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A COMONAD FOR GROTHENDIECK FIBRATIONS

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Abstract. We prove that cloven Grothendieck fibrations over a fixed base $\mathcal{B}$ are the pseudo-coalgebras for a lax idempotent 2-comonad on $\mathcal{C}at/\mathcal{B}$. We show this via an original observation that the known colax idempotent 2-monad for fibrations over a fixed base has a right 2-adjoint. As an important consequence, we obtain an original cofree construction of a fibration on a functor. We also give a new, conceptual proof of the fact that the forgetful 2-functor from split fibrations to cloven fibrations over a fixed base has both a left 2-adjoint and a right 2-adjoint, in terms of coherence phenomena of strictification of pseudo-(co)algebras. The 2-monad for fibrations yields the left splitting and the 2-comonad yields the right splitting. Moreover, we show that the constructions induced by these coherence theorems recover Giraud's explicit constructions of the left and the right splittings.

1. Introduction

Grothendieck introduced fibrations in the late 1950s for the purpose of studying descent in algebraic geometry. Fibred categories, as fibrations are also called, were then studied, among others, by Bénabou, Giraud [1971], Gray [1966], and Street [1974]. Bénabou developed the theory in the 1970s for the purpose of doing category theory over a general base topos and even more generally over categories with finite limits. Unfortunately, most of the material is unpublished. Copies of original manuscripts are available online at `www2.mathematik.tu-darmstadt.de/~streicher/FibCatTexts`, see also the notes by Streicher [2022]. Marta Bunge was a strong advocate for his approach, see *e.g.* [Bunge, 1979; Bunge and Paré, 1979; Bunge and Hermida, 2011].

An important well-known result is that cloven fibrations over a fixed base $\mathcal{B}$ are the pseudo-algebras for a colax idempotent 2-monad $\mathbf{M}$ on $\mathcal{C}at/\mathcal{B}$. Strict algebras for such 2-monad correspond to split fibrations. (Co)lax idempotent 2-monads were introduced by Kock in his PhD thesis [Kock, 1967], with examples coming from colimit completion processes. Zöberlein then studied the corresponding concept for pseudo-monads in his PhD thesis [Zöberlein, 1976]. The idea of a (co)lax idempotent 2-monad is to encode a property-like structure, see also [Kock, 1995]. Indeed objects of the base can have at most one structure of pseudo-algebra for such 2-monads.

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The free fibration on a functor was firstly constructed by Gray. In [Gray, 1966, Theorem 3.9], he produced a left adjoint to the forgetful functor from the category $\mathcal{SpFib}(\mathcal{B})$ of split fibrations over $\mathcal{B}$ and functors that preserve the cleavage on the nose into the slice category $\mathcal{Cat}/\mathcal{B}$. The monadicity of this forgetful functor has then been known to some extent for many years. A complete proof appeared recently in Chapter 9 of Johnson and Yau's book [2021]. The algebraic character of fibrations over functors was further clarified by Street, who introduced fibrations over a fixed base internal to a (suitable) 2-category as pseudo-algebras for a colax idempotent 2-monad, which generalizes the one for Grothendieck fibrations, see [Street, 1974, pp. 118, 122] where opfibrations and fibrations are respectively called 0-fibrations and 1-fibrations.

One of the main results of the paper is that cloven fibrations over a fixed base $\mathcal{B}$ are also the pseudo-coalgebras for a lax idempotent 2-comonad on $\mathcal{Cat}/\mathcal{B}$ (Theorem 3.4). We prove this by observing that the underlying 2-functor M of the 2-monad $\mathbf{M}$ can be expressed as the composition of two left adjoint 2-functors. So that M has a right 2-adjoint N. This observation is original. By an extension of the classical argument of Eilenberg and Moore [1965, Proposition 3.3], we can conclude that N underlies a lax idempotent 2-comonad $\mathbf{N}$ whose pseudo-coalgebras are isomorphic to the pseudo-algebras for $\mathbf{M}$, and thus coincide with cloven fibrations. We also give an explicit description of the 2-comonad $\mathbf{N}$ for fibrations in Remark 3.6, obtained via mating calculus.

The comonadicity result for Grothendieck fibrations has many important consequences. Notably, it provides in particular an original cofree construction of a fibration on a functor (Theorem 4.1). This gives a right adjoint to the forgetful functor from the category $\mathcal{SpFib}(\mathcal{B})$ of split fibrations over $\mathcal{B}$ and functors that preserve the cleavage on the nose into the the slice category $\mathcal{Cat}/\mathcal{B}$. Such right adjoint was devised by the second author in his MSc thesis under the supervision of the third author. We could not find previous traces of it in the literature.

Another main result is a new, conceptual proof of the fact that the forgetful 2-functor from split fibrations to cloven fibrations over a fixed base has both a left 2-adjoint and a right 2-adjoint (Theorem 5.4). We deduce this from the monadicity and the comonadicity results for Grothendieck fibrations. Both explicit left adjoint and right adjoint splittings of fibrations were introduced in [Giraud, 1971, Théorèmes 2.4.2 et 2.4.4], which is unfortunately hard to read. The right splitting, with a proof that the counit consists of equivalences, then appeared in Bénabou's lectures in 1974 at Montreal and in 1980 at Louvain-la-Neuve, with a construction based on his "fibered Yoneda lemma". The left splitting was brought to public attention again in [Kapulkin and Lumsdaine, 2021]. The two adjoints gained interest as they provide two ways of turning a fibration into an equivalent split one: a description of this can be found in [Streicher, 2022, Section 3].

We prove that both left and right splittings coincide with coherence phenomena of strictification of pseudo-(co)algebras, studied by Power [1989] and Lack [2002]. The 2-monad $\mathbf{M}$ yields the left adjoint splitting as the left adjoint to the forgetful from strict algebras to pseudo-algebras. The 2-comonad $\mathbf{N}$ yields the right adjoint splitting as the right adjoint to the forgetful from strict coalgebras to pseudo-coalgebras.

Moreover, we show that the recipes described in [Lack, 2002] and [Power, 1989] to strictify pseudo-(co)algebras recover the explicit constructions of the right and left splittings given by Giraud [1971] (Remark 5.5 and Remark 5.8). We believe this sheds new light on Giraud's explicit constructions.

Outline of the paper. In Section 2, we recall that cloven fibrations over a fixed base $\mathcal{B}$ are the pseudo-algebras for a colax idempotent 2-monad $\mathbf{M}$ on $\mathcal{C}at/\mathcal{B}$.

In Section 3, we prove that cloven fibrations over $\mathcal{B}$ are the pseudo-coalgebras for a lax idempotent 2-comonad $\mathbf{N}$ on $\mathcal{C}at/\mathcal{B}$. We show this via an original observation that the 2-monad $\mathbf{M}$ has a right 2-adjoint.

