Theory and Applications of Categories, Vol. 39, No. 6, 2023, pp. 172–185.

POINTED SEMIBIPRODUCTS OF MONOIDS

NELSON MARTINS-FERREIRA

ABSTRACT. A new notion of a (pointed) semibiproduct is introduced, which, in the case of groups amounts to an extension equipped with a set-theoretical section. When the section is a group homomorphism then a pointed semibiproduct is the same as a group split extension. The main result of the paper is a characterization of pointed semibiproducts of monoids using a structure that is a generalization of the action that is used in the definition of a semidirect product of groups.

1. Introduction

Biproducts were introduced by Mac Lane in his book *Homology* but can be traced back to his paper on *Duality for groups* [17]. Biproducts are useful to study split extensions in the context of abelian categories in the same way that semidirect products are appropriate to study group split extensions. Although these concepts have been thoroughly developed over the last decades in the context of protomodular and semi-abelian categories [1, 2, 3, 15], the notion of *relative biproduct* introduced by Mac Lane to study relative split extensions seems to have been forgotten (see [18], p. 263). On the other hand, much work has been done in extending the tools and techniques from groups [19] to monoids [6, 7, 8, 11, 12, 13, 16, 24, 27] and even more general settings [14]. However, as it has been observed several times, it is not a straightforward task to take a well-known result in the category of groups (or any other semi-abelian category) and materialize it in the category of monoids not to mention in more general situations. We will argue that a convenient reformulation of relative biproduct (called *semibiproduct*) can be used to study group and monoid extensions in a single unified frame work. Even though semidirect products are suitable to describe all group split extensions, they fail to capture those group extensions that do not split. The key observation to semibiproducts in reinterpreting relative biproducts (see [18], diagram (5.2), p. 263 and compare with diagram (2) in Definition 3.1) is that although an extension may fail to split as a monoid extension or as a group extension, it necessarily splits as an extension of pointed sets.

Funded by FCT/MCTES (PIDDAC) through the following Projects: Associate Laboratory ARISE LA/P/–0112/2020; UIDP/04044/2020; UIDB/04–044/2020; PAMI–ROTEIRO/0328/2013 (Nº 022158); MATIS (CENTRO-01-0145-FEDER-000014 - 3362); CENTRO-01-0247-FEDER-(069665,039969); POCI-01-0247-FEDER-(069603,039958,039863,024533); Generative thermodynamic; by CDRSP and ESTG from the Polytechnic of Leiria.

Received by the editors 2020-02-17 and, in final form, 2023-02-13.

Transmitted by Tim Van der Linden. Published on 2023-02-17.

2020 Mathematics Subject Classification: 18G50, 20M10, 20M32.

Key words and phrases: Semibiproduct, biproduct, semidirect product of groups and monoids, pointed semibiproduct, semibiproduct extension, pointed monoid action system, Schreier extension.

The main result (Theorem 6.2) establishes an equivalence of categories between pointed semibiproducts of monoids (Definition 3.2) and pointed monoid action systems (Definition 5.1). The 14 classes of non-isomorphic pointed semibiproducts of 2-element monoids are listed in Section 7. We start with some motivation in Section 2, introduce **Mag**-extended categories and semibiproducts in Section 3, restrict to the pointed case in Section 4 while studying some stability properties and pointing out some differences and similarities between groups, monoids and unitary magmas. From Section 5 on we work towards the main result and restrict our attention to monoids.

2. Motivation

It is well known that a split extension of groups

$$X \xrightarrow{k} A \xrightarrow{p} B,$$

with a specified section $s\colon B \to A$, can be completed into a diagram of the form

$$X \underset{k}{\overset{q}{\leftrightarrows}} A \underset{s}{\overset{p}{\rightleftarrows}} B,$$

in which $q\colon A \to X$ is the map uniquely determined by the formula $kq(a) = a - sp(a)$, $a \in A$. Furthermore, the information needed to reconstruct the split extension as a semidirect product is encoded in the map $\varphi\colon B \times X \to X$, uniquely determined as

$$\varphi(b, x) = q(s(b) + k(x)).$$

When writing the element $\varphi(b, x) \in X$ as $b \cdot x$ we see that $k(b \cdot x)$ is equal to $s(b) + k(x) - s(b)$ and that the group A is recovered as the semidirect product $X \rtimes_\varphi B$. In the event that the section s, while being a zero-preserving map, is not necessarily a group homomorphism, the classical treatment of group extensions prescribes a different procedure (see e.g. [26], p. 238). However, the results obtained here suggest that non-split extensions may be treated similarly to split extensions, and moreover the same approach is carried straightforwardly into the context of monoids. Indeed, when s is not a homomorphism, in addition to the map φ, we get a map $\gamma\colon B \times B \to X$, determined as $\gamma(b, b') = q(s(b) + s(b'))$ and the group A is recovered as the set $X \times B$ with group operation

$$(x, b) + (x', b') = (x + \varphi(b, x') + \gamma(b, b'), b + b')$$

defined for every $x, x' \in X$ and $b, b' \in B$. Note that X needs not be commutative. However, instead of simply saying that φ is an action and that γ is a factor system, we have to consider two maps φ and γ which in conjunction turn the set $X \times B$ into a group with a prescribed operation (Section 5). This is precisely what we call a semibiproduct of groups. Observe that when s is a homomorphism, it reduces to the usual notion of semidirect product.

