of scalars functor ${}^*f : {}_S\mathsf{Act} \longrightarrow {}_R\mathsf{Act}$ we see that $\lambda'_M : {}_RM \longrightarrow \mathrm{Hom}_S({}_SS_R, {}_SM)$ is a left R-isomorphism, and so is $\mathrm{Hom}_R({}_RR, \lambda'_M)$. Moreover, the diagram
$$\begin{array}{ccc}
{}_RM & \xrightarrow{\quad\lambda_M\quad} & \mathrm{Hom}_R({}_RR, {}_RM) \\
{\scriptstyle \lambda'_M}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_R({}_RR, \lambda'_M)} \\
\mathrm{Hom}_S({}_SS_R, {}_SM) & \xrightarrow{\lambda_{\mathrm{Hom}_S({}_SS_R, {}_SM)}} & \mathrm{Hom}_R({}_RR, \mathrm{Hom}_S({}_SS_R, {}_SM))
\end{array}$$
in ${}_R\mathsf{Act}$ commutes, because $\lambda : 1_{{}_R\mathsf{Act}} \Rightarrow \mathrm{Hom}_R({}_RR, -)$ is a natural transformation. The lower arrow is bijective because of Proposition 12.3(2) (since $\mathrm{Hom}_S({}_SS_R, {}_SM) \in {}_R\mathsf{CAct}$), so the upper arrow λ_M must also be bijective and therefore ${}_RM$ is in ${}_R\mathsf{CAct}$.

(2 ⇒ 1). We consider the character act $\mathbf{2}^S$ as a left S-act, which is closed by Lemma 11.4. We need to prove that $\mu_S : S \otimes_R R \longrightarrow S$ is bijective. By Lemma 11.3 it suffices to show that the mapping

$$- \circ \mu_S : \mathbf{2}^S \longrightarrow \mathbf{2}^{S \otimes_R R}$$

is bijective. Note that this mapping is the composite

$${}_R\left(\mathbf{2}^S\right) \xrightarrow{\lambda_{\mathbf{2}^S}} \mathrm{Hom}_R({}_RR, \mathbf{2}^S) = \mathrm{Hom}_R({}_RR, \mathsf{Set}(S_R, \mathbf{2})) \xrightarrow{\tau} \mathsf{Set}(S \otimes_R R, \mathbf{2}) = \mathbf{2}^{S \otimes_R R},$$

where τ is defined by

$$\tau(h)(s \otimes r) = h(r)(s)$$

for all $h : {}_RR \longrightarrow \mathsf{Set}(S_R, \mathbf{2})$, $s \in S$, $r \in R$. Indeed, for every $g \in \mathbf{2}^S$, $s \in S$ and $r \in R$, we compute

$$\begin{aligned}
(\tau \circ \lambda_{\mathbf{2}^S})(g)(s \otimes r) &= \tau(\lambda_{\mathbf{2}^S}(g))(s \otimes r) && \text{(composition of maps)}\\
&= \lambda_{\mathbf{2}^S}(g)(r)(s) && \text{(def. of } \tau)\\
&= (rg)(s) && \text{(def. of } \lambda_{\mathbf{2}^S})\\
&= (\dot{r}g)(s) && \text{(left } R\text{-action of } \mathbf{2}^S)\\
&= g(s\dot{r}) && \text{(by (4))}\\
&= g(sr) && \text{(right } R\text{-action of } S_R)\\
&= (g \circ \mu_S)(s \otimes r), && \text{(def. of } \mu_S)
\end{aligned}$$

so $\tau \circ \lambda_{\mathbf{2}^S} = - \circ \mu_S$. By the assumption, $\mathbf{2}^S \in {}_R\mathsf{CAct}$, and hence $\lambda_{\mathbf{2}^S}$ is bijective. The mapping τ is bijective because of the adjunction

$$S \otimes_R - \dashv \mathsf{Set}(S_R, -) : \mathsf{Set} \longrightarrow {}_R\mathsf{Act}.$$

It follows that $- \circ \mu_S$ is bijective, as needed.

(1 ⇒ 3). Let M_S be in FAct_S. Since $\mu : - \otimes_R R \Rightarrow 1_{\mathsf{Act}_R}$ is a natural transformation, the square

$$\begin{array}{ccc}
M \otimes_S S \otimes_R R & \xrightarrow{\mu_{M \otimes S}} & M \otimes_S S \\
{\scriptstyle \mu'_M \otimes R}\downarrow & & \downarrow{\scriptstyle \mu'_M} \\
M \otimes_R R & \xrightarrow{\mu_M} & M
\end{array}$$

commutes. The vertical arrows are bijective because μ'_M is a bijection. The upper arrow is bijective because of Proposition 12.3(3) ($M \otimes_S S_R \in \mathsf{FAct}_R$), so the lower arrow μ_M is also a bijection and therefore M_R is in FAct_R.

(3 ⇒ 1). If we apply condition 3 to the semigroup S that is a firm right S-act, we get that S_R is in FAct_R and therefore the homomophism f is right firm. ∎

As an application of this proposition we point out the following. Suppose that J is a firm ideal of a firm semigroup S such that the inclusion mapping $j : J \longrightarrow S$ is right firm. Proposition 12.4 asserts that the restriction of scalars functor j^* preserves firmness. By [18, Theorem 3.1], the categories FAct_S and FAct_J are equivalent. In other words: the semigroups S and J are Morita equivalent.

