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LAX COMMA 2-CATEGORIES AND ADMISSIBLE 2-FUNCTORS

In memory of Marta Bunge

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ABSTRACT. This paper is a contribution towards a two dimensional extension of the basic ideas and results of Janelidze's Galois theory. In the present paper, we give a suitable counterpart notion to that of *absolute admissible Galois structure* for the lax idempotent context, compatible with the context of *lax orthogonal factorization systems.* As part of this work, we study lax comma 2-categories, giving analogue results to the basic properties of the usual comma categories. We show that each morphism of a 2-category induces a 2-adjunction between lax comma 2-categories and comma 2-categories, playing the role of the usual *change-of-base functors.* With these induced 2-adjunctions, we are able to show that each 2-adjunction induces 2-adjunctions between lax comma 2-categories and comma 2-categories, which are our analogues of the usual lifting to the comma categories used in Janelidze's Galois theory. We give sufficient conditions under which these liftings are 2-premonadic and induce a lax idempotent 2-monad, which corresponds to our notion of 2-admissible 2-functor. In order to carry out this work, we analyse when a composition of 2-adjunctions is a lax idempotent 2-monad, and when it is 2-premonadic. We give then examples of our 2-admissible 2-functors (and, in particular, simple 2-functors), especially using a result that says that all admissible (2-)functors in the classical sense are also 2-admissible (and hence simple as well).

Contents

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Introduction

Categorical Galois theory, originally developed by Janelidze [17, 3], gives a unifying setting for most of the formerly introduced Galois type theorems, even generalizing most of them. It neatly gives a common ground for Magid's Galois theory of commutative rings, Grothendieck's theory of étale covering of schemes, and central extension of groups. Furthermore, since its genesis, Janelidze's Galois theory has found several developments, applications and examples in new settings (see, for instance, [5], [9], [34], [19, Theorem 4.2], and [28, Theorem 9.8]).

The most elementary observation on factorization systems and Janelidze's Galois theory is that, in the suitable setting of finitely complete categories, the notion of absolute admissible Galois structure coincides with that of a semi-left-exact reflective functor/adjunction (see, for instance, [3, Section 5.5] or [18]).

Motivated by the fact above and the theory of *lax orthogonal factorization systems* [8, 7, 10], we have started a project whose aim is to investigate a two dimensional extension of the basic ideas and results of (absolute) Janelidze's Galois theory. We deal herein with a key step of this endeavor, that is to say, we develop the basics in order to give a suitable counterpart notion to that of *absolute admissible Galois structure*.

We adopt the *usual* viewpoint that the 2-dimensional analogue of an idempotent monad (full reflective functor) is that of a lax idempotent monad (pre-Kock-Zöberlein 2-functor). Therefore the concept of an admissible Galois structure within our context should be a lax idempotent counterpart to the notion of *semi-left exact reflective functor*; namely, an appropriate notion of semi-left exact functor for the context of [10].

We study the lifting of 2-adjunctions to comma type 2-categories. We find two possible liftings which deserve interest. The underlying adjunction of the first type of lifting is the usual 1-dimensional case, while the other one, more relevant to our context, is a counterpart to the lifting of the 2-monad given in [10] by comma objects. The last one requires us to study the lax analogue notion for comma categories, the notion of *lax comma* 2-*categories* of the title.

We study the basic aspects of lax comma 2-categories. Among them, the 2-adjunction between the usual comma 2-category and the lax comma 2-category (for each object), and a counterpart for the usual change-of-base 2-functors, which comes into play as a fundamental aspect of our work and, specially, to introduce the definition of 2-*admissible* 2-*adjunction*.

With these analogues of the change-of-base 2-functors, we are able to introduce the lifting of each 2-adjunction to a 2-adjunction between the lax comma 2-category and the comma 2-category as a composition of 2-adjunctions. Namely, the composition of a straightforward lifting to the lax comma 2-categories with a change-of-base 2-functor induced by the appropriate component of the unit. Fully relying on the study of properties of compositions of 2-adjunctions, we investigate the properties of these liftings of the 2-adjunctions. Namely, we show under which conditions these liftings induce lax idempotent 2-monads (the simple 2-adjunctions of [10]), recovering one characterization given in [10] of their *simple* 2-*adjunctions*. We give also a characterization of the 2-functors whose introduced lifting is lax idempotent and 2-premonadic, the 2-*admissible* 2-*functors* within our context.

In Section 1 we recall basic aspects and terminology of 2-categories, such as 2-adjunctions and 2-monads, finishing the section giving aspects on *raris*, right-adjoint right-inverses (see Definition 1.2) within a 2-category. Taking the opportunity to fix notation, we also recall the universal properties of the main two dimensional limits used in our work in Section 2, that is to say, the definitions of conical 2-limits and comma objects.

In Section 3 we recall and show aspects on idempotent and lax idempotent 2-monads needed to our work on admissible and 2-admissible 2-functors, also introducing a characterization of the 2-adjunctions that induce lax idempotent 2-monads, called herein lax idempotent 2-adjunctions (see, for instance, Theorem 3.15).

In Section 4 we introduce the main concepts and results on composition of 2-adjunctions in order to introduce the notions of simple, admissible and 2-admissible 2-adjunctions (see, for instance, Definitions 4.4, 4.7, and 4.12). The results focus on characterizing and giving conditions under which the composition of 2-adjunctions is an idempotent/lax idempotent (full reflective/pre-Kock-Zöberlein) 2-adjunction (2-functor). Most of them are analogues for the simpler case of idempotent 2-adjunctions (see, for instance, Theorem 4.13 which characterizes when the composition of right 2-adjoints is pre-Kock-Zöberlein).

In Section 5 we recall the notion of lax comma 2-categories $\mathbb{A}//y$, for each 2-category $\mathbb{A}$ and object $y \in \mathbb{A}$ (see Definition 5.1). This notion has already appeared in the literature (see, for instance, [15, I,5], [14, § 6], [31, Exercise 5, p. 115] and [38, p. 305]). We, then, introduce the change-of-base 2-functors for lax comma 2-categories. More precisely, we show that, for each morphism $c : y \to z$ in a 2-category $\mathbb{A}$ with comma objects, we have an induced 2-adjunction

$$\begin{tikzcd}
\mathbb{A}//z \arrow[r, bend right, "c^{\Leftarrow}"'] \arrow[r, phantom, "\perp"] & \mathbb{A}/y . \arrow[l, bend right, "c\bar{!}"']
\end{tikzcd}$$

between the lax comma 2-category $\mathbb{A}//z$ and the comma 2-category $\mathbb{A}/y$. We give an explicit construction of this 2-adjunction: see Theorem 5.8.

Provided that $\mathbb{A}$ has pullbacks and comma objects, these induced 2-adjunctions, together with the classical change-of-base 2-functors, give the 2-adjunctions

$$\begin{tikzcd}
\mathbb{A}//z \arrow[r, bend right, "\mathrm{id}_z^{\Leftarrow}"'] \arrow[r, phantom, "\perp"] \arrow[rr, bend right=60, "c^{\Leftarrow}"'] & \mathbb{A}/z \arrow[l, bend right, "\mathrm{id}_z\bar{!}"'] \arrow[r, bend right, "c^*"'] \arrow[r, phantom, "\perp"] & \mathbb{A}/y \arrow[l, bend right, "c!"'] \arrow[ll, bend right=60, "c\bar{!}"']
\end{tikzcd}$$

in which the composition of $c! \dashv c^* : \mathbb{A}/z \to \mathbb{A}/y$ with $\mathrm{id}_z\bar{!} \dashv id_z^{\Leftarrow} : \mathbb{A}//z \to \mathbb{A}/z$ is, up to 2-natural isomorphism, the 2-adjunction $c\bar{!} \dashv c^{\Leftarrow} : \mathbb{A}//z \to \mathbb{A}/y$ (see Theorem 5.10). We finish Section 5 showing that, whenever it is well defined, $\mathrm{id}_y^{\Leftarrow}$ is pre-Kock-Zöberlein (Theorem 5.11).

The main point of Section 6 is to introduce our notions of admissibility and 2-admissibility (Definition 6.4), relying on the definitions previously introduced in Section 4. We also use the main results of Section 4 to characterize and give conditions under which a 2-functor is 2-admissible (see, for instance, Corollaries 6.10 and 6.11).

We finish Section 6 with a fundamental observation on admissibility and 2-admissibility, namely, Theorem 6.13. It says that, provided that $\mathbb{A}$ has comma objects, if $F \dashv G$ is admissible in the classical sense (called herein *admissible w.r.t. the basic fibration*), meaning that G itself is full reflective and the compositions

$$\eta_y^* \circ \check{G} : \mathbb{A}/F(y) \to \mathbb{B}/y$$

are full reflective for all y, then G is 2-admissible, which means that the compositions

$$\eta_y^{\Leftarrow} \circ \check{G} : \mathbb{A}//F(y) \to \mathbb{B}/y$$

are pre-Kock-Zöberlein for all objects y. We discuss examples of 2-admissible 2-functors (and hence also simple 2-functors) in Section 7. Most examples are about cocompletion of 2-categories, making use of Theorem 6.13.

1. Preliminaries

Let Cat be the cartesian closed category of categories in some universe. We denote the internal hom by

$$\mathsf{Cat}(-,-) : \mathsf{Cat}^{\mathrm{op}} \times \mathsf{Cat} \to \mathsf{Cat}.$$

A 2-category $\mathbb{A}$ herein is the same as a Cat-enriched category. We denote the enriched hom of a 2-category $\mathbb{A}$ by

$$\mathbb{A}(-,-) : \mathbb{A}^{\mathrm{op}} \times \mathbb{A} \to \mathsf{Cat}$$

which, again, is of course a 2-functor. As usual, the composition of 1-cells (morphisms) are denoted by $\circ$, $\cdot$, or omitted whenever it is clear from the context. The vertical composition of 2-cells is denoted by $\cdot$ or omitted when it is clear, while the horizontal composition is denoted by $*$. Recall that, from the vertical and horizontal compositions, we construct the fundamental operation of *pasting* [24, 35].

Finally, if $f : w \to x$, $g : y \to z$ are 1-cells of $\mathbb{A}$, given a 2-cell $\xi : h \Rightarrow h' : x \to y$, motivated by the case of $\mathbb{A} = \mathsf{Cat}$, we use interchangeably the notations

$$\mathrm{id}_g * \xi * \mathrm{id}_f \quad = \quad \begin{tikzcd} w \arrow[d, "f"] \\ x \arrow[d, bend right=60, "h'"'{name=L}] \arrow[d, bend left=60, "h"{name=R}] \\ y \arrow[d, "g"] \\ z \arrow[from=R, to=L, Rightarrow, "\xi"'] \end{tikzcd} \quad = \quad g\xi f \tag{1.0.1}$$

to denote the whiskering of ξ with f and g.