Almost every step in the treatment of groups is carried over into the context of monoids. However, while in groups all extensions are obtained as semibiproducts, in monoids we have to restrict our attention to those for which there exists a section s and a retraction q satisfying the condition $a = kq(a) + sp(a)$ for all $a \in A$ (Section 3). Consequently, in addition to the maps φ and γ obtained as in groups, a new map $\rho\colon X \times B \to X$, determined by $\rho(x, b) = q(k(x) + s(b))$ needs to be taken into consideration. Hence, the monoid A is recovered as the set $\{(x, b) \in X \times B \mid \rho(x, b) = x\}$ with operation

$$(x, b) + (x', b') = (\rho(x + \varphi(b, x') + \gamma(b, b'), b + b'), b + b') \tag{1}$$

which is defined for every $x, x' \in X$ and $b, b' \in B$.

3. Mag-extended categories and (pointed) semibiproducts

The notion of an extended category as introduced in [4] is a triple $(\mathbf{C}, E, \varepsilon)$ where $E\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{Set}$ is a functor and $\varepsilon\colon\ \hom_{\mathbf{C}} \to E$ is a natural transformation whose components are all injective. This allows to add *imaginary morphisms* to the morphisms of $\mathbf{C}$. From an object X to an object Y, we add the elements of $E(X, Y) \setminus \varepsilon_{X,Y}(hom(X, Y))$. Further details can be found in [4], p. 324; see also Example 2.1 in [5], p. 281.

In order to introduce the notion of a semibiproduct we need to have an extended category with *imaginary morphisms*, that we will simply call maps, for which a binary operation is defined for every parallel pair of maps (the operation will be denoted additively, although it is not assumed to be commutative nor even associative). This means that instead of an extended category we will need to work in a **Mag**-extended category where **Mag** denotes the category of magmas and magma homomorphisms. In general, any category $\mathbb{M}$ with a forgetful functor $U\colon \mathbb{M} \to \mathbf{Set}$ into the category of sets and maps gives rise to the notion of an $\mathbb{M}$-extended category. By an $\mathbb{M}$-extended category we mean a category $\mathbf{C}$ together with a bifunctor $\mathrm{map}\colon \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbb{M}$ and a natural inclusion $\varepsilon\colon \hom_{\mathbf{C}} \to U \circ \mathrm{map}$. For example, when $\mathbb{M} = \mathbf{Mag}$, if a category $\mathbf{C}$ is a concrete category over sets in which a meaningful map addition is available then a bifunctor map is obtained as follows. For every pair of objects (A, B) in $\mathbf{C}$, $\mathrm{map}(A, B)$ is the magma of underlying maps from object A to object B equipped with component-wise addition. In particular $\mathrm{map}(A, B)$ contains $\hom_{\mathbf{C}}(A, B)$ as a subset since

$$\varepsilon_{A,B}\colon \hom_{\mathbf{C}}(A, B) \to U(\mathrm{map}(A, B))$$

is required to be a natural inclusion. This means that the category **Mag** is a **Mag**-extended category with $\mathrm{map}(A, B)$ the magma of all maps from $U(A)$ to $U(B)$ (functions which need not be morphisms). If f is a magma homomorphism from A to B then $\varepsilon_{A,B}(f)$ is nothing but f considered as a map between the underlying sets of A and B. In the same way the categories of groups, abelian groups, monoids and commutative monoids are **Mag**-categories as well. However, there is a significant distinction between the **Ab**-enriched category of abelian groups, the linear category of commutative monoids and the **Mag**-extended categories of groups and monoids, which can also be considered, respectively, as **Grp**-extended and **Mon**-extended categories. If A is an object in an **Ab**-enriched category then $\hom(A, A)$ is a ring. If A is an object in a linear category then $\hom(A, A)$ is a semiring. In contrast, if A is a group (or a monoid) then $\hom(A, A)$ is a subset of the near-ring $\mathrm{map}(A, A)$.