In Section 4, we show that the comonadicity result of Section 3 provides in particular an original cofree construction of a fibration on a functor.

In Section 5, we give a new, conceptual proof of the two splittings of fibrations, in terms of coherence phenomena of strictification of pseudo-(co)algebras. We also show that the constructions induced by these coherence theorems recover Giraud's explicit construction of the left and right splitting.

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2. A colax idempotent monad for fibrations

In this section, we recall that Grothendieck fibrations over a fixed base $\mathcal{B}$ are the pseudo-algebras for a colax idempotent 2-monad on $\mathcal{C}at/\mathcal{B}$. The result has been known to some extent for many years. Gray [1966, Theorem 3.9(ii)] constructed a left adjoint to the inclusion into $\mathcal{C}at/\mathcal{B}$ of its sub-category $\mathcal{S}p\mathcal{F}ib(\mathcal{B})$ on split Grothendieck fibrations over $\mathcal{B}$ and functors that preserve the cleavage on the nose. It is quite straightforward to see that the canonical comparison functor into the (strict) algebras for the monad generated by Gray's adjunction is an isomorphism. By taking the 2-cells in $\mathcal{S}p\mathcal{F}ib(\mathcal{B})$ to be all 2-cells in $\mathcal{C}at/\mathcal{B}$, this upgrades to a 2-isomorphism between the 2-category of (strict) algebras and (strict) algebra morphisms and the 2-category $\mathcal{S}p\mathcal{F}ib(\mathcal{B})$. One can then see that the 2-category of pseudo-algebras is isomorphic to the 2-category $\mathcal{F}ib(\mathcal{B})$ of cloven fibrations and functors that preserve the cleavage up to isomorphism.

The monadicity result for fibrations played a crucial role in [Street, 1974], where fibrations in a 2-category are introduced precisely as the pseudo-algebras for a colax idempotent 2-monad which generalizes the one for Grothendieck fibrations. Recently, a complete proof of the monadicity result for fibrations appeared in Chapter 9 of [Johnson and Yau, 2021].

(Co)lax idempotent 2-monads were introduced by Kock in his PhD thesis [Kock, 1967], with examples coming from colimit completion processes. Zöberlein then studied the corresponding concept for pseudo-monads in his PhD thesis [Zöberlein, 1976]. Another useful reference is [Power et al., 2000].

2.1. Definition. [Kock, Zöberlein] *A 2-monad* $\mathbf{M} = (M, \mu, \eta)$ *on a 2-category* $\mathcal{K}$ *is called* colax idempotent (*or* coKZ) *if either of the following equivalent conditions holds:*

(i) *the multiplication* μ *is right adjoint left inverse of* ηM*;*

(ii) *any pseudo-algebra structure map* $\alpha\colon M(K) \to K$ *for* M *is right adjoint left inverse of the unit* η_K*.*

Notice that for a colax idempotent 2-monad $\mathbf{M}$ on $\mathcal{K}$, any object of $\mathcal{K}$ can have at most one structure of pseudo-algebra for $\mathbf{M}$ up to isomorphism. In this sense, a colax idempotent 2-monad encodes a property-like structure.

2.2. Remark. We would like to produce a free Grothendieck fibration on a functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$. So, for every object a in $\mathcal{A}$ and map $f\colon b \to F(a)$, we want to force the existence of a cartesian lifting of f to a. The idea is then to freely add all pairs $(b \xrightarrow{f} F(a), a)$, and thus to consider the comma category $\mathcal{B}/F$.[1] This can be compared for example to the construction of the free monoid on a set.

2.3. Proposition. [Gray, Street] *The 2-functor* $M\colon \mathcal{C}at/\mathcal{B} \longrightarrow \mathcal{C}at/\mathcal{B}$ *that maps a functor* $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ *into the functor on the left in the comma object diagram in* $\mathcal{C}at$

$$\begin{array}{ccc} \mathcal{B}/F & \xrightarrow{P_2} & \mathcal{A} \\ {\scriptstyle P_1}\big\downarrow & \nearrow\!\cdot & \big\downarrow{\scriptstyle F} \\ \mathcal{B} & \xrightarrow[\mathrm{Id}_{\mathcal{B}}]{} & \mathcal{B} \end{array}$$

extends to a colax idempotent 2-monad $\mathbf{M} = (M, \mu, \eta)$.

Proof. By the universal property of the comma object, the identity natural transformation induces the unit $\eta_F\colon \mathcal{A} \longrightarrow \mathcal{B}/F$ over $\mathcal{B}$. Explicitly,

$$\eta_F(a) = (F(a) \xrightarrow{\mathrm{id}} F(a), a).$$

Then the pasting

$$\begin{array}{ccccc} \mathcal{B}/P_1 & \xrightarrow{P_2'} & \mathcal{B}/F & \xrightarrow{P_2} & \mathcal{A} \\ {\scriptstyle P_1'}\big\downarrow & \nearrow\!\cdot & \big\downarrow{\scriptstyle P_1} & \nearrow\!\cdot & \big\downarrow{\scriptstyle F} \\ \mathcal{B} & \xrightarrow[\mathrm{Id}_{\mathcal{B}}]{} & \mathcal{B} & \xrightarrow[\mathrm{Id}_{\mathcal{B}}]{} & \mathcal{B} \end{array}$$

induces the multiplication $\mu_F\colon \mathcal{B}/P_1 \longrightarrow \mathcal{B}/F$ over $\mathcal{B}$. Explicitly,

$$\mu_F((b_1 \xrightarrow{f_1} b_2, (b_2 \xrightarrow{f_2} F(a), a))) = (b_1 \xrightarrow{f_2 f_1} F(a), a)$$

[1] We shall often write an object in a comma category $\mathcal{B}/F$ as a pair $(b \xrightarrow{f} F(a), a)$ instead of the appropriate triple $(b, a, b \xrightarrow{f} F(a))$ when no confusion arises.

and it acts on the arrows by pasting the two commutative squares.

Using the universal property of the comma object, it is straightforward to prove that (M, μ, η) is a colax idempotent 2-monad.