13. Compatibility conditions for semigroups

If $f : R \longrightarrow S$ is a firm semigroup homomoprphism between firm semigroups, then by Proposition 12.4 and its dual the restriction of scalars funtor $f^* : \mathsf{Act}_S \longrightarrow \mathsf{Act}_R$ restricts to functors

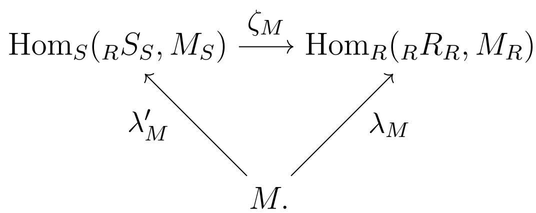
$$f^+ : \mathsf{FAct}_S \longrightarrow \mathsf{FAct}_R \quad \text{and} \quad f^\times : \mathsf{CAct}_S \longrightarrow \mathsf{CAct}_R.$$

In this section we will show that $(f^+, f^\times)$ is a pair of compatible concrete functors.

13.1. Proposition. [Compatibility **C1**] *Let R and S be firm semigroups. Suppose that $f : R \longrightarrow S, r \mapsto \dot{r}$, is a left firm semigroup homomorphism and let M_S be in Act_S. Then, in the category CAct_R, we have a right R-homomorphism*

$$\zeta_M : \mathrm{Hom}_S({}_RS_S, M_S) \longrightarrow \mathrm{Hom}_R({}_RR_R, M_R),$$

natural in M, such that the following diagram is commutative:



Furthermore, if we apply this construction to $M_S = S_S$, we get

$$\zeta_S : \mathrm{Hom}_S(S_S, S_S) \longrightarrow \mathrm{Hom}_R(R_R, S_R)$$

and $\zeta_S(\mathsf{id}_S)$ is precisely the semigroup homomorphism f.

Proof. Using the dual of Proposition 12.3(2) we know that the acts $\mathrm{Hom}_S({}_RS_S, M_S)$ and $\mathrm{Hom}_R({}_RR_R, M_R)$ are in CAct_R.

Let $\alpha \in \mathrm{Hom}_S({}_RS_S, M_S)$. We define a map $\zeta_M(\alpha) : R \longrightarrow M$ by

$$\zeta_M(\alpha)(r) = \alpha(\dot{r}). \tag{7}$$

With this definition, the last claim of the proposition is trivial because $\zeta_S(\mathsf{id}_S)(r) = \mathsf{id}_S(\dot{r}) = f(r)$ for all $r \in R$. Also, the triangle is commutative because, for every $m \in M$ and $r \in R$,

$$(\zeta_M \circ \lambda'_M)(m)(r) = \zeta_M(\lambda'_M(m))(r) = \lambda'_M(m)(\dot{r}) = m\dot{r} = mr = \lambda_M(m)(r).$$

We still have to check the following properties of ζ_M.

1. $\zeta_M(\alpha)$ is a right R-homomorphism. For every $u, r \in R$,

$$\zeta_M(\alpha)(ru) = \alpha(\dot{\overline{ru}}) = \alpha(\dot{r}\dot{u}) = \alpha(\dot{r})\dot{u} = \zeta_M(\alpha)(r)u.$$

2. ζ_M is a right R-homomorphism. For every $\alpha \in \mathrm{Hom}_S({}_RS_S, M_S)$, $r, u \in R$ we have

$$\begin{aligned}
\zeta_M(\alpha r)(u) &= (\alpha r)(\dot{u}) && (\text{def. of } \zeta_M)\\
&= \alpha(\dot{r}\dot{u}) && (R\text{-action of } \mathrm{Hom}_S({}_RS_S, M_S))\\
&= \alpha(\dot{\overline{ru}}) && (\dot{}\ \text{preserves multiplication})\\
&= \zeta_M(\alpha)(ru) && (\text{def. of } \zeta_M)\\
&= (\zeta_M(\alpha)r)(u). && (\text{right } R\text{-action of } \mathrm{Hom}_R({}_RR_R, M_R))
\end{aligned}$$

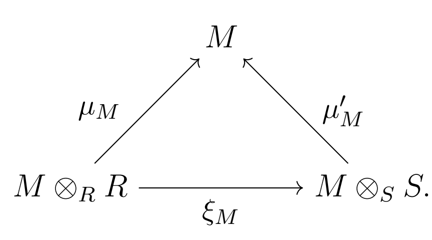
   This proves that $\zeta_M(\alpha r) = \zeta_M(\alpha)r$ for all $r \in R$ and all $\alpha \in \mathrm{Hom}_S({}_RS_S, M_S)$.

3. ζ_M is natural in M. Let $k : M_S \longrightarrow N_S$ be a morphism in Act_S. For every $\alpha \in \mathrm{Hom}_S({}_RS_S, M_S)$ and $r \in R$ we have

$$\begin{aligned}
\mathrm{Hom}_R({}_RR_R, k)(\zeta_M(\alpha))(r) &= (k \circ \zeta_M(\alpha))(r) && (\text{def. of } \mathrm{Hom}_R({}_RR_R, -))\\
&= k(\zeta_M(\alpha)(r)) && (\text{composition of maps})\\
&= k(\alpha(\dot{r})) && (\text{def. of } \zeta_M(\alpha))\\
&= \zeta_N(k\alpha)(r) && (\text{def. of } \zeta_N(k\alpha))\\
&= \zeta_N(\mathrm{Hom}_S({}_RS_S, k)(\alpha))(r). && (\text{def. of } \mathrm{Hom}_S({}_RS_S, -))
\end{aligned}$$

∎

13.2. Proposition. [Compatibility **C2**] *Let* $\dot{}\ : R \longrightarrow S$ *be a right firm semigroup homomorphism between firm semigroups and let* $M_S \in \mathsf{Act}_S$. *Then we have a right* R-*isomorphism* $\xi_M : M \otimes_R R \longrightarrow M \otimes_S S$ *natural in* M *such that the following diagram is commutative:*