3.1. DEFINITION. *Let* $(\mathbf{C}, \mathrm{map}, \varepsilon)$ *be a* **Mag**-*extended category. A* semibiproduct *is a tuple* (X, A, B, p, k, q, s) *represented as a diagram of the shape*

$$X \underset{k}{\overset{q}{\leftrightarrows}} A \underset{s}{\overset{p}{\rightleftarrows}} B \tag{2}$$

in which $p\colon A \to B$ *and* $k\colon X \to A$ *are morphisms in* $\mathbf{C}$, *whereas* $q \in \mathrm{map}(A, X)$ *and* $s \in \mathrm{map}(B, A)$. *Furthermore, the following conditions are satisfied:*

$$ps = 1_B \tag{3}$$
$$qk = 1_X, \tag{4}$$
$$kq + sp = 1_A. \tag{5}$$

There is an obvious abuse of notation in the previous conditions. This is justified because we will be mostly concerned with the case in which $\mathbf{C}$ is the category of monoids and $\mathrm{map}(A, B)$ is the set of zero-preserving maps. In more rigorous terms, the condition $ps = 1_B$ should have been written as $\mathrm{map}(1_B, p)(s) = \varepsilon_{B,B}(1_B)$ whereas condition $qk = 1_X$ should have been written as $\mathrm{map}(k, 1_X)(q) = \varepsilon_{X,X}(1_X)$. In the same way the condition $\mathrm{map}(1_A, k)(q) + \mathrm{map}(p, 1_A)(s) = \varepsilon_{A,A}(1_A)$ should replace $kq + sp = 1_A$.

We will now particularise the notion of semibiproduct to the pointed case and work in the concrete context of monoids as a **Mon**-extended category. Thus we introduce the category of pointed semibiproducts of monoids, denoted **Psb**.

3.2. DEFINITION. *A* pointed semibiproduct of monoids *is a tuple* (X, A, B, p, k, q, s) *that can also be represented as a diagram of the shape*

$$X \underset{k}{\overset{q}{\leftrightarrows}} A \underset{s}{\overset{p}{\rightleftarrows}} B \tag{6}$$

in which X, A *and* B *are monoids (not necessarily commutative),* p, k, *are monoid homomorphisms, while* q *and* s *are zero-preserving maps. Moreover, the following conditions are satisfied:*

$$ps = 1_B \tag{7}$$
$$qk = 1_X \tag{8}$$
$$kq + sp = 1_A \tag{9}$$
$$pk = 0_{X,B} \tag{10}$$
$$qs = 0_{B,X}. \tag{11}$$

A morphism in **Psb**, say from (X, A, B, p, k, q, s) to $(X', A', B', p', k', q', s')$, is a triple (f_1, f_2, f_3), displayed as

$$\begin{array}{ccccc} X & \underset{k}{\overset{q}{\leftrightarrows}} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} \\ X' & \underset{k'}{\overset{q'}{\leftrightarrows}} & A' & \underset{s'}{\overset{p'}{\rightleftarrows}} & B' \end{array} \tag{12}$$

in which f_1, f_2 and f_3 are monoid homomorphisms and moreover the following conditions are satisfied: $f_2k = k'f_1$, $p'f_2 = f_3p$, $f_2s = s'f_3$, $q'f_2 = f_1q$. Composition of morphisms is done as expected, that is, $(f_1, f_2, f_3) \circ (g_1, g_2, g_3) = (f_1g_1, f_2g_2, f_3g_3)$.

3.3. Theorem. *Let (X, A, B, p, k, q, s) be a pointed semibiproduct of monoids. For every $a, a' \in A$ the element $a + a' \in A$ can be written in terms of $q(a)$, $q(a')$, $p(a)$ and $p(a')$ as*

$$k(q(a) + q(sp(a) + kq(a')) + q(sp(a) + sp(a'))) + s(p(a) + p(a')). \tag{13}$$

Proof. We observe:

$$\begin{aligned} a + a' &= kqa + (spa + kqa') + spa' \qquad (kq + sp = 1) \\ &= kqa + kq(spa + kqa') + sp(spa + kqa') + spa' \\ &= kqa + kq(spa + kqa') + spa + spa' \qquad (ps = 1, pk = 0) \\ &= kqa + kq(spa + kqa') + kq(spa + spa') + sp(spa + spa') \\ &= kqa + kq(spa + kqa') + kq(spa + spa') + s(pa + pa') \\ &= k(qa + q(spa + kqa') + q(spa + spa')) + sp(a + a'). \end{aligned}$$

■

The previous result suggests a transport of structure from the monoid A into the set $X \times B$ as motivated with formula (1) in Section 2. However, as we will see, in order to keep an isomorphism with A we need to restrict the set $X \times B$ to those pairs (x, b) for which there exists $a \in A$ such that $x = q(a)$ and $b = p(a)$.

Furthermore, we observe that the condition $qs = 0_{B,X}$ has not been used and what is more, the proof can be carried out straightforwardly from the concrete category of monoids to any **Mon**-extended category via generalized elements.

4. Stability properties of pointed semibiproducts

From now on the category $\mathbf{C}$ is assumed to be either the category of groups or the category of monoids (occasionally we will refer to the category of unitary magmas) and $\mathrm{map}(A, B)$ is the magma of zero-preserving maps with component-wise addition.

As introduced in the previous section, a *pointed semibiproduct* is a semibiproduct satisfying two extra conditions, namely

$$pk = 0_{X,B}, \quad qs = 0_{B,X}. \tag{14}$$

However, as it is well known, in the case of groups this distinction is irrelevant.