Explicitly, $\eta M \dashv \mu$ with the unit of the adjunction being the identity and counit with component $u_F \colon \eta_{M(F)}\mu_F \dashrightarrow \mathrm{Id}_{M(MF)}$ the natural transformation given by the family of arrows in $\mathcal{B}/M(MF)$, which component at the index $(b_1 \xrightarrow{f_1} b_2, (b_2 \xrightarrow{f_2} F(a), a))$ is

$$
\begin{array}{ccc}
b_1 & \xrightarrow{\mathrm{id}_{b_1}} & b_1 \\
{\scriptstyle \mathrm{id}_{b_1}}\downarrow & & \downarrow{\scriptstyle f_1} \\
b_1 & \xrightarrow[f_1]{} & b_2
\end{array}
\qquad\qquad
\begin{array}{ccc}
b_1 & \xrightarrow{f_2 f_1} & F(a) \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \mathrm{id}_{F(a)}} \\
b_2 & \xrightarrow[f_2]{} & F(a)
\end{array}
$$

from $(b_1 \xrightarrow{\mathrm{id}} F(a), (b_1 \xrightarrow{f_2 f_1} F(a), a))$ to $(b_1 \xrightarrow{f_1} b_2, (b_2 \xrightarrow{f_2} F(a), a))$. ■

2.4. Remark. Clearly the functor underlying the 2-monad **M** is polynomial, see [Gambino and Kock, 2013].

2.5. Remark. The 2-monad **M** extends to a colax idempotent 2-monad on the 2-category $\mathcal{C}at^2$ of arrows of $\mathcal{C}at$, that applies $\mathcal{C}at/B$ into itself and commutes with change-of-base functors. However, such extension does not seem to work well with the rest of the theory. See for example Remark 2.7 and Remark 3.5.

The following result follows from the theory of fibrations internal to a 2-category as developed by Street [1974, 1980]. The argument we present is based on the well-known fact that fibrations can be characterized as those functors F such that the unit of **M** on F has a right adjoint in $\mathcal{C}at/B$, see for instance [Weber, 2007, Theorem 2.7]. This is an instance of the general phenomenon observed by Kock [1995], with the caveat that Kock only considers left adjoint right inverses (*lari*) to the unit since he is working with *lax*-idempotent monads (hence the left instead of right), and *normal* pseudo-algebras (hence the right inverse). About the latter observation, see also the discussion in [Street, 1974, p. 120] following the proof of Proposition 9.

As we already recalled at the beginning of the section, Johnson and Yau [2021] devote Chapter 9 to a detailed proof of the following theorem.

2.6. Theorem. [Street, Johnson–Yau] *The* 2*-category of pseudo-algebras for the colax idempotent 2-monad* **M** *on* $\mathcal{C}at/\mathcal{B}$ *is isomorphic to the* 2*-category* $\mathcal{F}ib(\mathcal{B})$ *of cloven Grothendieck fibrations over* $\mathcal{B}$ *and functors that preserve the cleavage up to isomorphism.*

The 2-category of strict algebras for **M** *is isomorphic to the 2-category* $\mathcal{S}p\mathcal{F}ib(\mathcal{B})$ *of split Grothendieck fibrations over* $\mathcal{B}$ *and functors that preserve the cleavage on the nose.*

Proof. By the characterization theorem for pseudo-algebras for a colax idempotent monad of [Kock, 1995] (see also Definition 2.1), a functor $F \colon \mathcal{A} \longrightarrow \mathcal{B}$ sustains a structure of pseudo-algebra for $\mathbf{M}$ if and only if the unit

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\eta_F} & \mathcal{B}/F \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle M(F)} \\ \mathcal{B} & \xrightarrow[\mathrm{Id}_{\mathcal{B}}]{} & \mathcal{B} \end{array}$$

has a right adjoint α in $\mathcal{C}at/\mathcal{B}$.[2] For every a in $\mathcal{A}$ and $f\colon b \to F(a)$ in $\mathcal{B}$, the object $\alpha(b{-}f{\to}a, F(a))$ is over b and the component of the counit of $\eta_F \dashv \alpha$ on $(b{-}f{\to}a, F(a))$ is a lifting of f to a, since the counit is vertical. The couniversality of the counit precisely translates as such lifting being cartesian.

The strict axioms for strict algebras translates as the cloven fibration being split. ∎

2.7. Remarks. (a) Recall from [Street, 1980] that a functor $F\colon \mathcal{A} \to \mathcal{B}$ is a *Street fibration* if, for every object a in $\mathcal{A}$ and arrow $f\colon b \to F(a)$, there are an isomorphism $h_a\colon b_a \to b$ and a cartesian arrow into a over the composite $fh_a\colon b_a \to F(a)$. Street fibrations can be characterized as pseudo-algebras for the "same" monad $\mathbf{M}$ of Grothendieck fibrations, but lifted to the pseudo-fibre $\mathcal{C}at/\!/\mathcal{B}$, see [Street, 1980]. The 2-category $\mathcal{C}at/\!/\mathcal{B}$ has the same objects as $\mathcal{C}at/\mathcal{B}$, 1-cells are triangles filled with a natural isomorphism, and 2-cells are those 2-cells in $\mathcal{C}at/\mathcal{B}$ which commute with the two natural isomorphisms. For the characterization of Street fibrations, it is enough to observe that the underlying functor of $\mathbf{M}$ lifts, and that unit and counit are still natural. The rest of the proof works the same as for Proposition 2.6.
(b) As $\mathbf{M}$ is a monad also on the whole 2-category $\mathcal{C}at^2$, we can also look at the pseudo-algebras there. It is easy to see that the strict algebras are again split Grothendieck fibrations. Following Kock [1995], pseudo-algebras for $\mathbf{M}$ on $\mathcal{C}at^2$ can also be characterized as those functors F such that the unit (Id, η_F) has a right adjoint in $\mathcal{C}at^2$. Let

$$\begin{array}{ccc} \mathcal{B}/F & \xrightarrow{A} & \mathcal{A} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle M(F)} \\ \mathcal{B} & \xrightarrow{B} & \mathcal{B} \end{array}$$

be such a right adjoint. In particular, the functor B is isomorphic to the identity on $\mathcal{B}$ via an isomorphism $\zeta\colon B \to \mathrm{Id}_{\mathcal{B}}$. Unfolding the other conditions one sees that for every object a in $\mathcal{A}$ and arrow $f\colon b \to F(a)$, there is a cartesian arrow into a over $f\zeta_b\colon B(b) \to F(a)$. It follows that a pseudo-algebra for $\mathbf{M}$ in $\mathcal{C}at^2$ is a Street fibration. However, the converse does not seem to hold. Indeed, in both cases, given a in $\mathcal{A}$ and $f\colon b \to F(a)$, we can choose an isomorphism h such that the composite fh has a cartesian lift to a. But in a Street fibration the choice of h depends on the object a, whereas in a pseudo-algebra in $\mathcal{C}at^2$ the choice is given uniformly for every object a by the isomorphism ζ_b.

[2]Indeed, Kock [1995] shows just a bijection between the objects.