Proof. By Corollary 12.2 and Proposition 12.3(3), respectively, we know that $M \otimes_R R$ and $M \otimes_S S$ are in FAct_R. We define $\xi_M : M \otimes_R R \longrightarrow M \otimes_S S$ by

$$\xi_M(m \otimes r) = m \otimes \dot{r}.$$

To see that ξ_M is well defined we note that the mapping

$$\overline{\xi_M} : M \times R \longrightarrow M \otimes_S S, \quad (m, r) \mapsto m \otimes \dot{r}$$

is R-balanced, because

$$\overline{\xi_M}(m, ur) = m \otimes \overset{\cdot}{\overline{ur}} = m \otimes \dot{u}\dot{r} = m\dot{u} \otimes \dot{r} = mu \otimes \dot{r} = \overline{\xi_M}(mu, r)$$

for all $m \in M$ and $u, r \in R$. The triangle is commutative because, for every $m \in M$ and $r \in R$,

$$(\mu'_M \circ \xi_M)(m \otimes r) = \mu'_M(m \otimes \dot{r}) = m\dot{r} = mr = \mu_M(m \otimes r).$$

We still have to check the following properties of ξ_M.

1. ξ_M is a right R-homomorphism. For any $m \in M$ and $r, u \in R$ we have

$$\begin{aligned}
\xi_M((m \otimes r)u) &= \xi_M(m \otimes ru) && \text{(right } R\text{-action of } M \otimes_R R) \\
&= m \otimes \overset{\cdot}{\overline{ru}} && \text{(def. of } \xi_M) \\
&= m \otimes \dot{r}\dot{u} && (\dot{}\ \text{preserves multiplication}) \\
&= (m \otimes \dot{r})\dot{u} && \text{(right } S\text{-action of } M \otimes_S S) \\
&= (m \otimes \dot{r})u && \text{(right } R\text{-action of } M \otimes_S S) \\
&= \xi_M(m \otimes r)u. && \text{(def. of } \xi_M)
\end{aligned}$$

2. ξ is natural in M. If $k : M_S \longrightarrow N_S$ is a morphism in Act_S and $m \otimes r \in M \otimes_R R$, then

$$\begin{aligned}
((k \otimes id_S) \circ \xi_M)(m \otimes r) &= (k \otimes id_S)(m \otimes \dot{r}) = k(m) \otimes \dot{r} = \xi_N(k(m) \otimes r) \\
&= (\xi_N \circ (k \otimes id_R))(m \otimes r).
\end{aligned}$$

■

Alternatively, Proposition 13.2 could be formulated as follows: if R and S are firm semigroups and $f : R \longrightarrow S$ is a right firm homomorphism, then the functor $- \otimes_S S_R : \mathsf{Act}_S \longrightarrow \mathsf{FAct}_R$ is naturally isomorphic to the composite functor

$$\mathsf{Act}_S \xrightarrow{f^*} \mathsf{Act}_R \xrightarrow{-\otimes_R R} \mathsf{FAct}_R.$$

13.3. Proposition. [Compatibility **C3**] *Let* $\dot{}\ : R \longrightarrow S$ *be a firm semigroup homomorphism between firm semigroups. Then, for any* $M_S \in \mathsf{FAct}_S$*, the following diagram of* R*-homomorphisms is commutative:*

$$\begin{array}{ccc}
\mathrm{Hom}_S(S_S, M_S) \otimes_R R & \xrightarrow{\xi_{\mathrm{Hom}_S(S_S, M_S)}} & \mathrm{Hom}_S(S_S, M_S) \otimes_S S \\
\Big\downarrow{\scriptstyle \zeta_M \otimes R} & & \Big\downarrow{\scriptstyle \varepsilon'_M} \\
\mathrm{Hom}_R(R_R, M_R) \otimes_R R & \xrightarrow{\varepsilon_M} & M
\end{array}.$$

Hence $\varepsilon f^+ \circ T\zeta = f^+\varepsilon' \circ \xi H'$.

PROOF. Note that ε_M is defined by $\varepsilon_M(\beta \otimes r) = \beta(r)$, and ε'_M is defined in Lemma 11.6. Let $\alpha \in \mathrm{Hom}_S(S_S, M_S)$ and $r \in R$. Then we have

$$\begin{aligned}
\varepsilon'_M(\xi_{\mathrm{Hom}_S(S_S,M_S)}(\alpha \otimes r)) &= \varepsilon'_M(\alpha \otimes \dot{r}) && (\text{def. of } \xi_{\mathrm{Hom}_S(S_S,M_S)})\\
&= \alpha(\dot{r}) && (\text{def. of } \varepsilon'_M)\\
&= \zeta_M(\alpha)(r) && (\text{def. of } \zeta_M)\\
&= \varepsilon_M(\zeta_M(\alpha) \otimes r) && (\text{def. of } \varepsilon_M)\\
&= \varepsilon_M((\zeta_M \otimes R)(\alpha \otimes r)). && (\text{def. of } - \otimes R \text{ over morphisms})
\end{aligned}$$

■

13.4. PROPOSITION. *The natural transformations $\xi : Tf^\times \Rightarrow f^+T'$ and $\zeta : f^\times H' \Rightarrow Hf^+$ defined in the previous propositions are natural isomorphisms.*

PROOF. We will prove that ξ_M is bijective for every M_S. For all $m \in M$ and $s \in S$ we can write $s = s'\dot{r}$ for some $s' \in S$ and $r \in R$, because S_R is firm and in particular $SR = S$. So, in $M \otimes_S S$, we have

$$m \otimes s = m \otimes s'\dot{r} = ms' \otimes \dot{r} = \xi_M (ms' \otimes r).$$

This proves that ξ_M is surjective.