Henceforth, we consider the 3-category of 2-categories, 2-functors, 2-natural transformations and modifications, denoted by 2-Cat. We refer to [24, 36] for the basics on 2-dimensional category theory, and, more particularly, to the definitions of adjunctions, monads and Kan extensions inside a 2-category. Moreover, we also extensively assume aspects of 2-monad theory. The pioneering reference is [1], while we mostly follow the terminology (and results) of [27].

In this paper, we consider the *strict* versions of 2-dimensional adjunctions and monads: the concepts coincide with the Cat-enriched ones. A 2-*adjunction*, denoted by

$$(F \dashv G, \varepsilon, \eta) : \mathbb{A} \to \mathbb{B},$$

consists of 2-functors

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend right, "G"'] & \mathbb{B} \arrow[l, bend right, "F"'] \end{tikzcd}$$

with 2-natural transformations $\varepsilon : FG \Longrightarrow \mathrm{id}_{\mathbb{A}}$ and $\eta : \mathrm{id}_{\mathbb{B}} \Longrightarrow GF$ playing the role of the *counit* and the *unit* respectively. More precisely, the equations of 2-natural transformations

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "G"] \arrow[d, equal] \arrow[dr, phantom, "\overset{\varepsilon}{\Longleftarrow}", pos=0.25] \arrow[dr, phantom, "\overset{\eta}{\Longleftarrow}", pos=0.75] & \mathbb{B} \arrow[dl, "F" description] \arrow[d, equal] \\
\mathbb{A} \arrow[r, "G"'] & \mathbb{B}
\end{tikzcd} = \mathrm{id}_G \quad \text{and} \quad
\begin{tikzcd}
\mathbb{A} \arrow[d, equal] \arrow[dr, phantom, "\overset{\varepsilon}{\Longleftarrow}", pos=0.25] \arrow[dr, phantom, "\overset{\eta}{\Longleftarrow}", pos=0.75] & \mathbb{B} \arrow[l, "F"'] \arrow[d, equal] \\
\mathbb{A} \arrow[ur, "G" description] & \mathbb{B} \arrow[l, "F"]
\end{tikzcd} = \mathrm{id}_F \qquad \text{(triangle identities)}$$

hold. We usually denote a 2-adjunction $(F \dashv G, \varepsilon, \eta) : \mathbb{A} \to \mathbb{B}$ by

$$\begin{tikzcd}
\mathbb{A} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp(\varepsilon,\eta)"] & \mathbb{B} \arrow[l, bend right, "F"']
\end{tikzcd}$$

or by $F \dashv G : \mathbb{A} \to \mathbb{B}$ for short, when the counit and unit are already given.

A 2-*monad* on a 2-category $\mathbb{B}$ is a triple $\mathcal{T} = (T, \mu, \eta)$ in which $T : \mathbb{B} \to \mathbb{B}$ is an endo-2-functor and μ, η are 2-natural transformations playing the role of the multiplication and the unit respectively. That is to say, μ and η are 2-natural transformations such that the equations

$$\begin{tikzcd}
\mathbb{B} \arrow[dr, phantom, "\overset{\mu}{\Longleftarrow}", pos=0.25] \arrow[dr, phantom, "\overset{\mu}{\Longleftarrow}", pos=0.75] & \mathbb{B} \arrow[l, "T"'] \\
\mathbb{B} \arrow[u, "T"] \arrow[r, "T"'] \arrow[ur, "T" description] & \mathbb{B} \arrow[u, "T"']
\end{tikzcd} = \begin{tikzcd}
\mathbb{B} & \mathbb{B} \arrow[l, "T"'] \\
\mathbb{B} \arrow[u, "T"] \arrow[r, "T"'] \arrow[ur, phantom, "\overset{\mu}{\Longleftarrow}", pos=0.75] \arrow[ur, phantom, "\overset{\mu}{\Longleftarrow}", pos=0.25] & \mathbb{B} \arrow[u, "T"'] \arrow[ul, "T" description]
\end{tikzcd} \qquad \text{(associativity of a 2-monad)}$$

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "T" description] \arrow[r, bend left=60, equal, ""{name=U, below}] \arrow[dr, "T"'] & \mathbb{B} \arrow[d, "T"] \arrow[Rightarrow, from=U, "\eta"] \\
& \mathbb{B} \arrow[ul, phantom, "\Downarrow \mu", pos=0.3]
\end{tikzcd} = \begin{tikzcd}
\mathbb{B} \arrow[r, bend left=60, equal, ""{name=U, below}] & \mathbb{B} \arrow[l, "T" description] \arrow[Rightarrow, from=U, "\eta"] \\
& \mathbb{B} \arrow[ul, "T"] \arrow[u, "T"'] \arrow[ul, phantom, "\Downarrow \mu", pos=0.7]
\end{tikzcd} = \mathrm{id}_T \qquad \text{(identity of a 2-monad)}$$

hold.

Since the notions above coincide with the Cat-enriched ones, it should be noted that the formal theory of monads applies to this case. More precisely, every 2-adjunction does induce a 2-monad, and we have the usual Eilenberg-Moore and Kleisli factorizations of a right 2-adjoint functor (*e.g* [36, Section 2] or [30, Section 3]), which give rise respectively to the notions of 2-monadic and Kleisli 2-functors. Furthermore, we also have (the enriched version of) Beck's monadicity theorem [11, Theorem II.2.1].

In this direction, we use expressions like *equivalence (or* 2*-equivalence)*, and *fully faithful* 2*-functor* to mean the (strict) Cat-enriched notions: that is to say, respectively, *equivalence* in the 2-category of 2-categories, and a 2-functor that is *locally an isomorphism.*

1.1. Lalis and ralis. To refer to adjunctions where the unit or counit is an identity, we adopt a terminology similar to the one introduced by Gray in [13, 0.3.B]. More precisely:

1.2. Definition. Assume that $(f \dashv g, v, n)$ is an adjunction in a 2-category $\mathbb{A}$.

- If the counit v is the identity 2-cell, $(f \dashv g, v, n)$ is called a *rari adjunction (or rari pair)*, or a *lali adjunction.*

  If there is a rari adjunction $f \dashv g$, the morphism f is called a *lali (left-adjoint left-inverse)*, while the morphism g is called a *rari (right-adjoint right-inverse).*

- If the unit n is the identity 2-cell, $(f \dashv g, v, n)$ is called a *rali adjunction*, or a *lari adjunction.*

  If there is a rali adjunction $f \dashv g$, the morphism f is called a *lari*, while the morphism g is called a *rali.*

Laris (ralis) are closed by composition, and have specific cancellation properties. We recall them below.

1.3. Lemma. *Assume that*

$$\begin{tikzcd}
w \arrow[r, bend right, "g"'] \arrow[r, phantom, "\perp(v,n)"] & x \arrow[l, bend right, "f"'] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\perp(v',n')"] & y \arrow[l, bend right, "f'"']
\end{tikzcd} \qquad (1.3.1)$$

are adjunctions in $\mathbb{A}$.

a) *Assuming that $f \dashv g$ is a lari adjunction: we have that $ff' \dashv g'g$ is a lari adjunction if, and only if, $f' \dashv g'$ is a lari adjunction as well.*

b) *Assuming that $f' \dashv g'$ is a lali adjunction: the adjunction $ff' \dashv g'g$ is a lali adjunction if, and only if, $f \dashv g$ is a lali adjunction as well.*

Proof. Assuming that n is an isomorphism, we have that the unit

$$\begin{tikzcd}
& & y \arrow[dl, "f'"'] \arrow[dddd, equal] \\
& x \arrow[dl, "f"'] \arrow[dd, equal] & \\
w \arrow[dr, "g"'] \arrow[r, phantom, "\overset{n}{\Longleftarrow}"] & {} \arrow[r, phantom, "\overset{n'}{\Longleftarrow}"] & {} \\
& x \arrow[dr, "g'"'] & \\
& & y
\end{tikzcd} \qquad (1.3.2)$$

of the composition $ff' \dashv g'g$ is invertible if, and only if, n' is invertible. This proves b) and, dually, we get a). ■

Of course, the situation is simpler when we consider isomorphisms. That is to say:

1.4. Corollary. *Assume that*

$$\begin{tikzcd} w \arrow[r, bend right, "g'"'] & x \arrow[l, bend right, "f'"'] \arrow[r, bend right, "g"'] & y \arrow[l, bend right, "f"'] \arrow[r, bend right, "g''"'] & z \arrow[l, bend right, "f''"'] \end{tikzcd} \tag{1.4.1}$$

are morphisms in $\mathbb{A}$ *such that* $(f')^{-1} = g'$ *and* $(f'')^{-1} = g''$. *There is a lali (rali) adjunction* $f' \cdot f \cdot f'' \dashv g'' \cdot g \cdot g'$ *if and only if there is a lali (rali) adjunction* $f \dashv g$.

Proof. If $f \dashv g$ is a lali (rali) adjunction, since $f' \dashv g'$ and $f'' \dashv g''$ are of course lali and rali adjunctions, it follows that the composite

$$f' \cdot f \cdot f'' \dashv g'' \cdot g \cdot g'$$

is a lali (rali) adjunction by Lemma 1.3.

Conversely, if $f' \cdot f \cdot f'' \dashv g'' \cdot g \cdot g'$ is a lali (rali) adjunction, since $g' \dashv f'$ and $g'' \dashv f''$ are lali and rali adjunctions, we get that the composite

$$g' \cdot f' \cdot f \cdot f'' \cdot g'' \dashv f'' \cdot g'' \cdot g \cdot g' \cdot f',$$

which is $f \dashv g$, is a lali adjunction. ■

But we also have a stronger cancellation property:

1.5. Theorem. [Left cancellation property] *Let* $f : x \to w, f' : y \to x$ *be morphisms of a* 2*-category* $\mathbb{A}$.

a) Assuming that $f : x \to w$ *is a lari: the composite* $ff' : y \to w$ *is a lari if, and only if,* $f' : y \to x$ *is a lari as well.*

b) Assuming that f *is a rari: the composite* ff' *is a rari if and only if* f' *is a rari.*

Proof. By Lemma 1.3, if f and f' are laris, the composite ff' is a lari as well.

Conversely, assume that f and ff' are laris. This means that there are adjunctions

$$\begin{tikzcd} w \arrow[r, bend right, "g"'] \arrow[r, phantom, "\perp_{(v,n)}"] & x \arrow[l, bend right, "f"'] \end{tikzcd} \qquad \begin{tikzcd} w \arrow[r, bend right, "\hat{g}"'] \arrow[r, phantom, "\perp_{(\hat{v},\hat{n})}"] & y \arrow[l, bend right, "ff'"'] \end{tikzcd}$$

in $\mathbb{A}$ such that $n = \mathrm{id}_{gf}$ and $\hat{n} = \mathrm{id}_{\hat{g}ff'}$.