Notice also that a normal pseudo-algebra in $\mathcal{C}at^2$, being a right adjoint right inverse to the unit, is in particular an arrow in $\mathcal{C}at/\mathcal{B}$ and, in fact, a right adjoint right inverse to the unit in $\mathcal{C}at/\mathcal{B}$. It follows that a normal pseudo-algebra in $\mathcal{C}at^2$ is a normal Grothendieck fibration. This fact suggests that (normal) Street fibrations are strictly more general than pseudo-algebras in $\mathcal{C}at^2$.

3. A lax idempotent comonad for fibrations

In this section, we prove an original result of comonadicity for Grothendieck fibrations. More precisely, we prove that fibrations over a fixed base $\mathcal{B}$ are also the pseudo-coalgebras for a lax idempotent 2-comonad on $\mathcal{C}at/B$. We show this by an original observation that the 2-monad $\mathbf{M}$ has a right 2-adjoint.

We will see in the following sections that the comonadicity theorem has important consequences. It provides in particular an original cofree construction of a fibration on a functor (Theorem 4.1). Moreover the monad and the comonad induce respectively the left splitting and the right splitting of fibrations (Theorem 5.4, Remark 5.5, Remark 5.8). The comonadicity of fibrations has also consequences for (higher) elementary topos theory.

To reach the comonadicity result for Grothendieck fibrations, we notice the following useful equivalent description of the 2-monad.

3.1. Proposition. *The 2-functor underlying the 2-monad* $\mathbf{M}$ *coincides with the composition*

$$\mathcal{C}at/\mathcal{B} \xrightarrow{\ \mathrm{cod}^*\ } \mathcal{C}at/\mathcal{B}^2 \xrightarrow{\ \mathrm{dom}_\bullet\ } \mathcal{C}at/\mathcal{B}$$

of the 2-functor cod^* *that calculates pullbacks along* $\mathrm{cod}\colon \mathcal{B}^2 \longrightarrow \mathcal{B}$ *and the 2-functor* $\mathrm{dom}_\bullet$ *of postcomposition with* $\mathrm{dom}\colon \mathcal{B}^2 \longrightarrow \mathcal{B}$.

Proof. The comma object in $\mathcal{C}at$

$$\begin{array}{ccc} \mathcal{B}/F & \xrightarrow{P_1} & \mathcal{B} \\ {\scriptstyle P_2}\downarrow & \swarrow & \downarrow{\scriptstyle \mathrm{Id}_\mathcal{B}} \\ \mathcal{A} & \xrightarrow[F]{} & \mathcal{B} \end{array}$$

from $\mathrm{Id}_\mathcal{B}$ to F is equivalent to the pullback of F along the lax limit of the arrow $\mathrm{Id}_\mathcal{B}$ (that acts as a replacement):

$$\begin{array}{ccccc} \mathcal{B}/F & \longrightarrow & \mathcal{B}^2 & \xrightarrow{\mathrm{dom}} & \mathcal{B} \\ {\scriptstyle P_2}\downarrow & & {\scriptstyle \mathrm{cod}}\downarrow & \swarrow & \downarrow{\scriptstyle \mathrm{Id}_\mathcal{B}} \\ \mathcal{A} & \xrightarrow[F]{} & \mathcal{B} & \xrightarrow[\mathrm{Id}_\mathcal{B}]{} & \mathcal{B} \end{array}$$

∎

3.2. Remark. The 2-monad $\mathbf{M}$ can thus be expressed as the composition of left adjoint 2-functors. This observation is original.

The adjunction $\mathrm{dom}_\bullet \dashv \mathrm{dom}^*$ is the usual change of base. And the functor cod^* has indeed a right 2-adjoint cod_* because cod is an opfibration. Hence it is 2-exponentiable as stated in Giraud [1964, Théorème 4.4] and in Conduché [1972, 2nd Proposition on p. 894], but for an explicit proof of the characterisation of exponentiable functors, see Street [1986] where such functors are called *powerful*. See also Street and Verity [2010, Theorem 2.16] for the extension to the case of internal categories in a cartesian closed category with pullbacks.

3.3. Proposition. *The 2-functor underlying the 2-monad* $\mathbf{M}$ *has a right 2-adjoint* N*, expressed as the composite*

$$\mathcal{C}at/\mathcal{B} \xrightarrow{\mathrm{dom}^*} \mathcal{C}at/\mathcal{B}^2 \xrightarrow{\mathrm{cod}_*} \mathcal{C}at/\mathcal{B}$$

Therefore N *underlies a lax idempotent 2-comonad* $\mathbf{N} = (N, \mu', \eta')$ *on* $\mathcal{C}at/\mathcal{B}$*.*

Proof. The mating calculus ensures that the double category of left adjoints in a 2-category is isomorphic to the double category of right adjoints, see [Kelly and Street, 1974]. It follows that the right 2-adjoint of a colax idempotent 2-monad underlies a lax idempotent 2-comonad. ■

For 1-dimensional monads, Eilenberg and Moore [1965] showed that the coalgebras for the right adjoint of a monad coincide with the algebras for the monad. Lauda [2006] then proved that this works for pseudo-monads as well, obtaining a 2-equivalence between the pseudo-coalgebras and the pseudo-algebras. In the present case, we can prove the following stricter result.

3.4. Theorem. *The* 2*-category of pseudo-coalgebras for the lax idempotent 2-comonad* $\mathbf{N}$ *on* $\mathcal{C}at/\mathcal{B}$ *is isomorphic to the* 2*-category* $\mathcal{F}ib(\mathcal{B})$ *of cloven Grothendieck fibrations over* $\mathcal{B}$ *and functors that preserve the cleavage up to isomorphism.*

The 2-category of strict coalgebras for $\mathbf{N}$ *is isomorphic to the 2-category* $\mathcal{S}p\mathcal{F}ib(\mathcal{B})$ *of split Grothendieck fibrations over* $\mathcal{B}$ *and functors that preserve the cleavage on the nose.*

Proof. The isomorphism between the double category of left adjoints in a 2-category and the double category of right adjoints [Kelly and Street, 1974] (via mating calculus) transforms the 2-category of pseudo-algebras for $\mathbf{M}$ into the the 2-category of pseudo-coalgebras for $\mathbf{N}$. Indeed, Lauda [2006] showed that a right pseudo-adjoint to a pseudo-monad in a Gray-category is a pseudo-comonad with pseudo-coalgebras 2-equivalent to the pseudo-algebras of the pseudo-monad. We notice that, when the pseudo-adjunction is a strict 2-adjunction and the interchange rule is strict, the 2-equivalence becomes a 2-isomorphism. We can then conclude by Theorem 2.6. ■

3.5. Remark. For the comonadicity of fibrations, it is essential to restrict to fibrations over a fixed base $\mathcal{B}$. The factorization of Proposition 3.1 indeed only holds for the 2-monad restricted to $\mathcal{C}at/\mathcal{B}$.