4.1. PROPOSITION. *Every semibiproduct of groups is pointed.*

PROOF. We have $pk = p1_A k = p(kq + sp)k = pkqk + pspk = pk + pk$. And $s = 1_A s = (kq + sp)s = kqs + sps = kqs + s$. Hence we may conclude $pk = 0$ and $kqs = 0$. Since k is a monomorphism $qs = 0$. ■

The previous proof also shows that a semibiproduct of monoids is pointed as soon as the monoid A admits right cancellation. Clearly, this is not a general fact.

4.2. PROPOSITION. *Let A be a monoid. The tuple $(A, A, A, 1_A, 1_A, 1_A, 1_A)$ is a semibiproduct of monoids if and only if A is an idempotent monoid.*

PROOF. Condition (5) in this case becomes $a = a + a$ for all $a \in A$. ■

Every pointed semibiproduct of monoids has an underlying exact sequence.

4.3. PROPOSITION. *Let (X, A, B, p, k, q, s) be a pointed semibiproduct of monoids. The sequence*

$$X \xrightarrow{k} A \xrightarrow{p} B$$

is an exact sequence.

PROOF. Let $f\colon Z \to A$ be a morphism such that $pf = 0$. Then the map $\bar{f} = qf$ is a homomorphism

$$\begin{aligned} qf(z+z') &= q(fz + fz') = q(kqf(z) + spf(z) + kqf(z') + spf(z')) \\ &= q(kqf(z) + 0 + kqf(z') + 0) \\ &= qk(qf(z) + qf(z')) = qf(z) + qf(z') \end{aligned}$$

and it is unique with the property $k\bar{f} = f$. Indeed, if $k\bar{f} = f$ then $qk\bar{f} = qf$ and hence $\bar{f} = qf$. This means that k is the kernel of p.

Let $g\colon A \to Y$ be a morphism and suppose that $gk = 0$. It follows that $g = gsp$,

$$g = g1_A = g(kq + sp) = gkq + gsp = 0 + gsp = gsp,$$

and consequently the map $\bar{g} = gs$ is a homomorphism, indeed

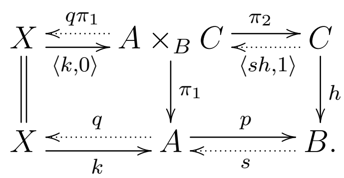
$$gs(b) + gs(b') = g(sb + sb') = gsp(sb + sb') = gs(b + b').$$

The fact that $\bar{g} = gs$ is the unique morphism with the property $\bar{g}p = g$ follows from $\bar{g}ps = gs$ which is the same as $\bar{g} = gs$. Hence p is the cokernel of k and the sequence is exact. ■

The following results show that pointed semibiproducts are stable under pullback and in particular split semibiproducts of monoids are stable under composition.

4.4. PROPOSITION. *Pointed semibiproducts of monoids are stable under pullback.*

PROOF. Let (X, A, B, p, k, q, s) be a pointed semibiproduct of monoids displayed as the bottom row in the following diagram which is obtained by taking the pullback of p along an arbitrary morphism $h\colon C \to B$, with induced morphism $\langle k, 0\rangle$ and map $\langle sh, 1\rangle$,



(15)

We have to show that the top row is a pointed semibiproduct of monoids. By construction we have $\pi_2\langle sh, 1\rangle = 1_C$, $\pi_2\langle k, 0\rangle = 0$, $q\pi_1\langle sh, 1\rangle = qsh = 0$, $q\pi_1\langle k, 0\rangle = qk = 1_X$. It remains to prove the identity

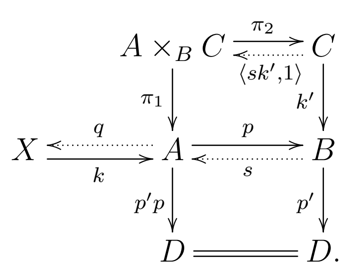
$$(a, c) = (kq(a), 0) + (sh(c), c) = (kq(a) + sh(c), c)$$

for every $a \in A$ and $c \in C$ with $p(a) = h(c)$, which follows from $a = kq(a) + sp(a) = kq(a) + sh(c)$. ∎

The previous results are stated at the level of monoids but are easily extended to unitary magmas. The particular case of semidirect products has been considered in [14] and the notion of composable pair of pointed semibiproducts is borrowed from there. We say that a pointed semibiproduct (X, A, B, p, k, q, s) can be composed with a pointed semibiproduct $(C, B, D, p', k', q', s')$ if the tuple

$$(A \times_B C, A, D, p'p, \pi_1, q'', ss'),$$

in which q'' is such that $\pi_1 q'' = kq + sk'q'p$ and $\pi_2 q'' = q'p$, is a pointed semibiproduct.



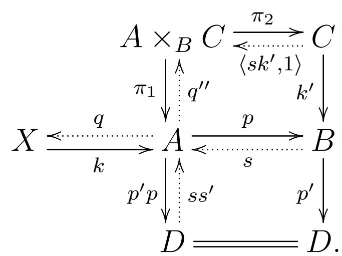
(16)

Note that in the case of groups the map q is uniquely determined as $q(a) = a - sp(a)$ for all $a \in A$. However this is not the case for monoids nor for unitary magmas.