We claim that

$$\left(f' \dashv \hat{g}f, \quad \begin{tikzcd} x \arrow[r, "f"] \arrow[d, equal] & w \arrow[r, "\hat{g}"] \arrow[d, equal] & y \arrow[d, "f'"] \\ x \arrow[from=1-1, to=2-2, phantom, "="] & w \arrow[l, "g"] \arrow[from=1-3, to=2-2, phantom, "\overset{\hat{v}}{\Longleftarrow}"] & x \arrow[l, "f"] \arrow[ll, bend left=60, equal, "\parallel"] \end{tikzcd} \quad , \mathrm{id}_{\hat{g}ff'} \right) \tag{1.5.1}$$

is a (lari) adjunction. In fact, the triangle identities follow from the facts that the equations $\hat{v}ff' = \mathrm{id}_{ff'}$ and $\hat{g}\hat{v} = \mathrm{id}_{\hat{g}}$ hold.

Finally, the statement b) is the codual of a). ■

On the one hand, the *left cancellation property* of Theorem 1.5 does not hold for lalis or ralis. For instance, in Cat, we consider the terminal category $\mathsf{1}$ and the category $\mathsf{2}$ with two objects and only one nontrivial morphism between them. The morphisms

$$\begin{tikzcd} \mathsf{1} & \mathsf{2} \arrow[l, "s^0"] & \mathsf{1} \arrow[ll, bend right, "s^0d^0"'] \end{tikzcd} \tag{1.5.2}$$

are lalis. But the inclusion $d^0 : \mathsf{1} \to \mathsf{2}$ of the terminal object of $\mathsf{2}$ is not a lali, since it does not have a right adjoint. On the other hand, the dual of Theorem 1.5 gives a right cancellation property for ralis and lalis.

1.6. Corollary. [Right cancellation property] *Let $f : x \to w, f' : y \to x$ be morphisms of a 2-category $\mathbb{A}$. If $f' : y \to x$ is a lali (rali): we have that $f : x \to w$ is a lali (rali) if, and only if, the composite $ff' : y \to w$ is a lali (rali) as well.*

2. Two dimensional limits

In this section, we recall basic universal constructions related to the results of this paper. Two dimensional limits are the same as weighted limits in the Cat-enriched context [11]. We refer, for instance, to [37] for the basics on 2-dimensional limits. We are particularly interested in *conical* 2-*(co)limits* and *comma objects*.

2.1. Conical 2-limits. Two dimensional conical (co)limits are just weighted (co)limits with a weight constantly equal to the terminal category $\mathsf{1}$. Henceforth, the words *(co)product*, *pullback/pushout* and *(co)equalizer* refer to the 2-dimensional versions of each of those (co)limits. For instance, if $a : x \to y$, $b : w \to y$ are morphisms of a 2-category

$\mathbb{A}$, assuming its existence, the *pullback* of b along a is an object $x \times_{(a,b)} w$ together with 1-cells $a^*(b) : x \times_{(a,b)} w \to x$ and $b^*(a) : x \times_{(a,b)} w \to w$ making the diagram

$$\begin{tikzcd}
x \times_{(a,b)} w \arrow[r, "b^*(a)"] \arrow[d, "a^*(b)"'] & w \arrow[d, "b"] \\
x \arrow[r, "a"'] & y
\end{tikzcd} \tag{2.1.1}$$

commutative, and satisfying the following universal property. For every object z and every pair of 2-cells

$$(\xi_0 : h_0 \Rightarrow h_0' : z \to x,\ \xi_1 : h_1 \Rightarrow h_1' : z \to w)$$

such that the equation

$$\begin{tikzcd}
z \arrow[r, bend left=50, "h_1'"{name=U}] \arrow[r, bend right=50, "h_1"'{name=D}] \arrow[d, "h_0"'] & w \arrow[d, "b"] \\
x \arrow[r, "a"'] & y
\arrow[from=D, to=U, Rightarrow, "\xi_1"]
\arrow[from=1-1, to=2-2, phantom, "\|"]
\end{tikzcd}
=
\begin{tikzcd}
z \arrow[r, "h_1'"] \arrow[d, bend right=50, "h_0"'{name=L}] \arrow[d, bend left=50, "h_0'"{name=R}] & w \arrow[d, "b"] \\
x \arrow[r, "a"'] & y
\arrow[from=L, to=R, Rightarrow, "\xi_0"]
\arrow[from=1-1, to=2-2, phantom, "="]
\end{tikzcd} \tag{2.1.2}$$

holds, there is a unique 2-cell $\xi : h \Rightarrow h' : z \to x \times_{(a,b)} w$ satisfying the equations

$$\mathrm{id}_{a^*(b)} * \xi = \xi_0 \text{ and } \mathrm{id}_{b^*(a)} * \xi = \xi_1.$$

2.2. REMARK. It is clear that the concept of *pullback* in locally discrete 2-categories coincides with the concept of (1-dimensional) *pullback* in the underlying categories.

Moreover, when a *pullback* exists in a 2-category, it is isomorphic to the (1-dimensional) *pullback* in the underlying category.

Finally, both the statements above are also true if *pullback* is replaced by any type of conical 2-limit with a locally discrete *shape* (domain).

2.3. COMMA OBJECTS. If $a : x \to y$, $b : w \to y$ are morphisms of a 2-category $\mathbb{A}$, the comma object of a along b, if it exists, is an object $a \downarrow b$ with the following universal property. There are 1-cells $a^{\Rightarrow}(b) : a \downarrow b \to x$ and $b^{\Leftarrow}(a) : a \downarrow b \to w$ and a 2-cell

$$\begin{tikzcd}
a \downarrow b \arrow[r, "a^{\Rightarrow}(b)"] \arrow[d, "b^{\Leftarrow}(a)"'] \arrow[dr, phantom, "\overset{\chi^{a\downarrow b}}{\Longleftarrow}"] & x \arrow[d, "a"] \\
w \arrow[r, "b"'] & y
\end{tikzcd} \tag{2.3.1}$$

such that:

1. For every triple $(h_0 : z \to x, h_1 : z \to w, \gamma : ah_0 \Rightarrow bh_1)$ in which h_0, h_1 are morphisms and γ is a 2-cell of $\mathbb{A}$, there is a unique morphism $h : z \to a \downarrow b$ such that the equations $h_0 = a^{\Rightarrow}(b) \cdot h$, $h_1 = b^{\Leftarrow}(a) \cdot h$ and

$$\begin{tikzcd}
z \arrow[dr, "h"] & & & & & \\
& a \downarrow b \arrow[r, "a^{\Rightarrow}(b)"] \arrow[d, "b^{\Leftarrow}(a)"'] \arrow[dr, phantom, "\overset{\chi^{a\downarrow b}}{\Longleftarrow}"] & x \arrow[d, "a"] \arrow[r, phantom, "="] & z \arrow[r, "h_0"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\overset{\gamma}{\Longleftarrow}"] & x \arrow[d, "a"] \\
& w \arrow[r, "b"'] & y & w \arrow[r, "b"'] & y
\end{tikzcd} \tag{2.3.2}$$

hold.

2. For every pair of 2-cells $(\xi_0 : h_0 \Rightarrow h_0' : z \to x,\ \xi_1 : h_1 \Rightarrow h_1' : z \to w)$ such that

$$\begin{tikzcd}
w \arrow[dd, "b"'] \arrow[rr, phantom, "\Uparrow \xi_1"] & & z \arrow[ll, bend right=40, "h_1'"'] \arrow[ll, bend left=40, "h_1"] \arrow[dd, "h_0"] & & w \arrow[dd, "b"'] \arrow[ddrr, phantom, "\overset{\chi^{a\downarrow b} * \mathrm{id}_{h'}}{\Longleftarrow}"] & & z \arrow[ll, "h_1'"'] \arrow[dd, bend right=50, "h_0'"'] \arrow[dd, bend left=50, "h_0"] \arrow[dd, phantom, "\overset{\xi_0}{\Longleftarrow}"] \\
& \Uparrow \chi^{a\downarrow b} * \mathrm{id}_h & & = & & & \\
y & & x \arrow[ll, "a"] & & y & & x \arrow[ll, "a"]
\end{tikzcd} \tag{2.3.3}$$

holds, there is a unique 2-cell $\xi : h \Rightarrow h' : z \to a \downarrow b$ such that $\mathrm{id}_{a^{\Rightarrow}(b)} * \xi = \xi_0$ and $\mathrm{id}_{b^{\Leftarrow}(a)} * \xi = \xi_1$.

2.4. Remark. If $\mathbb{A}$ is a locally discrete 2-category, the comma object of a morphism a along b has the same universal property of the pullback of a along b.

3. Lax idempotent 2-adjunctions

Herein, our standpoint is that the notion of *pre-Kock-Zöberlein* 2*-functor* is the 2-dimensional counterpart of the notion of *full reflective functor*. In this section, we recall the basic definitions and give basic characterizations, but we refer to [25, 33] and [4, Ch. 4] for fundamental aspects and examples of lax idempotent 2-monads.

3.1. Definition. [Lax idempotent 2-monad] A *lax idempotent* 2*-monad* is a 2-monad $\mathcal{T} = (T, \mu, \eta)$ such that we have a rari adjunction $\mu \dashv \eta * \mathrm{id}_T$.

An *idempotent* 2*-monad* is a 2-monad $\mathcal{T} = (T, \mu, \eta)$ such that μ is invertible or, in other words, it is a lax idempotent 2-monad such that $\mu \dashv \eta * \mathrm{id}_T$ is a rali adjunction as well.

More explicitly, a 2-monad $\mathcal{T} = (T, \mu, \eta)$ on a 2-category $\mathbb{B}$ is lax idempotent if there is a modification

$$\begin{tikzcd}
T^2 \arrow[rr, equal, "\mathrm{id}_{T^2}"] \arrow[dr, Rightarrow, "\mu"'] & \arrow[d, phantom, "\Downarrow \Gamma"] & T^2 \\
& T \arrow[ur, Rightarrow, "\eta T"'] &
\end{tikzcd}$$

such that, for each object $z \in \mathbb{B}$,

$$\begin{tikzcd}
T(z) \arrow[r, "\eta_{T(z)}"] & T^2(z) \arrow[rr, equal] \arrow[dr, "\mu_z"'] & \arrow[d, phantom, "\Downarrow \Gamma_z"] & T^2(z) \\
& & T(z) \arrow[ur, "\eta_{T(z)}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2(z) \arrow[rr, equal] \arrow[dr, "\mu_z"'] & \arrow[d, phantom, "\Downarrow \Gamma_z"] & T^2(z) \arrow[r, "\mu_z"] & T(z) \\
& T(z) \arrow[ur, "\eta_{T(z)}"'] & &
\end{tikzcd}$$

are respectively the identity 2-cells on $\eta_{T(z)}$ and on μ_z.