3.6. Remark. We can calculate the 2-comonad $\mathbf{N} = (\mathrm{cod}_*\mathrm{dom}^*, \mu', \eta')$ more explicitly. Indeed, as observed by Giraud [1964], the isomorphism between homsets that gives the adjunction $\mathrm{cod}^* \dashv \mathrm{cod}_*$ determines the explicit definition of cod_*, up to isomorphism. Notice that dom^* sends every functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ to the forgetful $F/\mathcal{B} \longrightarrow \mathcal{B}^2$. Since the fibre of cod over b in $\mathcal{B}$ is $\mathcal{B}/b$, we find that $N(F)$ is the projection on the first component $\mathcal{G}_F \longrightarrow \mathcal{B}$, where $\mathcal{G}_F$ is the category defined as follows. Its objects are pairs $\langle b, X\rangle$ where b is an object in $\mathcal{B}$ and X is a functor such that

$$\begin{array}{ccc} \mathcal{B}/b & \xrightarrow{\quad X \quad} & \mathcal{A} \\ & {\scriptstyle \partial_0}\searrow \quad \swarrow{\scriptstyle F} & \\ & \mathcal{B} & \end{array}$$

commutes. An arrow $\langle f, \alpha\rangle : \langle b, X\rangle \longrightarrow \langle b', X'\rangle$ consists of an arrow $f\colon b \longrightarrow b'$ in $\mathcal{B}$ and a natural transformation

$$\begin{array}{ccc} \mathcal{B}/b & \xrightarrow{\quad X \quad} & \mathcal{A} \\ & \alpha \overset{\cdot}{\searrow} & \\ {\scriptstyle f \circ_{\mathcal{B}} -}\searrow & & \nearrow {\scriptstyle X'} \\ & \mathcal{B}/b' & \end{array}$$

with vertical components.
N then sends every morphism $H\colon F \longrightarrow F'$ in $\mathcal{C}at/\mathcal{B}$ to $\langle \mathrm{Id}, H \circ -\rangle : \mathcal{G}_F \longrightarrow \mathcal{G}_{F'}$, which is over $\mathcal{B}$. And the action of N on 2-cells is analogous.

By the mating calculus, the counit η' of the 2-comonad $\mathbf{N}$ is the composite

$$\mathrm{cod}_*\mathrm{dom}^* \xrightarrow[\eta\,\mathrm{cod}_*\mathrm{dom}^*]{\cdot} \mathrm{dom}_\bullet\mathrm{cod}^*\mathrm{cod}_*\mathrm{dom}^* \xrightarrow[\mathrm{dom}_\bullet\,\xi\,\mathrm{dom}^*]{\cdot} \mathrm{dom}_\bullet\mathrm{dom}^* \xrightarrow[\zeta]{\cdot} \mathrm{id},$$

where ξ is the counit of $\mathrm{cod}^* \dashv \mathrm{cod}_*$, whose components are evaluation functors, and ζ is the counit of $\mathrm{dom}_\bullet \dashv \mathrm{dom}^*$. Given a functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$, the component of η' on F is then given by "evaluating at the identity"

$$\begin{array}{ccc} \mathcal{G}_F & \xrightarrow{\quad \mathrm{E} \quad} & \mathcal{A} \\ \langle b, X\rangle & \longmapsto & X(\mathrm{id}_b) \\ \Big\downarrow \langle f, \alpha\rangle & \longmapsto & \Big\downarrow X'(f\colon f \to \mathrm{id}_{b'}) \circ \alpha_{\mathrm{id}_b} \\ \langle b', X'\rangle & \longmapsto & X'(\mathrm{id}_{b'}) \\ X(\mathrm{id}_b) & \xrightarrow{\ \alpha_{\mathrm{id}_b}\ } & X'(f) \end{array}$$

The comultiplication μ' of the 2-comonad $\mathbf{N}$ has component μ'_F on $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ given by the functor $\mathcal{G}_F \longrightarrow \mathcal{G}_{N(F)}$ that sends $\langle b, X\rangle$ to $\left\langle b, \overline{X}\right\rangle$ with

$$\begin{array}{ccc} \mathcal{B}/b & \xrightarrow{\quad\overline{X}\quad} & \mathcal{G}_F \\ (h\colon b' \to b) & \longmapsto & \langle b', X \circ (h \circ -)\rangle \end{array}$$

After Remark 3.8, we will be able to see μ'_F as the coalgebra structure map of the split fibration $N(F)$.

One could also use the isomorphism between homsets, given by the adjunction $M \dashv N$, as guaranteed by Proposition 3.3, to determine N directly as follows.

Given b in $\mathcal{B}$, write $\mathsf{y}(b)\colon \mathcal{B}/b \longrightarrow \mathcal{B}$ for the split fibration given by the (restriction of the) domain functor from the comma category $\mathcal{B}/b$ to $\mathcal{B}$. That assignment extends to a functor

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\quad\mathsf{y}\quad} & \mathrm{SpFib}(\mathcal{B}) \\ b_1 & \longmapsto & \mathcal{B}/b_1 \\ \Big\downarrow f & \longmapsto & f \circ_{\mathcal{B}} - \Big\downarrow \\ b_2 & \longmapsto & \mathcal{B}/b_2 \end{array}$$

3.7. Proposition. *Fix a functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ in $\mathbf{Cat}/\mathcal{B}$. Homming from each $\mathsf{y}(b)$ into F in the comma 2-category $\mathbf{Cat}/\mathcal{B}$*

$$\begin{array}{ccc} \mathcal{B}^{op} & \xrightarrow{\quad\widehat{F}\quad} & \mathbf{Cat} \\ b & \longmapsto & (\mathbf{Cat}/\mathcal{B})(\mathsf{y}(b), F) \\ \mathcal{B}^{op}\Big\downarrow f & \longmapsto & - \circ_{\mathbf{Cat}/\mathcal{B}} \mathsf{y}(f) \Big\downarrow \\ b' & \longmapsto & (\mathbf{Cat}/\mathcal{B})(\mathsf{y}(b'), F) \end{array}$$

gives a strict indexed category over $\mathcal{B}$, and its fibration of points

$$\int \widehat{F}\colon \mathcal{G}_F \longrightarrow \mathcal{B}$$

is precisely $N(F)$.

Note that it follows directly from Proposition 3.7 that $N(F)$ is split, which we know already from Theorem 3.4 as $N(F)$ is a cofree strict coalgebra.