Suppose that $\xi_M(m_1 \otimes r_1) = \xi_M(m_2 \otimes r_2)$, where $m_1, m_2 \in M$ and $r_1, r_2 \in R$. Then $m_1 \otimes \dot{r}_1 = m_2 \otimes \dot{r}_2$ in $M \otimes_S S$. Applying the mapping μ'_M we obtain $m_1\dot{r}_1 = m_2\dot{r}_2$ in M_S. Then, for every $r \in R$,

$$(m_1 \otimes r_1)r = m_1 \otimes r_1 r = m_1\dot{r}_1 \otimes r = m_2\dot{r}_2 \otimes r = m_2 \otimes r_2 r = (m_2 \otimes r_2)r$$

in $M \otimes_R R$. By [11, Theorem 2.10], ξ_M is a monomorphism in FAct_R. As a surjective monomorphism, it must be an isomorphism due to Lemma 11.5.

Finally, ζ is a natural isomorphism because of the semigroup theoretic analogue of Lemma 6.4. ■

14. Main theorem for semigroups

We define pairs of compatible concrete functors similarly to the case of rings and we prove that they are in one-to-one correspondence with firm homomorphisms.

14.1. DEFINITION. *Let R and S be firm semigroups. A* **pair of compatible concrete functors** *is a pair of concrete functors*

$$F : \mathsf{FAct}_S \longrightarrow \mathsf{FAct}_R \quad \textit{and} \quad G : \mathsf{CAct}_S \longrightarrow \mathsf{CAct}_R$$

*which satisfies compatibility conditions **C1–C4**.*

Recall that if M_S is a right S-act, then $\mathrm{Hom}_S(S_S, M_S)$ is a right S-act with the action

$$(hs)(t) := h(st), \tag{8}$$

$h \in \mathrm{Hom}_S(S_S, M_S)$, $s, t \in S$. Also $\mathrm{Hom}_R(R_R, F(M)_R)$ is a right R-act with a similarly defined action. The main theorem for semigroups is the following.

14.2. THEOREM. *Given firm semigroups R and S, there is a bijection between the following sets:*

1. *The pairs of compatible concrete functors*

$$F : \mathsf{FAct}_S \longrightarrow \mathsf{FAct}_R \quad \text{and} \quad G : \mathsf{CAct}_S \longrightarrow \mathsf{CAct}_R.$$

2. *The pairs of compatible concrete functors*

$$F : {}_S\mathsf{FAct} \longrightarrow {}_R\mathsf{FAct} \quad \text{and} \quad G : {}_S\mathsf{CAct} \longrightarrow {}_R\mathsf{CAct}.$$

3. *The firm semigroup homomorphisms $R \longrightarrow S$.*

The bijection is given by the restriction of scalars.

PROOF. The condition (3) is symmetric, therefore we only need to see the bijection between (1) and (3). Let $f : R \longrightarrow S$ be a firm semigroup homomorphism. By Proposition 12.4, we have a concrete functor $f^+ : \mathsf{FAct}_S \longrightarrow \mathsf{FAct}_R$ and by its dual we have a concrete functor $f^\times : \mathsf{CAct}_S \longrightarrow \mathsf{CAct}_R$. The compatibility conditions have been verified in our previous propositions.

Conversely, suppose we have a pair of compatible concrete functors $F : \mathsf{FAct}_S \longrightarrow \mathsf{FAct}_R$ and $G : \mathsf{CAct}_S \longrightarrow \mathsf{CAct}_R$. We will give the proof in several steps.

1. If we apply the compatibility condition **C3** to the firm act S_S, $\mathsf{id}_S \in \mathrm{Hom}_S(S_S, S_S)$ and $r \in R$, then we obtain that

$$\tau'_S(\xi_{\mathrm{Hom}_S(S_S,S_S)}(\mathsf{id}_S \otimes r)) = \tau_S((\zeta_S \otimes R)(\mathsf{id}_S \otimes r)) = \zeta_S(\mathsf{id}_S)(r).$$

(Observe that $\mathsf{id}_S = G(\mathsf{id}_S)$, $\tau'_S = F(\tau'_S)$ and $S_R = F(S)_R$, because F and G are concrete functors.) This common value will be called $f(r)$ and the mapping $f := \zeta_S(\mathsf{id}_S) : R \longrightarrow S$ (which is actually a right R-homomorphism) will be the candidate to be the firm semigroup homomorphism.

2. For every $M_S \in \mathsf{FAct}_S$, $m \in M$ and $r \in R$, we are going to prove that $mr = mf(r)$.

Consider the S-homomorphism $\lambda'_M(m) : S_S \longrightarrow M_S$ given by $\lambda'_M(m)(s) = ms$ and the commutative diagram induced by the naturality of ε' and ξ:

$$\begin{array}{ccc}
G(\mathrm{Hom}_S(S_S, S_S)) \otimes_R R & \xrightarrow{\mathrm{Hom}_S(S,\lambda'_M(m)) \otimes R} & G(\mathrm{Hom}_S(S_S, M_S)) \otimes_R R \\
\downarrow{\scriptstyle \xi_{\mathrm{Hom}_S(S_S,S_S)}} & & \downarrow{\scriptstyle \xi_{\mathrm{Hom}_S(S_S,M_S)}} \\
\mathrm{Hom}_S(S_S, S_S) \otimes_S S & \xrightarrow{\mathrm{Hom}_S(S,\lambda'_M(m)) \otimes S} & \mathrm{Hom}_S(S_S, M_S) \otimes_S S \\
\downarrow{\scriptstyle \varepsilon'_S} & & \downarrow{\scriptstyle \varepsilon'_M} \\
S_S & \xrightarrow{\lambda'_M(m)} & M_S.
\end{array}$$