3.2. Remark. [Dualities and self-duality] The concepts of lax idempotent and idempotent 2-monads are actually notions that can be defined inside any 3-category (or, more generally, tricategory [12]). Therefore they have eight dual notions each (counting the concept itself).

However, the notions of lax idempotent and idempotent 2-monads are self-dual, that is to say, the dual notion coincides with itself. More precisely, a triple $\mathcal{T} = (T, \mu, \eta)$ is a (lax) idempotent 2-monad in the 3-category 2-Cat if and only if the corresponding triple is also a (lax) idempotent 2-monad in the 3-category $(2\text{-}\mathsf{Cat})^{\mathrm{op}}$.

Furthermore, the notion of idempotent 2-monad is self-3-dual, meaning that the notion does not change when we invert the directions of the 3-cells (which are, in our case, the modifications). However the 3-dual of the notion of lax idempotent 2-monad is that of colax idempotent 2-monad.

Finally, the notions obtained from the inversion of the directions of the 2-cells, that is to say, the codual (or 2-dual) concepts, are those of lax idempotent and idempotent 2-comonads.

Henceforth, throughout this section, we always assume that a 2-adjunction

$$\begin{tikzcd} \mathbb{A} \arrow[rr, bend right, "G"'] \arrow[rr, phantom, "\perp(\varepsilon,\eta)"] & & \mathbb{B} \arrow[ll, bend right, "F"'] \end{tikzcd}$$

is given, and we denote by $\mathcal{T} = (T, \mu, \eta)$ *the induced* 2*-monad* $(GF, G\varepsilon F, \eta)$ *on* $\mathbb{B}$*.*

3.3. Idempotency. There are several useful well-known characterizations of idempotent (2-)monads (see, for instance, [2, p. 196]).

3.4. Lemma. [Idempotent 2-monad] *The following statements are equivalent.*

i) $\mathcal{T}$ *is idempotent;*

ii) $T\eta$ *(or* ηT*) is an epimorphism;*

iii) μ *is a monomorphism;*

iv) $T\eta = \eta T$*;*

v) $a : T(x) \to x$ *is a* $\mathcal{T}$*-algebra structure if, and only if,* $a \cdot \eta_x = \mathrm{id}_x$*;*

vi) $a : T(x) \to x$ *is a* $\mathcal{T}$*-algebra structure if, and only if, a is the inverse of* η_x*;*

vii) the forgetful 2*-functor* $\mathcal{T}\text{-}\mathsf{Alg}_\mathsf{s} \to \mathbb{B}$ *between the* 2*-category of strict* $\mathcal{T}$*-algebras and strict* $\mathcal{T}$*-morphisms (with modifications as* 2*-cells) and the* 2*-category* $\mathbb{B}$ *is fully faithful (that is to say, locally an isomorphism).*

Proof. Since $\mu \cdot (\eta T) = \mu \cdot (T\eta) = \mathrm{id}_T$, we have the following chain of equivalences: μ is a monomorphism $\Leftrightarrow$ μ is invertible $\Leftrightarrow$ ηT or $T\eta$ is invertible $\Leftrightarrow$ ηT or $T\eta$ is an epimorphism. This proves the equivalence of the first three statements.

By the definition of monomorphism, iii) implies iv). Conversely, assuming that $T\eta = \eta T$, we have that $T^2\eta = T\eta T$ and, thus, we get that

$$(T\eta)\cdot\mu \quad = \quad \begin{tikzcd} \mathbb{A} \arrow[d, "T"'] \arrow[d, bend left=90, equal] \arrow[d, phantom, shift left=3ex, "\overset{\eta}{\Longleftarrow}"] & \\ \mathbb{A} \arrow[dd, "T"'] \arrow[dr, "T"] \arrow[dd, phantom, shift left=4ex, "\overset{\mu}{\Longleftarrow}"] & \\ & \mathbb{A} \arrow[dl, "T"] \\ \mathbb{A} & \end{tikzcd} \quad = \quad \begin{tikzcd} \mathbb{A} \arrow[d, "T"'] & \\ \mathbb{A} \arrow[dd, "T"'] \arrow[dr, "T"] \arrow[dr, bend left=60, equal] \arrow[dr, phantom, shift left=2ex, "\eta\Downarrow"] \arrow[dd, phantom, shift left=4ex, "\overset{\mu}{\Longleftarrow}"] & \\ & \mathbb{A} \arrow[dl, "T"] \\ \mathbb{A} & \end{tikzcd} \quad = \quad \mathrm{id}_{T^2}.$$

Therefore $T\eta$ is the inverse of μ and, hence, μ is a monomorphism.

Assuming one of the first four equivalent statements (and hence all of them), we have that, given a morphism $a : T(x) \to x$ such that $a \cdot \eta_x = \mathrm{id}_x$, the equation

$$\eta_x \cdot a = T(a) \cdot \eta_{T(x)} = T(a \cdot \eta_x) = \mathrm{id}_{T(x)}. \tag{3.4.1}$$

holds. Thus, since $\eta_{T(x)} \cdot \eta_x = T(\eta_x) \cdot \eta_x$ and $\mu = (T\eta)^{-1}$, we conclude that

$$a \cdot \mu_x = \left(\eta_{T(x)} \cdot \eta_x\right)^{-1} = \left(T\left(\eta_x\right) \cdot \eta_x\right)^{-1} = a \cdot T(a). \tag{3.4.2}$$

This proves that v) holds. Conversely, v) trivially implies iii) (and, hence, all of the first four equivalent statements), since, for each $x \in \mathbb{B}$, μ_x is a (free) $\mathcal{T}$-algebra structure for x. Moreover, by Equations (3.4.1) and (3.4.2), we conclude that the first four statements are also equivalent to vi).

Finally, recall that, for every 2-monad $\mathcal{T}$ on a 2-category $\mathbb{B}$, the forgetful functor $\mathcal{T}\text{-}\mathsf{Alg_s} \to \mathbb{B}$ between the 2-category of strict $\mathcal{T}$-algebras and strict $\mathcal{T}$-morphisms (with modifications as 2-cells) and the 2-category $\mathbb{B}$ is faithful. Assuming vi), in order to verify that the forgetful functor is full, it is enough to see that, for any morphism $f : x \to y$ of $\mathbb{B}$, if $a : T(x) \to x$, $b : T(y) \to y$ are $\mathcal{T}$-algebra structures, we have that the pasting

$$\begin{tikzcd}
T(x) \arrow[r, "a"] \arrow[d, equal] \arrow[dr, phantom, near start, "="] & x \arrow[d, "f"] \arrow[dl, "\eta_x" description] \\
T(x) \arrow[d, "T(f)"'] \arrow[r, phantom, "="] \arrow[dr, phantom, near end, "="] & y \arrow[d, equal] \arrow[dl, "\eta_y" description] \\
T(y) \arrow[r, "b"'] & y
\end{tikzcd}$$

is the identity 2-cell and, hence, the morphism f induces a morphism of algebras between (x, a) and (y, b).

Assuming vii), we get that, for any object $x \in \mathbb{B}$, $\eta_{T(x)}$ induces a morphism between the free $\mathcal{T}$-algebras $(T(x), \mu_x)$ and $\left(T^2(x), \mu_{T(x)}\right)$. That is to say,

$$\eta_{T(x)} \cdot \mu_x = \mu_{T(x)} \cdot T(\eta_{T(x)})$$

and, since the right side of the equation above is equal to the identity on $T^2(x)$, we conclude that μ_x is a split monomorphism. This proves that iii) holds. ■

A 2-adjunction induces an idempotent 2-monad if, and only if, the induced 2-comonad is also idempotent. More generally:

3.5. Proposition. *The following statements are equivalent.*

i) $\mathcal{T}$ *is idempotent;*

ii) *$F\eta$ (or ηG) is an epimorphism;*

iii) *εF (or $G\varepsilon$) is a monomorphism;*

iv) *The induced 2-comonad is idempotent.*

Proof. Since, by the triangle identities, we have that

$$(\varepsilon F)\cdot(F\eta) = \mathrm{id}_F \text{ and } (G\varepsilon)\cdot(\eta G) = \mathrm{id}_G,$$

we get that ii) implies that εF or $G\varepsilon$ is invertible and, therefore, $G\varepsilon F = \mu$ is invertible. Analogously, iii) implies i).

Moreover, if we assume that $\mathcal{T}$ is idempotent, by Lemma 3.4, we have that

$$GF\eta = \eta GF$$

which, together with one of the triangle identities, implies that

$$(F\eta)\cdot(\varepsilon F) \quad = \quad \begin{tikzcd}
& \mathbb{B} \arrow[dl, "F"'] \arrow[dd, equal] & \\
\mathbb{A} \arrow[dr, "G"'] & \arrow[l, Rightarrow, "\eta"'] & \\
& \mathbb{B} \arrow[d, "F"] & \\
& \mathbb{A} \arrow[dr, "G"] \arrow[dd, equal] & \\
& \arrow[l, phantom, ""] \Leftarrow^{\varepsilon} & \mathbb{B} \arrow[dl, "F"] \\
& \mathbb{A} &
\end{tikzcd} \quad = \quad \begin{tikzcd}
\mathbb{B} \arrow[d, "F"'] & & \\
\mathbb{A} \arrow[d, "G"'] & & \\
\mathbb{B} \arrow[d, "F"'] \arrow[ddrr, equal, bend left] & & \\
\mathbb{A} \arrow[drr, "G"] \arrow[dd, equal] & \arrow[l, Rightarrow, "\eta"'] & \\
& \arrow[l, Rightarrow, "\varepsilon"'] & \mathbb{B} \arrow[dll, "F"] \\
\mathbb{A} & &
\end{tikzcd} \quad = \quad \mathrm{id}_{FGF}.$$

This proves that i) implies ii) and iii). Therefore we proved that i), ii) and iii) are equivalent statements.

Finally, since condition iii) is codual and equivalent to condition ii), we conclude that i) is equivalent to its codual – that is to say, to condition iv). ■

Motivated by the result above, we say that a 2-adjunction is *idempotent* if it induces an idempotent 2-(co)monad.

3.6. Remark. If the 2-adjunction $F \dashv G : \mathbb{A} \to \mathbb{B}$ is such that the underlying category of $\mathbb{A}$ (or $\mathbb{B}$) is *thin*, then the induced 2-monad is idempotent by Proposition 3.5. In particular, seeing categories as locally discrete 2-categories and contravariant 2-functors as covariant ones defined in the dual of the respective domains, any *Galois connection* induces an idempotent (2-)(co)monad.