3.8. Remark. The pseudo-coalgebra structure map of a cloven fibration $p\colon \mathcal{E} \longrightarrow \mathcal{B}$ is, explicitly,

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad\alpha\quad} & \mathcal{G}_F \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle N(p)} & \\ & \mathcal{B} & \end{array}$$

where α sends E in $\mathcal{E}$ to the triangle

$$\begin{array}{ccc} \mathcal{B}/p(E) & \xrightarrow{(-)^*E} & \mathcal{E} \\ & {\scriptstyle \partial_0} \searrow \quad \swarrow {\scriptstyle p} & \\ & \mathcal{B} & \end{array}$$

with $(-)^*E$ calculating the domains of the chosen cartesian liftings to E, and is extended by cartesianity to a functor $\alpha\colon \mathcal{E} \longrightarrow \mathcal{G}_F$.

The rest of the pseudo-coalgebra structure of p is given by the isomorphisms $\alpha_{\eta'}$ and $\alpha_{\mu'}$ that regulate respectively liftings in the chosen cleavage of p of an identity and of a composite.

The well-known observation that the component at F of the comultiplication μ'_F of the comonad N is precisely the coalgebra structure map of the split fibration $N(F)$ translates, in the notation of Remark 3.6, as

$$\overline{X} = (-)^* \langle b, X \rangle \,.$$

That is, $\overline{X}$ shows the explicit chosen cartesian liftings for the split fibration $N(F)$, which can be read from Proposition 3.7.

4. The cofree fibration on a functor

An important consequence of the comonadicity result is the fact that the forgetful 2-functor from the 2-category $\mathcal{SpFib}(\mathcal{B})$ of split fibrations over a fixed $\mathcal{B}$ to $\mathcal{Cat}/\mathcal{B}$ also has a right adjoint. This result seems not to appear in the literature.

4.1. Theorem. *The forgetful 2-functor $U\colon \mathcal{SpFib}(\mathcal{B}) \longrightarrow \mathcal{Cat}/\mathcal{B}$ has a right 2-adjoint. The cofree fibration on a functor $F\colon \mathcal{A} \longrightarrow \mathcal{B}$ is the split fibration $N(F)\colon \mathcal{G}_F \longrightarrow \mathcal{B}$, described in Remark 3.6 and Proposition 3.7.*

Proof. The right 2-adjoint to U is given by the forgetful-cofree adjunction given by the 2-comonad $\mathbf{N}$. By Theorem 3.4, the strict coalgebras for the 2-comonad $\mathbf{N}$ are precisely the split fibrations. ∎

4.2. Remark. The fact that N gives a right adjoint to U could also be proved directly, showing that the counit η' is 2-universal. The component on p of the unit of the adjunction is given by the coalgebra structure map of the split fibration p, described in Remark 3.8. However a direct proof is considerably more involved than the one that just follows from the factorization of the 2-monad as the composite of two left adjoint 2-functors. It is indeed much easier to calculate the right adjoint to the monad $\mathbf{M}$ than the right adjoint to U.

4.3. Remark. We briefly describe a strategy for a direct proof of the comonadicity of U.

In order to prove that a coalgebra $(F\colon \mathcal{A} \longrightarrow \mathcal{B}, \alpha\colon F \to N(F))$ for the 2-comonad $\mathbf{N}$ is a split fibration, start from a diagram

$$\begin{array}{ccc} & & a \\ & & \big\downarrow F \\ b & \xrightarrow[f]{} & F(a) \end{array}$$

and construct a cartesian lifting of f to a. The idea is to apply α to such a diagram and compute the cartesian lifting with respect to the split fibration $N(F)$ of f to $\alpha(a) = \langle F(a), X\colon \mathcal{B}/(F(a)) \longrightarrow \mathcal{A}\rangle$. Next, applying the counit η'_F recovers the starting data of the diagram above and exhibits a lifting of f to a:

$$\begin{array}{ccc} \eta'_F(\langle b, X \circ (f \circ -)\rangle) & \xrightarrow{\eta'_F(\langle f, \mathrm{id}\rangle)} & a \\ \big\downarrow F & & \big\downarrow F \\ b & \xrightarrow[f]{} & F(a) \end{array}$$

Note that, by Remark 3.6,

$$\eta'_F(\langle b, X \circ (f \circ -)\rangle) = X(f).$$

One is left with proving directly that such a lifting of f to a is cartesian, whose proof is lengthy. The proof of Theorem 3.4 uses instead what was already known for the 2-monad $\mathbf{M}$.

5. Recovering the two ways to split a fibration

In this section, we present a new, conceptual proof of the fact that the forgetful 2-functor from split fibrations to cloven fibrations over a fixed base has both a left 2-adjoint and a right 2-adjoint. Both explicit left and right splittings of fibrations were introduced in [Giraud, 1971, I.2.4].

We prove that both left and right splittings coincide with coherence phenomena of strictification of pseudo-(co)algebras. We show that such strictification adjoints are guaranteed for the monad $\mathbf{M}$ and the comonad $\mathbf{N}$ by the theorems of Lack [2002]. The 2-monad $\mathbf{M}$ yields the left adjoint splitting as the left adjoint to the forgetful from strict algebras to pseudo-algebras. The 2-comonad $\mathbf{N}$ yields the right adjoint splitting as the right adjoint to the forgetful from strict coalgebras to pseudo-coalgebras.

Moreover, we show that the recipes described in [Lack, 2002; Power, 1989] to strictify pseudo-(co)algebras concretely, recover the explicit constructions of right and left splittings given by Giraud [1971]. This sheds new light on Giraud's explicit constructions.

A conceptual proof of the two splittings of fibrations is obtained via a result of [Lack, 2002, Theorem 3.2], which we shall apply in its dual form, as we recall in the following.

5.1. Theorem. [Lack] *If T is a 2-comonad on a 2-category $\mathcal{K}$ admitting descent objects, and T preserves them, then the inclusion T-$CoAlg_s \longrightarrow Ps$-$T$-$CoAlg$ of strict coalgebras into pseudo-coalgebras has a right adjoint, and the components of the counit are equivalences in Ps-T-Alg. In particular this is the case if $\mathcal{K}$ has iso-inserters and equifiers, and T preserves these.*

The following two propositions show that the 2-monad $\mathbf{M}$ and the 2-comonad $\mathbf{N}$ satisfy the assumptions of Theorem 5.1, guaranteeing the existence of the strictification adjoints.

5.2. Proposition. *$\mathcal{C}at/\mathcal{B}$ has all weighted 2-colimits, created by the domain functor into $\mathcal{C}at$. In particular, it has all codescent objects.*

Moreover $\mathcal{C}at/\mathcal{B}$ also has all iso-inserters and equifiers, thus all descent objects.

Proof. $\mathcal{C}at/\mathcal{B}$ is the 2-category of strict coalgebras for the 2-comonad $- \times \mathcal{B}$ on $\mathcal{C}at$. So the forgetful into $\mathcal{C}at$ creates all weighted 2-colimits. Since $\mathcal{C}at$ is cocomplete as a 2-category, $\mathcal{C}at/\mathcal{B}$ is cocomplete as well. As shown in [Lack, 2002], codescent objects are particular weighted 2-colimits.