4.5. PROPOSITION. *A pointed semibiproduct of monoids (X, A, B, p, k, q, s) can be composed with $(C, B, D, p', k', q', s')$, another pointed semibiproduct of monoids, if and only if the map s is equal to the map $sk'q' + ss'p'$.*

Proof. Let us observe that the tuple $(A \times_B C, A, D, p'p, \pi_1, q'', ss')$ is a pointed semibiproduct if and only if $\pi_1 q'' + ss'p'p = 1_A$. Indeed, the kernel of the composite $p'p$ is obtained by taking the pullback of p along k', the kernel of p', as illustrated in diagram (16).

In order to obtain a pointed semibiproduct we complete de diagram with a map q'' such that $\pi_1 q'' = kq + sk'q'p$ and $\pi_2 q'' = q'p$ as illustrated



(17)

The map q'' is well defined, $p(kq + sk'q'p) = pkq + psk'q'p = k'q'p$. Moreover, $p'pss' = 1_D$, $p'p\pi_1 = p'k'\pi_2 = 0$, $q''ss' = 0$ and we observe

$$\begin{aligned} q''\pi_1 &= \langle kq + sk'q'p, q'p\rangle \pi_1 \\ &= \langle kq\pi_1 + sk'q'p\pi_1, q'p\pi_1\rangle \\ &= \langle kq\pi_1 + sk'q'k'\pi_2, q'k'\pi_2\rangle \\ &= \langle kq\pi_1 + sp\pi_1, \pi_2\rangle \\ &= \langle \pi_1, \pi_2\rangle = 1_{A\times_B C}. \end{aligned}$$

It remains to analyse the condition $\pi_1 q'' + ss'p'p = 1_A$. If $s = sk'q' + ss'p'$ then we have $\pi_1 q'' + ss'p'p = kq + sk'q'p + ss'p'p$ and hence $kq + sp = 1_A$. Conversely, having $\pi_1 q'' + ss'p'p = 1_A$ we get $kq + sk'q'p + ss'p'p = 1_A$ and $kqs + sk'q'ps + ss'p'ps = s$ so $sk'q' + ss'p' = s$. ■

Note that associativity is used to convert $(kq+sk'q'p)+ss'p'p$ into $kq+(sk'q'p+ss'p'p)$. Moreover, if the map s is a homomorphism then condition $s = sk'q' + ss'p'$ is trivial. A pointed semibiproduct (X, A, B, p, k, q, s) in which the map s is a homomorphism is called a pointed split semibiproduct. This means that pointed split semibiproducts of monoids are stable under composition.

In spite of the fact that the previous results have been presented in the concrete categories of groups and monoids (and that Propositions 4.3 and 4.4 are also valid for unitary magmas), it is clear that Propositions 4.1, 4.4, 4.5 are still valid at the level of **UMag**-extended categories, or **Mon**-extended categories when associativity is required, or **Grp**-extended categories when inverses are required as well.

5. The category of pointed monoid action systems

The purpose of this section is to introduce the category of pointed monoid action systems, which will be denoted as **Act**. This category is obtained by requiring the existence of a categorical equivalence between **Act** and **Psb** (see Theorem 6.2).

5.1. Definition. *A* pointed monoid action system *is a five-tuple*

$$(X, B, \rho, \varphi, \gamma)$$

in which X and B are monoids, $\rho\colon X \times B \to X$, $\varphi\colon B \times X \to X$, $\gamma\colon B \times B \to X$ are maps such that the following conditions are satisfied for every $x \in X$ and $b, b' \in B$:

$$\rho(x,0) = x, \quad \rho(0,b) = 0 \tag{18}$$

$$\varphi(0,x) = x, \quad \varphi(b,0) = 0 \tag{19}$$

$$\gamma(b,0) = 0 = \gamma(0,b) \tag{20}$$

$$\rho(x,b) = \rho(\rho(x,b),b) \tag{21}$$

$$\varphi(b,x) = \rho(\varphi(b,x),b) \tag{22}$$

$$\gamma(b,b') = \rho(\gamma(b,b'),b+b') \tag{23}$$

and moreover the following condition holds for every $x, x', x'' \in X$ and $b, b', b'' \in B$,

$$\begin{aligned}&\rho(\rho(x+\varphi(b,x')+\gamma(b,b'),b+b')+\varphi(b+b',x'')+\gamma(b+b',b''),b''') = \\ &= \rho(x+\varphi(b,\rho(x'+\varphi(b',x'')+\gamma(b',b''),b'+b''))+\gamma(b,b'+b''),b''')\end{aligned} \tag{24}$$

where $b''' = b + b' + b''$.