If the 2-adjunction $F \dashv G$ is idempotent and G is 2-monadic, G is called a *full reflective 2-functor*. This terminology is justified by the well-known characterization below.

3.7. Proposition. [Full reflective 2-functor] *The following statements are equivalent.*

i) G is a full reflective 2-functor;

ii) $F \dashv G$ is idempotent and G is 2-premonadic;

iii) G is fully faithful;

iv) ε is invertible.

Proof. Recall that a 2-functor is 2-premonadic if the (Eilenberg-Moore) comparison 2-functor is fully faithful (that is to say, locally an isomorphism).

We have that i) trivially implies ii). Moreover, since the forgetful 2-functor $\mathcal{T}\text{-}\mathsf{Alg}_{\mathsf{s}} \to \mathbb{B}$ is fully faithful whenever $\mathcal{T}$ is idempotent, we have that ii) implies iii).

Since, for every pair of objects $w, x \in \mathbb{A}$, the diagram

$$\begin{tikzcd}
\mathbb{A}(w,x) \arrow[r, "\mathbb{A}(\varepsilon_w, x)"] \arrow[rr, bend right, "G"'] & \mathbb{A}(FG(w),x) \arrow[r, "\cong"] & \mathbb{B}(G(w),G(x))
\end{tikzcd}$$

commutes, iii) and iv) are equivalent.

Assuming iv), we have in particular that ε is a split epimorphism and G reflects isomorphisms, hence, G is 2-monadic (see Proposition at [32, p. 236]). Furthermore, clearly, we also get that $G\varepsilon$ is a (split) monomorphism, which implies that $F \dashv G$ is idempotent by Proposition 3.5. Therefore iv) implies i). ■

The dual notion of full reflective 2-functor in 2-Cat is called *full co-reflective* 2*-functor*. As a consequence of Proposition 3.7, we have:

3.8. Corollary. *If $F \dashv G$ is such that F is full co-reflective and G is full reflective, then $F \dashv G$ is a* 2*-adjoint equivalence.*

3.9. Remark. [Idempotent 2-adjunction vs. full reflective 2-functor] It should be noted that there are non-2-monadic idempotent 2-adjunctions. Remark 3.6 gives a way of constructing easy examples. For instance, given a 2-category $\mathbb{A}$, the unique 2-functor $\mathbb{A} \to \mathsf{1}$ has a left 2-adjoint if and only if $\mathbb{A}$ has an initial object. Assuming that $\mathbb{A}$ has an initial object and $\mathbb{A}$ is not (2-)equivalent to $\mathsf{1}$, the 2-functor $\mathbb{A} \to \mathsf{1}$ is not a reflective 2-functor, although the 2-adjunction is idempotent.

More generally, by Corollary 3.8 any full reflective 2-functor which is not an equivalence gives an example of an idempotent 2-adjunction such that the left 2-adjoint is not 2-comonadic. Dually, any non-equivalence full co-reflective 2-functor gives an idempotent 2-adjunction such that the right 2-adjoint is not a full reflective 2-functor.

3.10. KLEISLI VS. IDEMPOTENT ADJUNCTIONS. Recall that a 2-adjunction $F \dashv G$ *is Kleisli if the Kleisli comparison* 2*-functor* is an equivalence. This fact holds if, and only if, F is essentially surjective on objects. Moreover, a Kleisli 2-adjunction is always premonadic, since the Kleisli 2-category is equivalent to the full sub-2-category of free algebras of the 2-category $\mathcal{T}\text{-}\mathsf{Alg}_\mathsf{s}$ of the strict algebras of the induced 2-monad.

It should be noted that, by Proposition 3.7, we have that, whenever a 2-adjunction $F \dashv G$ is idempotent, *G is* 2*-premonadic if and only if G is* 2*-monadic.* Therefore by Lemma 3.11 below, this means that, whenever $\mathcal{T}$ is idempotent, the Kleisli 2-category is (2-)equivalent to the 2-category $\mathcal{T}\text{-}\mathsf{Alg}_\mathsf{s}$.

3.11. LEMMA. *The following statements are equivalent.*

i) The Kleisli 2*-category w.r.t. $\mathcal{T}$ is* 2*-equivalent to the* 2*-category of (strict) $\mathcal{T}$-algebras.*

ii) If $F' \dashv G'$ induces $\mathcal{T}$, then G' is 2*-premonadic if, and only if, G' is* 2*-monadic.*

By Proposition 3.7, we conclude the following well-known result:

3.12. COROLLARY. *An idempotent* 2*-adjunction $F \dashv G$ is* 2*-monadic if, and only if, it is Kleisli.*

3.13. LAX IDEMPOTENCY. For this part, we assume the definition of strict algebras and lax $\mathcal{T}$-morphisms between them, which can be found, for instance, in [29, Definition 2.2].

Given a 2-monad $\mathcal{T}$, we denote by $\mathcal{T}\text{-}\mathsf{Alg}_\ell$ the 2-category of strict algebras, lax $\mathcal{T}$-morphisms and modifications. In this case, $\mathcal{T}\text{-}\mathsf{Alg}_\mathsf{s}$ is the locally full sub-2-category of $\mathcal{T}\text{-}\mathsf{Alg}_\ell$ consisting of strict $\mathcal{T}$-algebras and strict $\mathcal{T}$-morphisms between them.

Theorem 3.14 is a well-known characterization of lax idempotent 2-monads [25]. We refer to [33, 23] for the proofs.

3.14. THEOREM. [Lax idempotent 2-monad] *The following statements are equivalent.*

i) $\mathcal{T}$ is lax idempotent;

ii) $\mathrm{id}_T * \eta \dashv \mu$ is a rali adjunction;

iii) $a : T(x) \to x$ is a $\mathcal{T}$-algebra structure if, and only if, there is a rari adjunction $a \dashv \eta_x$;

iv) $a : T(x) \to x$ is a $\mathcal{T}$-pseudoalgebra structure if, and only if, there is an adjunction $a \dashv \eta_x$;

v) the forgetful 2*-functor $\mathcal{T}\text{-}\mathsf{Alg}_\ell \to \mathbb{B}$ between the* 2*-category of strict $\mathcal{T}$-algebras and lax $\mathcal{T}$-morphisms and the* 2*-category $\mathbb{B}$ is fully faithful.*

Similarly to the idempotent case, a 2-adjunction induces a lax idempotent 2-monad if and only if it induces a lax idempotent 2-comonad. Furthermore, we give below a lax idempotent analogue of Proposition 3.5.

3.15. Theorem. [Lax idempotent 2-adjunction] *The following statements are equivalent.*

i) $\mathcal{T}$ *is lax idempotent;*

ii) $G\varepsilon \dashv \eta G$ *is a lali adjunction;*

iii) $F\eta \dashv \varepsilon F$ *is a rali adjunction;*

iv) The induced 2*-comonad is lax idempotent.*

Proof. By Lemma 3.14, it is clear that ii) or iii) implies i). Conversely, assuming i), we have by Lemma 3.14 that $\mathrm{id}_{GF} * \eta \dashv \mathrm{id}_G * \varepsilon * \mathrm{id}_F$. By *doctrinal adjunction* (*e.g.* [22]), we conclude that $F(\eta_x) \dashv \varepsilon_{F(x)}$ for every x of $\mathbb{B}$. Finally, again, by doctrinal adjunction, we conclude that $\mathrm{id}_F * \eta \dashv \varepsilon * \mathrm{id}_F$. This proves that i) implies iii).

Analogously, by doctrinal adjunction, we get that i) implies ii). Hence we proved that the first three statements are equivalent.

Since the condition ii) is codual and equivalent to iii), we get that i) is equivalent to its codual – which means iv). ■

3.16. Definition. [pre-Kock-Zöberlein 2-functor] If the induced 2-monad $\mathcal{T}$ is lax idempotent, the 2-adjunction $F \dashv G$ is *lax idempotent.* In this case if, furthermore, G is 2-premonadic, G is called a *pre-Kock-Zöberlein* 2*-functor.* Finally, if it is also 2-monadic, G is a *Kock-Zöberlein* 2*-functor.*

3.17. Proposition. *Assume that* $F \dashv G : \mathbb{A} \to \mathbb{B}$ *is lax idempotent. The following statements are equivalent.*

i) G *is a pre-Kock-Zöberlein* 2*-functor;*

ii) For each object $x \in \mathbb{A}$*,* ε_x *is a regular epimorphism;*

iii) For each object $x \in \mathbb{A}$*,*

$$\begin{tikzcd} FGFG(x) \arrow[r, bend left, "\varepsilon_{FG(x)}"] \arrow[r, bend right, "FG(\varepsilon_x)"'] & FG(x) \arrow[r, "\varepsilon_x"] & x \end{tikzcd} \tag{3.17.1}$$

is a coequalizer.

Proof. The result follows directly from the well-known characterization of (2-)premonadic (2-)functors due to Beck (see, for instance, [32, p. 226]). ■

3.18. Theorem. *Assume that $F \dashv G : \mathbb{A} \to \mathbb{B}$ is lax idempotent. The following statements are equivalent.*

i) G is a Kock-Zöberlein 2-functor;

ii) G creates absolute coequalizers;

iii) G is a pre-Kock-Zöberlein 2-functor, and, whenever η_y is a rari, there is $x \in \mathbb{A}$ such that $G(x) \cong y$.

Proof. The result follows from Proposition 3.17, and the characterization of algebra structures for lax idempotent 2-monads recalled in Theorem 3.14. ∎

3.19. Remark. [Algebras and free algebras] Corollary 3.12 says that a 2-functor G is Kleisli if and only if it is monadic, whenever $F \dashv G$ induces an idempotent 2-monad. This is not the case when $\mathcal{T}$ is only lax idempotent. The reference [16] provides several counterexamples in this direction. Moreover, in our context, in Section 5, Theorem 5.11 also provides several examples: more precisely, given any 2-category $\mathbb{A}$ and object $z \in \mathbb{A}$, the 2-adjunction between the *lax comma 2-category $\mathbb{A}//z$ (see Definition 5.1)* and the corresponding comma 2-category $\mathbb{A}/z$ usually is a Kleisli 2-adjunction which is not 2-monadic.