We calculate:

$$\begin{aligned}
mf(r) &= \lambda'_M(m)(f(r)) && \text{(def. of } \lambda'_M(m))\\
&= \lambda'_M(m)(\varepsilon'_S(\xi_{\mathrm{Hom}_S(S_S,S_S)}(\mathsf{id}_S \otimes r))) && \text{(def. of } f(r))\\
&= \varepsilon'_M(\xi_{\mathrm{Hom}_S(S_S,M_S)}(\lambda'_M(m) \otimes r)) && \text{(comm. of the diagram above)}\\
&= \varepsilon_M((\zeta_M \otimes R)(\lambda'_M(m) \otimes r)) && \text{(condition } \mathbf{C3})\\
&= \varepsilon_M(\zeta_M(\lambda'_M(m)) \otimes r) && \text{(def. of } \zeta_M \otimes R)\\
&= \zeta_M(\lambda'_M(m))(r) && \text{(def. of } \varepsilon_M)\\
&= \lambda_{F(M)}(m)(r) && \text{(condition } \mathbf{C1})\\
&= mr. && \text{(def. of } \lambda_{F(M)}(m))
\end{aligned}$$

3. For every $M_S \in \mathsf{CAct}_S$, $m \in M$ and $r \in R$, we are going to prove that $mr = mf(r)$.

Consider the canonical S-isomorphism $\eta'_M : M \longrightarrow \mathrm{Hom}_S(S, M \otimes_S S)$ given by $\eta'_M(m)(s) = m \otimes s$. Applying the naturality of ζ to the morphism $\eta'_M(m) : S_S \longrightarrow M \otimes_S S$ in FAct_S we obtain the commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_S(S,S) & \xrightarrow{\mathrm{Hom}_S(S,\eta'_M(m))} & \mathrm{Hom}_S(S, M \otimes_S S)\\
\Big\downarrow \zeta_S & & \Big\downarrow \zeta_{M\otimes_S S}\\
\mathrm{Hom}_R(R, F(S)) & \xrightarrow{\mathrm{Hom}_R(R,\eta'_M(m))} & \mathrm{Hom}_R(R, F(M \otimes_S S))
\end{array}.$$

The commuativity of this diagram over the element $\mathsf{id}_S \in \mathrm{Hom}_S(S,S)$ gives

$$\zeta_{M\otimes_S S}(\eta'_M(m)) = \eta'_M(m) \circ \zeta_S(\mathsf{id}_S) = \eta'_M(m) \circ f \tag{9}$$

because of the definition of f made in step 1. From condition **C4** we know that $\zeta_{M\otimes_S S} \circ G(\eta'_M) = \mathrm{Hom}_R(R, \xi_M) \circ \eta_{G(M)}$. Then we have

$$\begin{aligned}
mf(r) &= \mu'_M(m \otimes f(r)) && \text{(def. of } \mu'_M)\\
&= \mu'_M(\eta'_M(m)(f(r))) && \text{(def. of } \eta'_M(m))\\
&= \mu'_M(\zeta_{M\otimes_S S}(\eta'_M(m))(r)) && \text{(by (9))}\\
&= \mu'_M(\mathrm{Hom}_R(R,\xi_M)(\eta_{G(M)}(m))(r)) && \text{(condition } \mathbf{C4})\\
&= \mu'_M(\xi_M(\eta_{G(M)}(m)(r))) && \text{(def. of } \mathrm{Hom}_R(R,\xi_M))\\
&= \mu'_M(\xi_M(m \otimes r)) && \text{(def. of } \eta_M)\\
&= \mu_{G(M)}(m \otimes r) && \text{(condition } \mathbf{C2})\\
&= mr. && \text{(def. of } \mu_{G(M)})
\end{aligned}$$

4. f is a semigroup homomorphism. For all $r, r' \in R$ we have

$$\begin{aligned} f(rr') &= f(r)r' && (f = \zeta_S(\mathsf{id}_S) \text{ is a right } R\text{-homomorphism}) \\ &= f(r)f(r'). && (\text{step 2 applied to } S_S \in \mathsf{FAct}_S) \end{aligned}$$

5. f is a firm semigroup homomorphism. We have seen in steps 2 and 3 that the functors $F : \mathsf{FAct}_S \longrightarrow \mathsf{FAct}_R$ and $G : \mathsf{CAct}_S \longrightarrow \mathsf{CAct}_R$ are the restriction of scalars functors of the semigroup homomorphism f. Hence Proposition 12.4 and its left-right dual version say that f is a left and right firm semigroup homomorphism.

Finally we prove the one-to-one correspondence. Denote the set in (1) by X and the set in (3) by Y. Let (F, G) be a pair of compatible concrete functors. By $X \longrightarrow Y$ it is mapped to the semigroup homomorphism $f = \zeta_S(\mathsf{id}_S)$. Since F and G are the restriction of scalars funtors induced by f, we receive back the pair (F, G) with the map $Y \longrightarrow X$. Thus the composite $X \longrightarrow Y \longrightarrow X$ is the identity map.