A morphism of pointed monoid action systems, say from a pointed monoid action system $(X, B, \rho, \varphi, \gamma)$ to $(X', B', \rho', \varphi', \gamma')$ is a pair (f, g) of monoid homomorphisms, with $f\colon X \to X'$ and $g\colon B \to B'$ such that for every $x \in X$ and $b, b' \in B$

$$f(\rho(x,b)) = \rho'(f(x),g(b)), \tag{25}$$

$$f(\varphi(b,x)) = \varphi'(g(b),f(x)), \tag{26}$$

$$f(\gamma(b,b')) = \gamma'(g(b),g(b')). \tag{27}$$

The composition of morphisms between pointed monoid action systems is as expected, namely $(f, g) \circ (f', g') = (ff', gg')$.

5.2. Theorem. *There exists a functor $R\colon \mathbf{Act} \to \mathbf{Mon}$ such that for every morphism in $\mathbf{Act}$, say $(f, g)\colon (X, B, \rho, \varphi, \gamma) \to (X', B', \rho', \varphi', \gamma')$, the diagram*

$$\begin{array}{ccccc} X & \underset{\langle 1,0\rangle}{\overset{\pi_X}{\rightleftarrows}} & R(X,B,\rho,\varphi,\gamma) & \underset{\langle 0,1\rangle}{\overset{\pi_B}{\rightleftarrows}} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle R(f,g)} & & \downarrow{\scriptstyle g} \\ X' & \underset{\langle 1,0\rangle}{\overset{\pi_X}{\rightleftarrows}} & R(X',B',\rho',\varphi',\gamma') & \underset{\langle 0,1\rangle}{\overset{\pi_B}{\rightleftarrows}} & B \end{array} \tag{28}$$

is a morphism in $\mathbf{Psb}$.

The functor R realizes a pointed monoid action system $(X, B, \rho, \varphi, \gamma)$ as a synthetic semibiproduct diagram

$$X \underset{\langle 1,0\rangle}{\overset{\pi_X}{\rightleftarrows}} R \underset{\langle 0,1\rangle}{\overset{\pi_B}{\rightleftarrows}} B \tag{29}$$

in which $R = R(X, B, \rho, \varphi, \gamma) = \{(x, b) \in X \times B \mid \rho(x, b) = x\}$ is equipped with the binary synthetic operation

$$(x, b) + (x', b') = (\rho(x + \varphi(b, x') + \gamma(b, b'), b + b'), b + b') \tag{30}$$

which is well defined for every $x, x' \in X$ and $b, b' \in B$ due to condition (21) and is associative due to condition (24). It is clear that π_B is a monoid homomorphism and due to conditions (18)–(20) we see that the maps $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are well defined and moreover $\langle 1, 0\rangle$ is a monoid homomorphism. Finally, we observe that a pair $(x, b) \in X \times B$ is in R if and only if $(x, b) = (x, 0) + (0, b)$.

Further details on the more general situation of (not necessarily pointed) semibiproducts of semigroups rather than monoids can be found in the preprint [22].

6. The equivalence

In order to establish a categorical equivalence between **Act** and **Psb** we need a procedure to associate a pointed monoid action system to every pointed semibiproduct of monoids in a functorial manner.

6.1. Theorem. *Let (X, A, B, p, k, q, s) be an object in* **Psb**. *The system $(X, B, \rho, \varphi, \gamma)$ with*

$$\rho(x, b) = q(k(x) + s(b)) \tag{31}$$

$$\varphi(b, x) = q(s(b) + k(x)) \tag{32}$$

$$\gamma(b, b') = q(s(b) + s(b')) \tag{33}$$

is an object in **Act**. *Moreover, if (f_1, f_2, f_3) is a morphism in* **Psb** *then (f_1, f_3) is a morphism in* **Act**.

Proof. To see that the system $(X, B, \rho, \varphi, \gamma)$ is a well defined object in **Act** we recall that q and s are zero-preserving maps and hence conditions (18)–(20) are satisfied. Conditions (21)–(23) are obtained by applying the map q to both sides of equations

$$k(x) + s(b) = kq(k(x) + s(b)) + s(b)$$
$$s(b) + k(x) = kq(s(b) + k(x)) + s(b)$$
$$s(b) + s(b') = kq(s(b) + s(b')) + s(b + b')$$

which hold because (X, A, B, p, k, q, s) is a pointed semibiproduct of monoids. Condition (24) follows from Theorem 3.3 with $a = k(x) + s(b)$, $a' = k(x') + s(b') + k(x'') + s(b'')$ on the one hand whereas on the other hand $a = k(x) + s(b) + k(x') + s(b')$, $a' = k(x'') + s(b'')$. Moreover, the pair (f_1, f_3) is a morphism of actions as soon as the triple (f_1, f_2, f_3) is a morphism of semibiproducts, indeed we have

$$\begin{aligned} f_1(\rho(x, b)) &= f_1 q(k(x) + s(b)) = q' f_2(k(x) + s(b)) \\ &= q'(k' f_1(x) + s' f_3(b))) = \rho'(f_1(x), f_3(b)) \end{aligned}$$

and similarly for φ and γ thus proving conditions (25)–(27). ∎

The previous result describes a functor from the category of pointed semibiproducts of monoids into the category of pointed monoid action systems, let us denote it by $P\colon \mathbf{Psb} \to \mathbf{Act}$. The synthetic construction of Theorem 5.2 produces a functor in the other direction, let us denote it $Q\colon \mathbf{Act} \to \mathbf{Psb}$. We will see that $PQ = 1$ whereas $QP \cong 1$. It is clear that both constructions preserve composition and identity morphisms.