Finally, it should be noted that:

3.20. Lemma. *If $\mathbb{A}$ and $\mathbb{B}$ are locally discrete, we have that $F \dashv G$ is lax idempotent (G is pre-Kock-Zöberlein) if and only if $F \dashv G$ is idempotent (G is full reflective).*

Proof. It is enough to note that a 2-monad defined on a locally discrete 2-category is lax idempotent if and only if it is idempotent. The rest follows from Proposition 3.7. More particularly, it follows from the fact that 2-premonadicity and 2-monadicity are equivalent properties for idempotent 2-adjunctions. ∎

4. Composition of 2-adjunctions

Throughout this section,

$$\begin{tikzcd}
\mathbb{A} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp(\varepsilon,\eta)"] & \mathbb{B} \arrow[l, bend right, "F"'] \arrow[loop above, "\mathcal{T}"] \arrow[r, bend right, "J"'] \arrow[r, phantom, "\perp(\delta,\rho)"] & \mathbb{C} \arrow[l, bend right, "H"']
\end{tikzcd} \tag{4.0.1}$$

are given 2-adjunctions, and $\mathcal{T} = (T, \mu, \eta) = (GF, G\varepsilon F, \eta)$ is the 2-monad induced by the 2-adjunction $F \dashv G$. Recall that we have the composition of 2-adjunctions above given by

$$\begin{tikzcd}
\mathbb{A} \arrow[r, bend right, "J\circ G"'] \arrow[r, phantom, "\perp(\varepsilon\cdot(F\delta G),\ (J\eta H)\cdot\rho)"] & \mathbb{C} \arrow[l, bend right, "F\circ H"'] \arrow[loop right, "\mathcal{R}"]
\end{tikzcd} \tag{4.0.2}$$

where $\mathcal{R} = (R, v, \alpha)$ denotes the 2-monad induced by $FH \dashv JG$.

4.1. Idempotent 2-adjunctions. If J and G are full reflective 2-functors, JG is a full reflective 2-functor and, in particular, $FH \dashv JG$ induces an idempotent 2-monad. However, if $F \dashv G$ and $H \dashv J$ are only idempotent 2-adjunctions, we cannot conclude that the composite is idempotent. For instance, consider the 2-adjunctions

$$\begin{tikzcd}
\mathsf{CmpHaus} \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathsf{Top} \arrow[l, bend right] \arrow[r, bend right] \arrow[r, phantom, "\perp"] & \mathsf{Set} \arrow[l, bend right]
\end{tikzcd} \tag{4.1.1}$$

in which Top is the locally discrete 2-category of topological spaces and continuous functions, $\mathsf{CmpHaus}$ is the full sub-2-category of compact Hausdorff spaces, and the right adjoints are the usual forgetful functors. Both 2-adjunctions are idempotent, but the composition induces the ultrafilter (2-)monad which is not idempotent.

Proposition 4.2 characterizes when the composition of the 2-adjunctions is idempotent. It corresponds to the characterization of the simple (reflective) functors in the 1-dimensional case.

4.2. Proposition. *Assume that $F \dashv G$ is idempotent. The following statements are equivalent.*

i) $FH \dashv JG$ is idempotent;

ii) $JGF\delta G$ (or $F\delta GFH$) is a monomorphism;

iii) $FH\alpha$ (or αJG) is an epimorphism.

Proof. Since $F \dashv G$ is idempotent, $G\varepsilon$, εF, $F\eta$ and ηG are invertible.

By Proposition 3.5, the 2-adjunction $FH \dashv JG$ is idempotent if, and only if,

$$JG\left(\varepsilon \cdot (F\delta G)\right) = (JG\varepsilon) \cdot (JT\delta G), \text{ or } \left(\varepsilon \cdot (F\delta G)\right) FH = (\varepsilon FH) \cdot (F\delta TH),$$

is a monomorphism. Therefore, since $JG\varepsilon$ and εFH are invertible, we get that $FH \dashv JG$ is idempotent if, and only if, $JT\delta G$, or $F\delta TH$, is a monomorphism. This proves that i) is equivalent to ii).

Finally, i) is equivalent to iii) by Proposition 3.5. ■

4.3. Corollary. *If J is full reflective and $F \dashv G$ is idempotent, then the composition is idempotent.*

Proof. In this case, since δ is invertible, we have that $JGF\delta G$ is an isomorphism and, hence, a monomorphism. ■

4.4. Definition. [Admissible 2-functor] The 2-adjunction $F \dashv G$ is *admissible* w.r.t. $H \dashv J$ if JG is a full reflective 2-functor.

If G is full reflective, and the composition JG is full reflective, we generally cannot conclude that J is full reflective. More precisely, in this case, we have:

4.5. Proposition. *Assuming that G is full reflective, the horizontal composition $F\delta G$ is invertible if and only if the 2-adjunction $F \dashv G$ is admissible w.r.t. $H \dashv J$.*

Proof. Since ε is invertible (by Proposition 3.7), we get that $(F\delta G)$ is invertible if and only if the counit $\varepsilon\,(F\delta G)$ of $FH \dashv JG$ is invertible. By Proposition 3.7, this fact completes the proof. ■

4.6. Lax idempotent 2-adjunctions. We turn our attention now to analogous results for the lax idempotent case. The main point is to investigate when the composition of the 2-adjunctions is lax idempotent and premonadic.

4.7. Definition. [Simplicity] The 2-adjunction $F \dashv G$ is *simple* w.r.t. $H \dashv J$ if the composition $FH \dashv JG$ is lax idempotent.

As a consequence of the characterization of lax idempotent 2-adjunctions, we get:

4.8. Theorem. [Simplicity] *Assume that G is locally fully faithful. The 2-adjunction $F \dashv G$ is simple w.r.t. $H \dashv J$ if and only if*

$$(\mathrm{id}_{TH} * \alpha) \dashv (\mu * \mathrm{id}_H) \cdot (\mathrm{id}_T * \delta * \mathrm{id}_{TH})$$

is a rali adjunction.

Proof. By Theorem 3.15, we conclude that the 2-adjunction $FH \dashv JG$ is lax idempotent if and only if

$$(FH\alpha) \dashv (\varepsilon FH) \cdot (F\delta * TH)$$

is a rali adjunction. Since G is locally fully faithful, we have the rali adjunction above if, and only if, there is a rali adjunction $TH\alpha \dashv (\mu H) \cdot (T\delta TH)$. ■

The characterization of Theorem 4.8 turns out to be difficult to apply for most of the examples, since it involves several units and counits of the given 2-adjunctions. Therefore it seems useful to have suitable sufficient conditions to get simplicity.

4.9. Theorem.

a) *Assume that $JGF\delta G$ is invertible: $FH \dashv JG$ is lax idempotent if and only if there is a lali adjunction $JG\varepsilon \dashv J\eta G$.*

b) *Assume that $F\delta GFH$ is invertible: $FH \dashv JG$ is lax idempotent if and only if there is a rali adjunction $F\eta H \dashv \varepsilon FH$.*

Proof. We assume that $JGF\delta G$ is invertible. The other case is entirely analogous and, in fact, dual (3-dimensional codual).

By hypothesis, there is a 2-natural transformation $\vartheta : JGFG \Longrightarrow JGFHJG$ which is the inverse of $JGF\delta G$. Therefore, since

$$(JGF\delta G)\cdot(\alpha JG) \quad = \quad \begin{tikzcd}
\mathbb{A} \arrow[rrrr, "G"] &&&& \mathbb{B} \arrow[d, "J"] \arrow[ddll, equal, bend right] \\
&&& \overset{\delta}{\Longleftarrow} & \mathbb{C} \arrow[dll, "H"'] \arrow[dddd, equal] \\
&& \mathbb{B} \arrow[dll, "F"'] \arrow[dd, equal] && \\
\mathbb{A} \arrow[drr, "G"'] & \overset{\eta}{\Longleftarrow} && \overset{\rho}{\Longleftarrow} & \\
&& \mathbb{B} \arrow[drr, "J"'] && \\
&&&& \mathbb{C}
\end{tikzcd} \quad = \quad J\eta G, \tag{4.9.1}$$

we conclude that

$$\vartheta\cdot(J\eta G) = \vartheta\cdot(JGF\delta G)\cdot(\alpha JG) = \alpha JG. \tag{4.9.2}$$

Therefore we have the following situation

$$\begin{tikzcd}
JG \arrow[r, Rightarrow, bend right, "J\eta G"'] \arrow[rr, Rightarrow, bend right=40, "\alpha_{JG}"'] & JGFG \arrow[l, Rightarrow, bend right, "JG\varepsilon"'] \arrow[r, Rightarrow, bend right, "\vartheta"'] & JGFHJG \arrow[l, Rightarrow, bend right, "JGF\delta G"'] \arrow[ll, Rightarrow, bend right=40, "JG(\varepsilon(F\delta G))"']
\end{tikzcd} \tag{4.9.3}$$

in which $\vartheta^{-1} = JGF\delta G$. This is the hypothesis of Corollary 1.4 and, thus, there is a lali adjunction

$$JG\,(\varepsilon\cdot(F\delta G)) \dashv \alpha_{JG}$$

if, and only if, there is a lali adjunction $JG\varepsilon \dashv J\eta G$. By Theorem 3.15, this completes the proof. ∎

4.10. Corollary. *Assume that $F \dashv G$ is lax idempotent.*

a) If $JGF\delta G$ is invertible, then $FH \dashv JG$ is lax idempotent.

b) If $F\delta GFH$ is invertible, then $FH \dashv JG$ is lax idempotent.

Proof. In fact, if $F \dashv G$ is lax idempotent, we have in particular that there are a rali adjunction $F\eta H \dashv \varepsilon FH$ and a lali adjunction $JG\varepsilon \dashv J\eta G$. Therefore the result follows from Theorem 4.9. ∎

It should be noted that the 2-adjunctions in (4.1.1) show in particular that $FH \dashv JG$ might not be lax idempotent, even if $F \dashv G$ and $H \dashv J$ are. However, analogously to the idempotent case (see Corollary 4.3), we have a nicer situation whenever J is full reflective.

4.11. Corollary. *If J is full reflective, then $F \dashv G$ is lax idempotent if, and only if, $FH \dashv JG$ is lax idempotent.*

Proof. Assuming that J is full reflective, we get that δ is invertible and, thus, $JGF\delta G$ is invertible.

If $F \dashv G$ is lax idempotent, we get that the composite is lax idempotent by Corollary 4.10. Conversely, if $FH \dashv JG$ is lax idempotent, by Theorem 4.9, there is a lali adjunction

$$JG\varepsilon \dashv J\eta G.$$

Since J is locally an isomorphism, this implies that there is a lali adjunction $G\varepsilon \dashv \eta G$ which proves that $F \dashv G$ is lax idempotent by Theorem 3.15. ∎

4.12. Definition. [2-admissibility] The 2-adjunction $F \dashv G$ is 2-*admissible* w.r.t. $H \dashv J$ if the composition $FH \dashv JG$ is lax idempotent and premonadic (that is to say, JG is pre-Kock-Zöberlein).