We prove that also the composite $Y \longrightarrow X \longrightarrow Y$ is the identity map. Let $f : R \longrightarrow S$ be a firm semigroup homomorphism in Y. It is mapped to a pair (F, G) of compatible restrictions of $f^* : \mathsf{Act}_S \longrightarrow \mathsf{Act}_R$. This pair is mapped to a firm semigroup homomorphism $g := \zeta_S(\mathsf{id}_S)$. We need to show that $f = g$. Take $r \in R$ and factorise it as $r = r_1 r_2$ where $r_1, r_2 \in R$. (Since R is firm, it is also factorisable.) Then

$$\begin{aligned} f(r) &= f(r_1 r_2) && (r = r_1 r_2) \\ &= \mathsf{id}_S(f(r_1)f(r_2)) && (f \text{ is a semigroup homomorphism}) \\ &= (\mathsf{id}_S \cdot f(r_1))(f(r_2)) && (\text{by } (8)) \\ &= (\mathsf{id}_S \cdot r_1)(f(r_2)) && (\text{right } R\text{-action of } \mathrm{Hom}_S(S_S, S_S)) \\ &= \zeta_S(\mathsf{id}_S \cdot r_1)(r_2) && (\text{by } (7)) \\ &= (\zeta_S(\mathsf{id}_S) r_1)(r_2) && (\zeta \text{ is an } R\text{-homomorphism}) \\ &= \zeta_S(\mathsf{id}_S)(r_1 r_2) && (R\text{-action of } \mathrm{Hom}_R(R_R, S_R)) \\ &= g(r). && (g = \zeta_S(\mathsf{id}_S),\ r = r_1 r_2) \end{aligned}$$

■

15. The category of firm semigroups

Obviously, the identity morphism id_S of a semigroup S is right firm (or firm) if and only if S is firm. To obtain a category, we must show that right firm homomorphisms compose.

15.1. Lemma. *The composition of right (left) firm semigroup homomorphisms between firm semigroups is a right (left) firm semigroup homomorphism.*

PROOF. Let $f : R \longrightarrow S$ and $g : S \longrightarrow T$ be right firm semigroup homomorphisms. The composition $g \circ f : R \longrightarrow T$ is a semigroup homomorphism. By Proposition 12.3,

$$M_T \in \mathsf{FAct}_T \implies g^*(M_T) \in \mathsf{FAct}_S \implies (f^* \circ g^*)(M_T) = f^*(g^*(M_T)) \in \mathsf{FAct}_R.$$

But $f^* \circ g^* = (g \circ f)^*$, so the implication $3 \Rightarrow 1$ in Proposition 12.3 yields that $g \circ f$ is a right firm homomorphism. Using a symmetric argument, we get the result on the left. ∎

This lemma means that we may consider a category, where objects are all firm semigroups and morphisms are right firm (or left firm, or firm) homomorphisms. This category will be considered in the next theorem.

15.2. THEOREM. *The category of monoids (with monoid homomorphisms) is a full subcategory of the category of firm semigroups with right firm semigroup homomorphisms.*

PROOF. Suppose that R and S are monoids and $f : R \longrightarrow S$ is a monoid homomorphism. Then, clearly, $S = Sf(R)$, and $sf(r) = s'f(r')$ implies

$$s \otimes r = sr \otimes 1_R = sf(r) \otimes 1_R = s'f(r') \otimes 1_R = s'r' \otimes 1_R = s' \otimes r'$$

in $S \otimes_R R$ for all $s, s' \in S$ and $r, r' \in R$. By Lemma 11.2, f is a right firm homomorphism. Thus the category of monoids is a subcategory of the category of firm semigroups.

To prove that it is a full subcategory, consider monoids R and S and a right firm semigroup homomorphism $f : R \longrightarrow S$. We have to check that $f(1_R) = 1_S$. In order to prove that, notice that $1_S \in S = Sf(R)$, so there exist $r \in R$ and $s \in S$ such that $1_S = sf(r)$. Therefore we have

$$1_S = sf(r) = sf(r1_R) = sf(r)f(1_R) = 1_S f(1_R) = f(1_R).$$

∎

Different sources define a reflective subcategory $\mathcal{A}$ of a category $\mathcal{B}$ in different ways. Some of them (like [3, Definition 3.5.2]) require $\mathcal{A}$ to be a full subcategory of $\mathcal{B}$, some (like [13, page 91] or [1, Definition 4.16]) do not. In our situation we obtain a reflective subcategory which is full.

15.3. COROLLARY. *The category of firm semigroups with right firm semigroup homomorphisms is a full reflective subcategory of* Mon.

PROOF. For every firm semigroup R we can consider the monoid R^1 with externally adjoined identity 1 and the inclusion mapping $\iota : R \longrightarrow R^1$. If $f : R \longrightarrow S$ is a right firm semigroup homomorphism, then putting $g(1) := 1_S$ and $g(r) := f(r)$ for every $r \in R$ we obtain a monoid homomorphism $g : R^1 \longrightarrow S$, which is right firm by Theorem 15.2. Clearly $g\iota = f$ and g is unique with this property. ∎

16. Examples of firm homomorphisms

In this section we will give some examples of firm homomorphisms between semigroups. We will use Lemma 11.2 for checking right firmness.

16.1. Example. Every bijective semigroup homomorphism $f : R \longrightarrow S$ between firm semigroups is firm.

Due to surjectivity, $S = SS = Sf(R)$. Now suppose that $sf(r) = s'f(r')$, where $s, s' \in S$ and $r, r' \in R$. Using surjectivity, we can find $r_1, r_2 \in R$ such that $s = f(r_1)$ and $s' = f(r_2)$. Since f is a semigroup homomorphism, we have $f(r_1 r) = f(r_2 r')$, and injectivity of f yields $r_1 r = r_2 r'$. Using that R is firm, we conclude that $r_1 \otimes r = r_2 \otimes r'$ in $R \otimes_R R$. Applying the mapping $f \otimes R : R \otimes_R R \longrightarrow S \otimes_R R$ we obtain the equality $f(r_1) \otimes r = f(r_2) \otimes r'$ in $S \otimes_R R$, which is precisely $s \otimes r = s' \otimes r'$, as needed. Thus f is right firm. A dual argument shows that it is left firm.