The iso-inserter of two functors $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$ over $\mathcal{B}$ is given by the category whose objects are all pairs (C, ϕ_C) with C in $\mathcal{C}$ and $\phi_C\colon F(C) \cong G(C)$ an isomorphism in $\mathcal{D}$ over the identity, and whose morphisms $(C, \phi_C) \to (C', \phi_{C'})$ are morphisms $f\colon C \to C'$ in $\mathcal{C}$ such that $G(f)\phi_C = \phi_{C'}F(f)$. We are thus restricting the usual inserter in $\mathcal{C}at$ taking only those ϕ_C that are vertical.

Equifiers in $\mathcal{C}at/\mathcal{B}$ are just calculated in $\mathcal{C}at$. By [Lack, 2002], $\mathcal{C}at/\mathcal{B}$ has then descent objects as well, as they can be produced via an iso-inserter followed by two equifiers. ■

5.3. Proposition. *The 2-monad $\mathbf{M}$ preserves codescent objects. The 2-comonad $\mathbf{N}$ preserves descent objects.*

Proof. By Proposition 3.3, $M \dashv N$. So M preserves all weighted 2-colimits and N preserves all weighted 2-limits. ■

We can now present a new, conceptual proof for the following theorem, which firstly appeared in [Giraud, 1971].

5.4. Theorem. *The forgetful 2-functor $V\colon \mathcal{S}p\mathcal{F}ib(\mathcal{B}) \longrightarrow \mathcal{F}ib(\mathcal{B})$ from split fibrations over $\mathcal{B}$ to cloven fibrations over $\mathcal{B}$ has both a left 2-adjoint L and a right 2-adjoint R.*

Moreover, the components of the unit of the adjunction $L \dashv V$ are equivalences in $\mathcal{F}ib(\mathcal{B})$. And the components of the counit of $V \dashv R$ are equivalences in $\mathcal{F}ib(\mathcal{B})$.

Proof. Thanks to Proposition 5.2 and Proposition 5.3, Lack [2002] (see Theorem 5.1) guarantees that the forgetful 2-functor from strict algebras to pseudo-algebras for $\mathbf{M}$ has a left 2-adjoint, such that the components of the unit are equivalences between pseudo-algebras. Dually, the forgetful 2-functor from strict coalgebras to pseudo-coalgebras for $\mathbf{N}$ has a right 2-adjoint, such that the components of the counit are equivalences between pseudo-coalgebras. Both forgetful 2-functors coincide (up to isomorphism) with V, by Theorem 2.6 and Theorem 3.4. ■

Since adjoints are unique up to isomorphism, the two splitting adjoints of Theorem 5.4 need to be realized by the explicit constructions of Giraud [1971]. But we can do more than this. Lack [2002] also gives a concrete recipe to calculate the strictification of pseudo-algebras, in terms of codescent objects. Dually, for the strictification of pseudo-coalgebras in terms of descent objects. In the rest of this section, we show that our conceptual proof of Theorem 5.4 also produces the explicit splitting constructions of Giraud [1971].

5.5. Remark. Using the explicit construction of descent objects in $\mathcal{Cat}/\mathcal{B}$ given in Proposition 5.2, one can recover the explicit construction of the right splitting of fibrations of Giraud [1971]. Let $p\colon \mathcal{E} \longrightarrow \mathcal{B}$ be a cloven fibration, with pseudo-coalgebra structure for the 2-comonad $\mathbf{N}$ given by a map $\alpha\colon p \to N(p)$ and isomorphisms $\alpha_{\eta'}$ and $\alpha_{\mu'}$ (see Remark 3.8). Following the concrete recipe of Lack [2002] to strictify pseudo-coalgebras, we have that $R(p)$ is the descent object in $\mathcal{SpFib}(\mathcal{B})$ of the coherence data

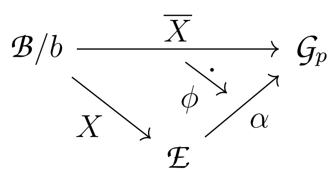
$$N(p) \;\underset{N\alpha}{\overset{\mu'_p}{\underset{\xleftarrow{N\eta'_p}}{\rightrightarrows}}}\; N^2(p) \;\underset{N^2\alpha}{\overset{\mu'_{Np}}{\underset{\xrightarrow{N\mu'_p}}{\rightrightarrows}}}\; N^3(p)$$

By Proposition 5.2 and Proposition 5.3, such descent object is calculated in $\mathcal{Cat}/\mathcal{B}$, as an inserter followed by two equifiers. The first step is to calculate the iso-inserter of

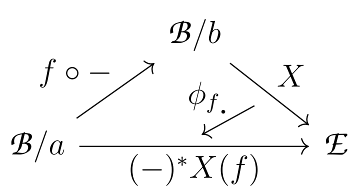
$$N(p) \underset{N\alpha}{\overset{\mu'_p}{\rightrightarrows}} N^2(p)$$

By Proposition 5.2, such iso-inserter is given by the category whose objects are all pairs $(\langle b, X\rangle, \phi)$ with $\langle b, X\rangle$ in $\mathcal{G}_p$ and $\phi\colon \mu'_p(\langle b, X\rangle) \cong (N\alpha)(\langle b, X\rangle)$ an isomorphism in $\mathcal{G}_{Np}$ over the identity, and whose morphisms $(\langle b, X\rangle, \phi) \to (\langle b', X'\rangle, \phi')$ are morphisms $\langle h, \lambda\rangle$ in $\mathcal{G}_p$ such that $(N\alpha)(\langle h, \lambda\rangle) \circ \phi = \phi' \circ \mu'_p(\langle h, \lambda\rangle)$.

In the notation of Remark 3.6, ϕ is a natural isomorphism



over $\mathcal{B}$. For every $f\colon a \to b$ in $\mathcal{B}$, the component ϕ_f of ϕ on f is given by a natural isomorphism



over $\mathcal{B}$. For every $u\colon a' \to a$ in $\mathcal{B}$, the component $\phi_{f,u}$ of ϕ_f on u is given by an isomorphism

$$\phi_{f,u}\colon X(fu) \cong u^* X(f)$$

in $\mathcal{E}$ over $\mathrm{id}_{a'}$. Naturality of ϕ_f on u yields in particular a commutative triangle

$$\begin{array}{ccc} X(fu) & \xrightarrow{X(u)} & X(f) \\ & {\scriptstyle \phi_{f,u}} \searrow \quad \nearrow {\scriptstyle \mathrm{Cart}(u, X(f))} & \\ & u^* X(f) & \end{array}$$

where $\mathrm{Cart}(u, X(f))$ is the chosen cartesian lifting of u to $X(f)$. This triangle subsumes all the other conditions ϕ needs to satisfy, by cartesianity arguments.