As a consequence of Proposition 3.17 and Theorem 4.8, we have:

4.13. Theorem. [2-admissibility] *Assume that G is pre-Kock-Zöberlein. The 2-adjunction $F \dashv G$ is 2-admissible w.r.t. $H \dashv J$ if, and only if, the two conditions below hold.*

- *$TH\alpha \dashv (\mu * \mathrm{id}_H) \cdot (\mathrm{id}_T * \delta * \mathrm{id}_{TH})$ is a lari adjunction (or, equivalently, $F \dashv G$ is simple w.r.t. $H \dashv J$);*
- *For each object $z \in \mathbb{C}$, $(\varepsilon \cdot (F\delta G))_z$ is a regular epimorphism.*

Whenever a (2-)category has kernel pairs, the composition of a regular epimorphism with a split epimorphism is always a regular epimorphism *c.f.* [21]. Therefore we also have that:

4.14. Corollary. *If $\mathbb{A}$ has kernel pairs, $F \dashv G$ is simple w.r.t. $H \dashv J$, and $F\delta G$ is a split epimorphism, we conclude that $F \dashv G$ is 2-admissible w.r.t. $H \dashv J$.*

Proof. It follows directly from Theorem 4.13 and the observation above. ∎

Since the composition of a regular epimorphism with an isomorphism is always a regular epimorphism, we get:

4.15. Corollary. *If $F\delta G$ is an isomorphism and G is pre-Kock-Zöberlein, then $F \dashv G$ is 2-admissible w.r.t. $H \dashv J$. In particular, if J is full reflective and G is pre-Kock-Zöberlein, we conclude that JG is pre-Kock-Zöberlein.*

Proof. Since $F\delta G$ is invertible, we get that $JGF\delta G$ is invertible. Therefore, by Corollary 4.10, we get the simplicity. Moreover $\varepsilon \cdot (F\delta G)$ is a regular epimorphism since ε is a regular epimorphism and $(F\delta G)$ is invertible. ■

5. Lax comma 2-categories and change-of-base 2-functors

The notion of lax comma 2-categories is well-known and has been considered in the literature in many contexts (see, for instance, [15, I,5], [14, § 6], [31, Exercise 5, p. 115] or [38, p. 305]). We recall the definition in an elementary manner below, following the perspective of our setting. The main aim is to introduce the respective notions of change-of-base 2-functors.

In a 2-category $\mathbb{A}$ with products, given any object $y \in \mathbb{A}$, the endofunctor

$$(y \times -) : \mathbb{A} \to \mathbb{A}$$

has a unique comonadic structure. In this context, the usual 2-category of strict coalgebras $(y\times -)\text{-}\mathsf{CoAlg}_{\mathsf{s}}$ is isomorphic to the comma 2-category $\mathbb{A}/y$. Moreover, the 2-category $(y\times -)\text{-}\mathsf{CoAlg}_{\ell}$ of strict $(y\times -)$-coalgebras and lax morphisms is what we call the *lax comma 2-category* $\mathbb{A}//y$.[1] More generally, we explicitly define the lax comma categories below.

5.1. Definition. [Lax comma 2-category] Given an object y of a 2-category $\mathbb{A}$, we denote by $\mathbb{A}//y$ the 2-category defined by the following.

- The objects are pairs (w, a) in which w is an object of $\mathbb{A}$ and

$$w \xrightarrow{a} y$$

is a morphism of $\mathbb{A}$.

- A morphism in $\mathbb{A}//y$ between objects (w, a) and (x, b) is a pair

$$\left(w \xrightarrow{f} x \;,\; \begin{tikzcd} w \arrow[rr, "f"] \arrow[dr, "a"'] & \arrow[d, phantom, "\overset{\gamma}{\Longleftarrow}"] & x \arrow[dl, "b"] \\ & y & \end{tikzcd} \right)$$

---

[1] See, for instance, [27, Def. 4.1].

in which $f : w \to x$ is a morphism of $\mathbb{A}$ and γ is a 2-cell of $\mathbb{A}$.

If $(f, \gamma) : (w, a) \to (x, b)$ and $(g, \chi) : (x, b) \to (z, c)$ are morphisms of $\mathbb{A}//y$, the composition is defined by $(g \circ f, \gamma \cdot (\chi * \mathrm{id}_f))$, that is to say, the composition of the morphisms g and f with the pasting

$$\begin{tikzcd}
w \arrow[rr, "f"] \arrow[rrdd, "a"'] & & x \arrow[rr, "g"] \arrow[dd, "b"] & & z \arrow[lldd, "c"] \\
& {\overset{\gamma}{\Longleftarrow}} & & {\overset{\chi}{\Longleftarrow}} & \\
& & y & &
\end{tikzcd}$$

of the 2-cells χ and γ. Finally, with the definitions above, the identity on the object (w, a) is of course the morphism $(\mathrm{id}_w, \mathrm{id}_a)$.

– A 2-cell between morphisms (f, γ) and (f', γ') is given by a 2-cell $\zeta : f \Rightarrow f'$ such that the equation

$$\begin{tikzcd}
w \arrow[rr, bend left, "f"] \arrow[rr, bend right, "f'"'] \arrow[rr, phantom, "\zeta \Downarrow"] \arrow[dd, "a"'] & & x \arrow[lldd, "b"] & & w \arrow[rr, "f"] \arrow[dd, "a"'] & & x \arrow[lldd, "b"] \\
& {\overset{\gamma'}{\Longleftarrow}} & & = & & {\overset{\gamma}{\Longleftarrow}} & \\
y & & & & y & &
\end{tikzcd}$$

holds.

The 2-category $\mathbb{A}//y$ is called the *lax comma* 2*-category* of $\mathbb{A}$ over y, while the 2-category $\mathbb{A}^{\mathrm{co}}//y$ is called the *colax comma* 2*-category* of $\mathbb{A}$ over y.

The concept of (co)lax comma 2-category, possibly under other names, has already appeared in the literature. See, for instance, [31, Exercise 5, p. 115] or [38, p. 305]. As for our choice of the direction of the 2-cells for the notion of lax comma 2-categories, although we do not follow [38, p. 305], our choice is compatible with the usual definition of lax natural transformation.

5.2. Definition. [(Strict) comma 2-category] Given an object y of a 2-category $\mathbb{A}$, we denote by $\mathbb{A}/y$ the *comma* 2*-category* over y, defined to be the locally full *wide* sub-2-category of $\mathbb{A}//y$ in which a morphism from (w, a) to (x, b) is a morphism

$$(f, \chi) : (w, a) \to (x, b)$$

such that χ is the identity 2-cell.

5.3. Remark. We have an inclusion 2-functor $\mathbb{A}/y \to \mathbb{A}//y$ obviously defined. The morphisms in the image of this inclusion are called *strict* (or *tight*) morphisms of $\mathbb{A}//y$. The 2-category $\mathbb{A}//y$ endowed with this inclusion forms an enhanced 2-category, or, more precisely, an $\mathfrak{F}$-category as defined in [26].

5.4. Classical (strict) change-of-base functor. Assuming that $\mathbb{A}$ has pullbacks, given any morphism $c : y \to z$ of a 2-category $\mathbb{A}$, it is well known that it induces a 2-adjunction

$$\begin{tikzcd}
\mathbb{A}/z \arrow[r, bend right, "c^*"'] \arrow[r, phantom, "\perp"] & \mathbb{A}/y \arrow[l, bend right, "c!"']
\end{tikzcd} \tag{5.4.1}$$

between the (strict) comma 2-categories in which the right 2-adjoint is called the *change-of-base* 2*-functor* induced by the morphism c (see, for instance, [20]). Recall that c^* is defined by the pullback along c, and the left adjoint is defined by the composition with c, the so called direct-image 2-functor $c!(w, a) = (w, ca)$.

In the present section, we give the analogue for lax comma 2-categories, that is to say, the *change-of-base* 2*-functors for the lax comma* 2*-categories*, given in Proposition 5.7. Firstly, we recall the classical case:

5.5. Proposition. [Change-of-base 2-functor] *Let* $\mathbb{A}$ *be a* 2*-category with pullbacks. If* $c : y \to z$ *is any morphism, we get a* 2*-adjunction*

$$\begin{tikzcd}
\mathbb{A}/z \arrow[r, bend right, "c^*"'] \arrow[r, phantom, "\perp"] & \mathbb{A}/y \arrow[l, bend right, "c!"']
\end{tikzcd} \tag{5.5.1}$$

in which c^* *is defined by the pullback along* c*. Explicitly, the assignment of objects of* c^* *is given by*

$$(w, a) \mapsto (w \times_{(a,c)} y, c^*(a) : w \times_{(a,c)} y \to y)$$

while the action of c^* *on morphisms is given by*

$$\left(w \xrightarrow{f} x,\, \mathrm{id}_a\right) : (w, a) \to (x, b) \;\mapsto\; \left(w \times_{(a,c)} y \xrightarrow{c^*(f,\mathrm{id}_a)} x \times_{(b,c)} y,\, \mathrm{id}_{c^*(a)}\right) : c^*(a) \to c^*(b) \tag{5.5.2}$$

in which all the squares of

$$\begin{tikzcd}
w \times_{(a,c)} y \arrow[r] \arrow[d, "c^*(f)"'] \arrow[dd, bend right=60, "c^*(a)"'] & w \arrow[d, "f"] \arrow[dd, bend left=60, "a"] \\
x \times_{(b,c)} y \arrow[r, "b^*(c)"] \arrow[d, "c^*(b)"'] & x \arrow[d, "b"] \\
y \arrow[r, "c"'] & z
\end{tikzcd} \tag{5.5.3}$$

are pullbacks. Finally, the image of a 2-cell $\zeta : f \Rightarrow f' : (w,a) \to (x,b)$ is defined by the unique 2-cell $c^*(\zeta)$ such that the equations

$$\begin{tikzcd}
w \times_{(a,c)} y \arrow[d, bend right=50, "c^*(f')"'] \arrow[d, bend left=50, "c^*(f)"] \arrow[d, phantom, "\overset{c^*(\zeta)}{\Longleftarrow}"] & & w \times_{(a,c)} y \arrow[d, "a^*(c)"] & & w \times_{(a,c)} y \arrow[d, bend right=50, "c^*(f')"'] \arrow[d, bend left=50, "c^*(f)"] \arrow[d, phantom, "\overset{c^*(\zeta)}{\Longleftarrow}"] & & w \times_{(a,c)} y \arrow[dd, bend right=20, "c^*(a)"'] \arrow[dd, bend left=20, "c^*(a)"] \arrow[dd, phantom, "="] \\
x \times_{(b,c)} y \arrow[d, "b^*(c)"] & = & w \arrow[d, bend right=50, "f'"'] \arrow[d, bend left=50, "f"] \arrow[d, phantom, "\overset{\zeta}{\Longleftarrow}"] & \text{and} & x \times_{(b,c)} y \arrow[d, "c^*(b)"] & = & \\
x & & x & & y & & y
\end{tikzcd} \tag{5.5.4}$$

hold.