16.2. Example. Consider the semigroup $S = (\mathbb{Z}, +)$ and its subsemigroup $R = \mathbb{N} = \{1, 2, \ldots\}$. It is shown in [18, Example 3.1] that the act $\mathbb{Z}_{\mathbb{N}}$ (with the action $(a, n) \mapsto a+n$) is firm. Therefore the inclusion mapping $f : \mathbb{N} \longrightarrow \mathbb{Z}$ is a firm semigroup homomorphism.

16.3. Example. If R is a monoid which is a subsemigroup of a semigroup S and $S = SR$, then the inclusion mapping $f : R \longrightarrow S$ is right firm. Indeed,

$$sr = s'r' \implies s \otimes r = sr \otimes 1_R = s'r' \otimes 1_R = s' \otimes r'$$

in $S \otimes_R R$.

For example, if R is any monoid and T is any semigroup, then we obtain a semigroup with such properties if we take $S := T \sqcup R$, define

$$rt = tr := t$$

for all $r \in R$, $t \in T$, and preserve the multiplication of R and T.

16.4. Example. Let R be a semigroup with common weak right local units, i.e.

$$(\forall r, r' \in R)(\exists u \in R)(r = ru \ \text{ and } \ r' = r'u).$$

(For example, $(\mathbb{Z}, \min)$ is such a semigroup.) Then every surjective homomorphism $f : R \longrightarrow S$ is right firm. Since R is factorisable, also S is factorisable, and hence $S = SS = Sf(R)$. If $sf(r) = s'f(r')$ and u is as above, then

$$s \otimes r = s \otimes ru = sf(r) \otimes u = s'f(r') \otimes u = s' \otimes r'u = s' \otimes r'$$

in $S \otimes_R R$.

16.5. Example. Let $S = (\{2, 3, 4, \ldots\}, \cdot)$ and let $R = S^2$ be the subsemigroup of positive composite numbers. Since $2 \in S \setminus SR$, we have $S \neq SR$ and the inclusion mapping $f : R \longrightarrow S$ is not firm.

The next proposition allows to produce firm semigroup homomorphisms from a monoid homomorphism. It is a semigroup theoretic version of Proposition 10.5.

16.6. Proposition. *Let $f : R \longrightarrow S$ be a monoid homomorphism. Take some nonempty sets I, Λ and consider the semigroup $\overline{R} = I \times R \times \Lambda$ with the multiplication*

$$(i, r, \lambda)(i', r', \lambda') = (i, rr', \lambda').$$

and a similar semigroup $\overline{S} = I \times S \times \Lambda$. The mapping

$$\overline{f} : \overline{R} \longrightarrow \overline{S}, \quad (i, r, \lambda) \mapsto (i, f(r), \lambda)$$

is a firm semigroup homomorphism.

Proof. Note that $\overline{R}$ and $\overline{S}$ are Rees matrix semigroups whose sandwich matrix has the identity element of a corresponding monoid at each position.

Now $\overline{S} = \overline{S} f(\overline{R})$, because, for every $(i, s, \lambda) \in \overline{S}$,

$$(i, s, \lambda) = (i, s1_S, \lambda) = (i, s, \lambda)(i, 1_S, \lambda) = (i, s, \lambda) f(i, 1_R, \lambda).$$

To prove that $\mu_{\overline{S}_{\overline{R}}}$ is injective, we suppose that $(i, s, \lambda) f(j, r, \kappa) = (i', s', \lambda') f(j', r', \kappa')$, i.e. $(i, sf(r), \kappa) = (i', s'f(r'), \kappa')$. Then $i = i'$, $\kappa = \kappa'$ and $sf(r) = s'f(r')$. In $\overline{S} \otimes_{\overline{R}} \overline{R}$ we compute:

$$\begin{aligned}
(i, s, \lambda) \otimes (j, r, \kappa) &= (i, s, \lambda) \otimes (j, r, \kappa)(i, 1_R, \kappa) \\
&= (i, s, \lambda)(j, f(r), \kappa) \otimes (i, 1_R, \kappa) \\
&= (i, sf(r), \kappa) \otimes (i, 1_R, \kappa) \\
&= (i', s'f(r'), \kappa') \otimes (i, 1_R, \kappa') \\
&= (i', s', \lambda')(j', f(r'), \kappa') \otimes (i, 1_R, \kappa') \\
&= (i', s', \lambda') \otimes (j', f(r'), \kappa')(i, 1_R, \kappa') \\
&= (i', s', \lambda') \otimes (j', r', \kappa').
\end{aligned}$$

■

Observe that if either I or Λ has more than one element, then $\overline{R}$ and $\overline{S}$ are not monoids.

16.7. Example. Let R be a left zero semigroup (i.e. a semigroup satisfying the identity $xy = x$). Consider the semigroup $S := R^{\mathbf{0}}$ obtained from R by adjoining an external zero $\mathbf{0}$. We will prove that the inclusion mapping $f : R \longrightarrow S$ is right firm.

The equality $S = SR$ is rather clear. Suppose that $sr = s'r'$, where $s, s' \in S$ and $r, r' \in R$. We have three possibilities.

1. If $s \in R$, then also $s' \in R$ and, moreover, $s = s'$. Hence, in $S \otimes_R R$,

$$s \otimes r = ss \otimes r = s \otimes sr = s \otimes s'r' = ss' \otimes r' = s's' \otimes r' = s' \otimes r'.$$

2. If $s' \in R$, then a similar argument works.

3. If $s, s' \notin R$, then $s = s' = \mathbf{0}$ and in $S \otimes_R R$ we have

$$\mathbf{0} \otimes r = \mathbf{0}r \otimes r = \mathbf{0} \otimes rr = \mathbf{0} \otimes rr' = \mathbf{0}r \otimes r' = \mathbf{0} \otimes r'.$$

Therefore f is right firm.