6.2. Theorem. *The categories* **Psb** *and* **Act** *are equivalent.*

Proof. Theorem 6.1 tells us that the assignment $P(X, A, B, p, k, q, s) = (X, B, \rho, \varphi, \gamma)$ and $P(f_1, f_2, f_3) = (f_1, f_3)$ is a functor from **Psb** to **Act** whereas Theorem 5.2 gives a functor Q in the other direction. It is clear that $Q(X, B, \rho, \varphi, \gamma)$ is the synthetic realization $(X, R, B, \pi_B, \langle 1, 0\rangle, \pi_X, \langle 0, 1\rangle)$ displayed in (29) and hence it is a pointed semibiproduct. Moreover $Q(f, g) = (f, R(f, g), g)$ with $R(f, g)$ illustrated as in (28) and defined as $R(f, g)(x, b) = (f(x), g(b))$ is clearly a morphism of semibiproducts.

We observe that $PQ(X, B, \rho, \varphi, \gamma) = (X, B, \rho, \varphi, \gamma)$ due to conditions (22) and (23). This proves $PQ = 1$, in order to prove $QP \cong 1$ we need to specify natural isomorphisms α and β as illustrated

$$\begin{array}{ccc} A & \underset{\beta_A}{\overset{\alpha_A}{\rightleftarrows}} & RP(X, A, B, p, k, q, s) \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle R(f_1, f_3)} \\ A' & \underset{\beta_{A'}}{\overset{\alpha_{A'}}{\rightleftarrows}} & RP(X', A', B', p', k', q', s') \end{array} \tag{34}$$

and show that they are compatible with diagrams (12) and (28). Indeed it is a routine calculation to check that $\alpha(a) = (q(a), p(a))$ and $\beta(x, b) = k(x) + s(b)$ are well defined natural isomorphisms compatible with semibiproducts. ∎

The particular case of groups can be found in more detail in the preprint [22].

7. Examples

Here we list all the possible pointed semibiproducts of monoids (X, A, B, p, k, q, s) in which X and B are monoids with two elements. This particular case is interesting because it gives a simple list with all the possible components of an action system $(X, B, \rho, \varphi, \gamma)$. The equivalence of Theorem 6.2 then gives us an easy way of checking all the possibilities. Let us denote by M and G the two monoids with two elements, M being the idempotent monoid while G being the group, both expressed in terms of multiplication tables as

$$M = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}, \quad G = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Note that we are using multiplicative notation so that $2 \cdot 2 = 2$ in M, whereas in G we have $2 \cdot 2 = 1$. Due to restrictions (18)–(20) we have the following two possibilities for each component ρ, φ and γ:

$$\rho_0 = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix},$$

$$\varphi_0 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \varphi_1 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix},$$

$$\gamma_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

The following list shows all the possible 14 cases of pointed semibiproducts of monoids (X, A, B, p, q, k, q, s) in which X and B are either M or G via the equivalence of Theorem 6.2.

1. $(G, G, \rho_0, \varphi_0, \gamma_0)$
2. $(G, G, \rho_0, \varphi_0, \gamma_1)$
3. $(G, M, \rho_0, \varphi_0, \gamma_0)$
4. $(G, M, \rho_0, \varphi_0, \gamma_1)$
5. $(G, M, \rho_0, \varphi_1, \gamma_0)$
6. $(G, M, \rho_1, \varphi_1, \gamma_0)$
7. $(M, G, \rho_0, \varphi_0, \gamma_0)$
8. $(M, G, \rho_0, \varphi_0, \gamma_1)$
9. $(M, G, \rho_1, \varphi_1, \gamma_1)$
10. $(M, M, \rho_0, \varphi_0, \gamma_0)$
11. $(M, M, \rho_0, \varphi_0, \gamma_1)$
12. $(M, M, \rho_0, \varphi_1, \gamma_0)$
13. $(M, M, \rho_0, \varphi_1, \gamma_1)$
14. $(M, M, \rho_1, \varphi_1, \gamma_0)$

Note that the cases with γ_0 correspond to split extensions while the cases with ρ_0 correspond to Schreier extensions. The cases with ρ_1 correspond to $R = \{(1,1), (1,2), (2,1)\}$ since $(2,2)$ fails to be in R because $\rho_1(2,2) = 1 \neq 2$. If interpreting φ as an action then the map φ_0 is the trivial action whereas φ_1 is a non-trivial action.