So an arbitrary object of the iso-inserter is equivalently given by $\langle b, X\rangle$ equipped with chosen isomorphisms $\phi_{f,u}$ that satisfy the commutative triangle above. Under the axiom of choice, this is equivalent to restrict to those $\langle b, X\rangle$ in $\mathcal{G}_p$ such that X is a cartesian functor, *i.e.* a functor that preserves the cleavage up to isomorphism.

The naturality-like condition that the morphisms of the iso-inserter need to satisfy is equally subsumed under the commutative triangle above. So that morphisms in the iso-inserter are simply all morphisms $\langle h, \lambda\rangle$ in $\mathcal{G}_p$.

In order to produce a descent object, we should then take two equifiers to force the following two conditions (the second one is a cocycle condition):

$$\phi_{f,\mathrm{id}} = \alpha_{\eta'}\colon X(f) \cong (\mathrm{id})^* X(f)$$

$$\begin{array}{ccc} X(fuv) & \xrightarrow{\phi_{f,uv}} & (uv)^* X(f) \\ {\scriptstyle \phi_{fu,v}} \downarrow & & \wr\| \, \alpha_{\mu'} \\ v^* X(fu) & \xrightarrow[v^* \phi_{f,u}]{} & v^* u^* X(f) \end{array}$$

where the isomorphisms $\alpha_{\eta'}$ and $\alpha_{\mu'}$ are those of the pseudo-coalgebra structure α on p, *i.e.* those given by the chosen cleavage of p (see Remark 3.8). Both conditions are however true for all $(\langle b, X\rangle, \phi)$, as they are subsumed by the commutative triangle above, by cartesianity arguments. So the iso-inserter calculated above is already the descent object we needed.

We conclude that the concrete recipe of Lack [2002] to strictify pseudo-coalgebras translates as taking $R(p)$ to be the restriction of $N(p)$ to those objects $\langle b, X\rangle$ with X that preserves the cleavage up to isomorphism (actually with chosen isomorphisms), without any further condition on morphisms. This is precisely the right splitting construction of Giraud [1971].

5.6. Remark. We note another way to recover the right adjoint splitting of fibrations. By Theorem 3.4, the pseudo-coalgebra structure α of a cloven fibration p is a right adjoint right inverse of the counit $\eta'_F\colon N(F) \longrightarrow F$. This α is a homomorphism of fibrations which embeds F into the split fibration $N(F)$. Closing under isomorphism the image of F into $N(F)$ provides a split fibration equivalent to F.

This recovers also the right splitting construction of Giraud [1971], as by Remark 3.8 such image is made of triangles

$$\begin{array}{ccc} \mathcal{B}/p(E) & \xrightarrow{(-)^*E} & \mathcal{E} \\ & {\scriptstyle \partial_0}\searrow \qquad \swarrow{\scriptstyle p} & \\ & \mathcal{B} & \end{array}$$

and $(-)^*E$ preserves the cleavage up to isomorphism.

In the following Remark 5.8, we recover the explicit construction of the left splitting of fibrations given in [Giraud, 1971]. Notice that the explicit construction of codescent objects in $\mathcal{C}at$ is much more convoluted than the one of descent objects. So it is better to use the concrete recipe of strictification of pseudo-coalgebras given by Power [1989], later refined by Lack [2002]. Such an approach uses enhanced factorization systems. We recall Lack's result [2002, Theorem 4.10], built on results of Power [1989].

5.7. Theorem. [Lack, Power] *If $\mathcal{K}$ is a 2-category with an enhanced factorization system $(\mathcal{E}, \mathcal{M})$ having the property that if j in $\mathcal{M}$ and $jk \cong 1$ then $kj \cong 1$, and if T is a 2-monad on $\mathcal{K}$ for which Tf in $\mathcal{E}$ whenever f in $\mathcal{E}$, then the inclusion T-$Alg_s \longrightarrow Ps$-T-Alg has a left adjoint, and the components of the unit of the adjunction are equivalences in Ps-T-Alg.*

While it seems that Remark 5.6 could as well be inscribed in this idea, it is hard to capture the essential image in a strict factorization system.

We now use Lack (and Power)'s concrete proof of Theorem 5.7 to recover Giraud's explicit construction of the left splitting of fibrations.

5.8. Remark. Recall that a factorization system on a 2-category is *enhanced* when squares which are pseudo morphisms in $\mathcal{C}at^2$ with domain in $\mathcal{E}$ and codomain in $\mathcal{M}$—so they commute up to an invertible 2-cell α—can be filled with a unique diagonal such that the upper triangle commutes strictly and the lower one commutes up to a unique invertible 2-cell which coincides with α on the whole square.

The factorization system on $\mathcal{C}at$ where $\mathcal{E}$ consists of the bijective-on-objects functors and $\mathcal{M}$ consists of the fully-faithful functors, is enhanced. This lifts to a factorization system on the 2-category $\mathcal{C}at/\mathcal{B}$, which is easily seen to be enhanced.

The other two assumptions of Theorem 5.7 are easily verified for the 2-monad $\mathbf{M}$. Indeed a fully faithful functor with a right quasi-inverse is clearly an equivalence. And it is easily seen from the definition of the 2-monad $\mathbf{M}$ (see Proposition 2.3) that M preserves bijective-on-objects functors. In fact, the action of M on morphisms in $\mathcal{C}at/\mathcal{B}$, induced by the universal property of the comma object, is almost trivial.

Power's insight is that an enhanced factorization system can be used to extract a strict algebra from a pseudo one. In particular, the underlying object of the strict algebra can be computed factoring the pseudo-algebra map.

Let $p\colon \mathcal{E} \longrightarrow \mathcal{B}$ be a cloven fibration, with pseudo-algebra structure for the 2-monad $\mathbf{M}$ given by a map $\alpha\colon M(p) \rightarrow p$ and isomorphisms α_η and α_μ that regulate respectively liftings in the chosen cleavage of p of an identity and of a composite. Using (the proof of) Theorem 5.7, we obtain that the underlying object of the left splitting $L(p)$ of p is the full image of the pseudo-algebra map $\alpha\colon M(p) \rightarrow p$. That is, the objects are the same of $M(p)$, but an arrow from (f, a) to (f', a') is an arrow $\alpha(f, a) \rightarrow \alpha(f', a')$ in $\mathcal{E}$ and thus an arrow $f^*a \rightarrow (f')^*a'$ in $\mathcal{E}$. This is precisely the left splitting construction of Giraud [1971].

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