We will also show that f is not left firm if R contains at least two different elements r and r'. We have the equality $r\mathbf{0} = r'\mathbf{0}$. Suppose that $r \otimes \mathbf{0} = r' \otimes \mathbf{0}$ in $R \otimes_R S$. Then

$$\begin{array}{rclcrcl} r & = & r_1 u_1 & \qquad & u_1 \mathbf{0} & = & v_1 s_1 \\ r_1 v_1 & = & r_2 u_2 & & u_2 s_1 & = & v_2 s_2 \\ & \dots & & & & \dots & \\ r_n v_n & = & r' u_{n+1} & & u_{n+1} s_n & = & \mathbf{0} \end{array}$$

for some $r_i \in R$, $s_i \in S$ and $u_i, v_i \in R^1$. Since R is a left zero semigroup, we conclude that $r = r_1 = r_2 = \ldots = r_n = r'$, a contradiction. Therefore f is not left firm.

With the help of Example 16.7 we can construct examples of right firm ring homomorphisms that are not left firm.

16.8. Example. Consider the two element left zero semigroup $R = \{r_1, r_2\}$, that is, $r_1 r_1 = r_1 r_2 = r_1$ and $r_2 r_2 = r_2 r_1 = r_2$. It is the same semigroup that we have in Example 16.3. Let the semigroup $S = R^{\mathbf{0}}$ be obtained from R by adjoining an external zero $\mathbf{0}$. Consider the inclusion mapping $f : \mathbb{Z}[R] \longrightarrow \mathbb{Z}[S]$ of semigroup rings. We will prove that f is right firm. If $a_1 r_1 + a_2 r_2 + a_3 \mathbf{0} \in \mathbb{Z}[S]$ $(a_1, a_2, a_3 \in \mathbb{Z})$, then

$$a_1 r_1 + a_2 r_2 + a_3 \mathbf{0} = a_1 r_1 r_1 + a_2 r_2 r_1 + a_3 \mathbf{0} r_1 = (a_1 r_1 + a_2 r_2 + a_3 \mathbf{0}) r_1,$$

proving that $\mathbb{Z}[S] = \mathbb{Z}[S] \cdot \mathbb{Z}[R]$. Suppose that $\sum_i x_i y_i = 0$ in the module $\mathbb{Z}[S]_{\mathbb{Z}[R]}$, where $x_i \in \mathbb{Z}[S]$ and $y_i \in \mathbb{Z}[R]$. Then

$$\sum_i (x_i \otimes y_i) = \sum_i (x_i \otimes y_i r_1) = \sum_i (x_i y_i \otimes r_1) = \sum_i (x_i y_i) \otimes r_1 = 0 \otimes r_1 = 0$$

in the tensor product $\mathbb{Z}[S] \otimes_{\mathbb{Z}[R]} \mathbb{Z}[R]$. Thus f is a right firm homomorphism of rings.

We will prove that f is not left firm by showing that the mapping

$$\nu : \mathbb{Z}[R] \otimes_{\mathbb{Z}[R]} \mathbb{Z}[S] \longrightarrow \mathbb{Z}[S], \quad y \otimes x \mapsto yx$$

is not injective. Note that $\nu(1r_1 \otimes 1\mathbf{0}) = (1r_1)(1\mathbf{0}) = 1\mathbf{0} = (1r_2)(1\mathbf{0}) = \nu(1r_2 \otimes 1\mathbf{0})$ in the left module ${}_{\mathbb{Z}[R]}\mathbb{Z}[S]$. It suffices to show that $1r_1 \otimes 1\mathbf{0} \neq 1r_2 \otimes 1\mathbf{0}$ in the tensor product $\mathbb{Z}[R] \otimes_{\mathbb{Z}[R]} \mathbb{Z}[S]$.

To this end we consider the tensor product $R \otimes_R S$ of the R-acts R_R and ${}_R S$ and the free abelian group $\mathbb{Z}^{(R \otimes_R S)}$ on the set $R \otimes_R S$. A straightforward verification shows that the mapping

$$\overline{\phi} \colon \mathbb{Z}[R] \times \mathbb{Z}[S] \longrightarrow \mathbb{Z}^{(R \otimes_R S)}, \quad \left(\sum_i a_i r_i, \sum_j b_j s_j \right) \mapsto \sum_{i,j} a_i b_j (r_i \otimes s_j)$$

is $\mathbb{Z}[R]$-balanced. By the universal property of the tensor product there exists an abelian group homomorphism $\phi\colon \mathbb{Z}[R] \otimes_{\mathbb{Z}[R]} \mathbb{Z}[S] \longrightarrow \mathbb{Z}^{(R\otimes_R S)}$ such that $\phi(y \otimes x) = \overline{\phi}(y, x)$ for every $y \in \mathbb{Z}[R]$ and $x \in \mathbb{Z}[S]$. In particular,

$$\phi(1r_1 \otimes 1\mathbf{0}) = \overline{\phi}(1r_1, 1\mathbf{0}) = r_1 \otimes \mathbf{0} \neq r_2 \otimes \mathbf{0} = \overline{\phi}(1r_2, 1\mathbf{0}) = \phi(1r_2 \otimes 1\mathbf{0})$$

where the inequality is shown in Example 16.7. Hence the needed inequality $1r_1 \otimes 1\mathbf{0} \neq 1r_2 \otimes 1\mathbf{0}$ follows.

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