8. Conclusion

A new tool has been introduced for the study of monoid extensions from which a new notion of action has emerged in order to establish the categorical equivalence of Theorem 6.2. A clear drawback to this approach is the necessity of handling morphisms and maps at the same level. We have solved the problem by extending the hom-functor through an appropriate profunctor (Definition 3.1) in the fashion of imaginary morphisms [4, 5, 25]. Other possible solutions would consider maps as an extra structure in higher dimensions [9, 20, 21]. A further development of categorical frameworks in which to study semibiproducts seems desirable due to several important cases occurring in different settings. For example, semibiproduct extensions can be studied in the context of preordered monoids

[23] and preordered groups [10], where the maps q and s are required to be monotone maps rather than zero-preserving maps. The context of topological monoids [13] should also be worthwhile studying with q and s required to be continuous maps.

References

[1] F. Borceux, D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Mathematics and Its Applications 566, Kluwer 2004.

[2] D. Bourn, *From Groups to Categorial Algebra : Introduction to Protomodular and Mal'tsev Categories*, Compact Textbooks in Mathematics, Birkhäuser 2017.

[3] D. Bourn and G. Janelidze , *Protomodularity, descent, and semidirect products*, Theory and Applications of Categories, **4**(2) (1998) 37–46.

[4] D. Bourn and Z. Janelidze, *Subtractive Categories and Extended Subtractions*, Appl. Categor. Struct. **17** (2009) 317–343.

[5] D. Bourn and Z. Janelidze, *Categorical (binary) difference terms and protomodularity*, Algebra Universalis **66** (2011) 277–316.

[6] D. Bourn, N. Martins-Ferreira, A. Montoli and M. Sobral, *Schreier split epimorphisms in monoids and in semirings*, Textos de Matemática Série B **45** (Departamento de Matemática, Universidade de Coimbra) 2013.

[7] D. Bourn, N. Martins-Ferreira, A. Montoli, M. Sobral, *Schreier split epimorphisms between monoids*, Semigroup Forum **88** (2014) 739–752.

[8] D. Bourn, N. Martins-Ferreira, A. Montoli and M. Sobral, *Monoids and pointed S-protomodular categories*, Homology, Homotopy and Applications **48**(1) (2016) 151–172.

[9] R. Brown, *Possible connections between wiskered categories and groupoids, Leibniz algebras, automorphism structures and local-to-global questions*, Journal of Homotopy and Related Structures **1**(1) (2010) 1–13.

[10] M. M. Clementino, N. Martins-ferreira and A. Montoli, *On the categorical behaviour of preordered groups*, J. Pure Appl. Algebra **223** (2019) 4226–4245.

[11] P. Faul, *A characterization of weakly Schreier extensions of monoids*, J. Pure Appl. Algebra **225**(2) (2021) article no. 106489.

[12] I. Fleischer, *Monoid extension theory*, J. Pure Appl. Algebra **21** (1981) 15l–159.

[13] J. J. Ganci, *Schreier extensions of topological semigroups*, PhD, Louisiana State University (1975).

[14] M. Gran, G. Janelidze and M. Sobral, *Split extensions and semidirect products of unitary magmas*, Comment. Math. Univ. Carolin. **60**(4) (2019) 509–527.

[15] G. Janelidze, L. Márki, W. Tholen, Semi-abelian categories, J. Pure Appl. Algebra **168** (2002) 367–386.

[16] J. Leech, *Extending groups by monoids*, Journal of Algebra **74** (1982) 1–19.

[17] S. MacLane, *Duality for groups*, Bull. Amer. Math. Soc. **56**(6) (1950) 485–516.

[18] S. Mac Lane, *Homology*, Springer-Verlag, Berlin, 1963.

[19] S. Mac Lane, *Cohomology Theory in Abstract Groups. III* Annals of Mathematics, Second Series, **50**(3) (1949) 736–761.

[20] N. Martins-Ferreira, *On the notion of pseudocategory internal to a category with a 2-cell structure*, Tbil. Math. J. **8** (1) (2015) 107–141.

[21] N. Martins-Ferreira, *Semi-biproducts in monoids*, arXiv: 2002.05985v1 (2020).

[22] N. Martins-Ferreira, *On semibiproducts of magmas and semigroups*, arXiv:2208.12704 (2022).

[23] N. Martins-Ferreira and M. Sobral, *Schreier split extensions of preordered monoids*, Journal of Logical and Algebraic Methods in Programming **120** (2021).

[24] N. Martins-Ferreira, A. Montoli, M. Sobral and A. Patchkoria, *On the classification of Schreier extensions of monoids with non-abelian kernel*, Forum Math. **32** (3) (2020) 607–623.

[25] A. Montoli, D. Rodelo and T. Van der Linden, *Intrinsic Schreier split extensions*, Appl. Categ. Structures **28** (2020) 517–538.

[26] D. G. Northcott, *An introduction to Homological Algebra*, Cambridge University Press, 1960.

[27] C. Wells, *Extension theories for monoids*, Semigroup Forum **16** (1978) 13–35.

Center for Rapid and Sustainable Product Development
Department of Mathematics - ESTG
Polytechnic of Leiria, Portugal
Email: martins.ferreira@ipleiria.pt

This article may be accessed at http://www.tac.mta.ca/tac/