By considering the comma object, every morphism $c : y \to z$ in a 2-category $\mathbb{A}$ induces a 2-functor $c^{\Leftarrow} : \mathbb{A}//z \to \mathbb{A}/y$. This gives what we call, herein, the (lax) *change-of-base* 2-*functor*. We start by defining the appropriate analogous of direct image 2-functor in our setting, that is to say, the 2-functor $c\bar{!} : \mathbb{A}/y \to \mathbb{A}//z$ induced by each morphism $c : y \to z$ of $\mathbb{A}$.

5.6. Definition. [Direct image] If $c : y \to z$ is any morphism of a 2-category $\mathbb{A}$, we define the commutative diagram

$$\begin{tikzcd}
\mathbb{A}//z & \mathbb{A}/z \arrow[l] & \mathbb{A}/y \arrow[l, "c!"'] \arrow[ll, bend right=40, "c\bar{!}"']
\end{tikzcd} \tag{5.6.1}$$

in which the unlabeled arrow is the obvious inclusion, and

$$c! : \mathbb{A}/y \to \mathbb{A}/z$$

is defined by

$$(x,a)\mapsto(x,ca),\ (f,\mathrm{id})\mapsto(f,\mathrm{id}_c*\mathrm{id}),\ \zeta\mapsto\zeta,$$

that is to say, the usual direct image 2-functor.

Theorem 5.8 shows that, in the presence of suitable comma objects, for each morphism $c:y\to z$ in 2-category, the direct image 2-functor $c\bar{!}:\mathbb{A}/y\to\mathbb{A}//z$ has a right 2-adjoint – the (lax) change-of-base 2-functor $c^{\Leftarrow}$ defined below.

5.7. Definition. $[c^{\Leftarrow}]$ Let $\mathbb{A}$ be any 2-category, and $c:y\to z$ a morphism of $\mathbb{A}$. Assume that $\mathbb{A}$ has comma objects along c. We denote by

$$c^{\Leftarrow}:\mathbb{A}//z\to\mathbb{A}/y$$

the 2-functor defined by the comma object along the morphism c. Explicitly, the action on objects of $c^{\Leftarrow}$ is given by

$$(x,b)\mapsto(b\downarrow c,c^{\Leftarrow}(b):b\downarrow c\to y) \tag{5.7.1}$$

in which

$$\begin{tikzcd} b\downarrow c \arrow[r, "b^{\Rightarrow}(c)"] \arrow[d, "c^{\Leftarrow}(b)"'] \arrow[dr, phantom, "\overset{\chi^{b\downarrow c}}{\Longleftarrow}"] & x \arrow[d, "b"] \\ y \arrow[r, "c"'] & z \end{tikzcd} \tag{5.7.2}$$

is the comma object as in 2.3, while the action on morphisms is given by

$$\left(w \xrightarrow{f} x\,,\ \begin{tikzcd} w \arrow[rr, "f"] \arrow[dr, "a"'] & \overset{\gamma}{\Longleftarrow} & x \arrow[dl, "b"] \\ & z & \end{tikzcd} \right) \mapsto \left(a\downarrow c \xrightarrow{c^{\Leftarrow}(f,\gamma)} b\downarrow c,\quad \mathrm{id}_{c^{\Leftarrow}(a)}\right) \tag{5.7.3}$$

in which $c^{\Leftarrow}(f,\gamma)$, sometimes only denoted by $c^{\Leftarrow}(f)$, is the unique morphism of $\mathbb{A}$ such that the equations

$$b^{\Rightarrow}(c)\cdot c^{\Leftarrow}(f,\gamma)=f\cdot a^{\Rightarrow}(c),\qquad c^{\Leftarrow}(b)\cdot c^{\Leftarrow}(f,\gamma)=c^{\Leftarrow}(a),$$

$$\begin{tikzcd}
a\downarrow c \arrow[rrd, bend left, "f\cdot a^{\Rightarrow}(c)"] \arrow[rd, "c^{\Leftarrow}(f,\gamma)"] \arrow[rdd, bend right, "c^{\Leftarrow}(a)"'] \arrow[rd, phantom, bend left=20, "="] \arrow[rdd, phantom, bend right=10, "="] & & \\
& b\downarrow c \arrow[r, "b^{\Rightarrow}(c)"] \arrow[d, "c^{\Leftarrow}(b)"'] \arrow[rd, phantom, "\overset{\chi^{b\downarrow c}}{\Longleftarrow}"] & x \arrow[d, "b"] \\
& y \arrow[r, "c"'] & z
\end{tikzcd}
=
\begin{tikzcd}
a\downarrow c \arrow[r, "a^{\Rightarrow}(c)"] \arrow[dd, "c^{\Leftarrow}(a)"'] \arrow[rdd, phantom, "\overset{\chi^{a\downarrow c}}{\Longleftarrow}"] & w \arrow[rd, "f"] \arrow[dd, "a"] & \\
& & x \arrow[ld, "b"] \arrow[l, phantom, "\overset{\gamma}{\Longleftarrow}"] \\
y \arrow[r, "c"'] & z &
\end{tikzcd}
\qquad (5.7.4)$$

hold. Finally, if $\zeta : f \Rightarrow f' : (w,a) \to (x,b)$ is a 2-cell between morphisms (f,γ) and (f',γ') in $\mathbb{A}//z$, the 2-cell $c^{\Leftarrow}(\zeta)$ is the unique 2-cell such that the equations

$$\begin{tikzcd}
a\downarrow c \arrow[d, bend right=50, "c^{\Leftarrow}(f',\gamma')"'] \arrow[d, bend left=50, "c^{\Leftarrow}(f,\gamma)"] \arrow[d, phantom, "\overset{c^{\Leftarrow}(\zeta)}{\Longleftarrow}"] \\
b\downarrow c \arrow[d, "b^{\Rightarrow}(c)"] \\
x
\end{tikzcd}
=
\begin{tikzcd}
a\downarrow c \arrow[d, "a^{\Rightarrow}(c)"] \\
w \arrow[d, bend right=50, "f'"'] \arrow[d, bend left=50, "f"] \arrow[d, phantom, "\overset{\zeta}{\Longleftarrow}"] \\
x
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
a\downarrow c \arrow[d, bend right=50, "c^{\Leftarrow}(f',\gamma')"'] \arrow[d, bend left=50, "c^{\Leftarrow}(f,\gamma)"] \arrow[d, phantom, "\overset{c^{\Leftarrow}(\zeta)}{\Longleftarrow}"] \\
b\downarrow c \arrow[d, "c^{\Leftarrow}(b)"] \\
y
\end{tikzcd}
=
\begin{tikzcd}
a\downarrow c \arrow[d, bend right=30, "c^{\Leftarrow}(a)"'] \arrow[d, bend left=30, "c^{\Leftarrow}(a)"] \arrow[d, phantom, "="] \\
y
\end{tikzcd}
\qquad (5.7.5)$$

hold.

5.8. Theorem. *Let $\mathbb{A}$ be any 2-category, and $c : y \to z$ a morphism in $\mathbb{A}$. If $\mathbb{A}$ has comma objects along c, then we have a 2-adjunction*

$$\begin{tikzcd}
\mathbb{A}//z \arrow[r, bend right, "c^{\Leftarrow}"'] \arrow[r, phantom, "\perp"] & \mathbb{A}/y \arrow[l, bend right, "c\bar{!}"']
\end{tikzcd} . \qquad (5.8.1)$$

Proof. We define below the counit, denoted by δ, and the unit, denoted by ρ, of the 2-adjunction $c\bar{!} \dashv c^{\Leftarrow}$.

For each object

$$\left(x, x \xrightarrow{b} z\right)$$

of $\mathbb{A}//z$, we have the comma object

$$\begin{tikzcd}
b\downarrow c \arrow[r, "b^{\Rightarrow}(c)"] \arrow[d, "c^{\Leftarrow}(b)"'] & x \arrow[d, "b"] \arrow[dl, phantom, "\overset{\chi^{b\downarrow c}}{\Longleftarrow}" description] \\
y \arrow[r, "c"'] & z
\end{tikzcd} \tag{5.8.2}$$

as in (5.7.2). We define the counit on (x,b), denoted by $\delta_{(x,b)}$, to be the morphism between $c\bar{!}c^{\Leftarrow}(x,b)$ and (x,b) in $\mathbb{A}//z$ given by the pair $(b^{\Rightarrow}(c),\chi^{b\downarrow c})$.

Moreover, for each object

$$\left(w, w \xrightarrow{a} y\right)$$

in $\mathbb{A}/y$, we have the comma object

$$\begin{tikzcd}
ca\downarrow c \arrow[r, "(ca)^{\Rightarrow}(c)"] \arrow[d, "c^{\Leftarrow}c\bar{!}(a)"'] & w \arrow[d, "ca"] \arrow[dl, phantom, "\overset{\chi^{ca\downarrow c}}{\Longleftarrow}" description] \\
y \arrow[r, "c"'] & z
\end{tikzcd} \tag{5.8.3}$$

in $\mathbb{A}$. By the universal property of the comma object, there is a unique morphism $\rho'_{(w,a)}$ of $\mathbb{A}$ such that the equations

$$\begin{tikzcd}
w \arrow[dr, "\rho'_{(w,a)}"] & & & & & \\
& ca\downarrow c \arrow[r, "(ca)^{\Rightarrow}(c)"] \arrow[d, "c^{\Leftarrow}(ca)"'] & w \arrow[d, "ca"] \arrow[dl, phantom, "\overset{\chi^{ca\downarrow c}}{\Longleftarrow}" description] & = & w \arrow[r, "\mathrm{id}_w"] \arrow[d, "a"'] & w \arrow[d, "ca"] \arrow[dl, phantom, "=" description] \\
& y \arrow[r, "c"'] & z & & y \arrow[r, "c"'] & z
\end{tikzcd} \tag{5.8.4}$$

$$(ca)^{\Rightarrow}(c)\cdot\rho'_{(w,a)}=\mathrm{id}_w \qquad \text{and} \qquad c^{\Leftarrow}c\bar{!}(a)\cdot\rho'_{(w,a)}=a$$

hold.

By the equation above, the pair $(\rho'_{(w,a)},\mathrm{id}_a)$ gives a morphism between (w,a) and $(ca\downarrow c, c^{\Leftarrow}c\bar{!}(a))$ in $\mathbb{A}/y$. We claim that the component $\rho_{(w,a)}$ of the unit of $c\bar{!}\dashv c^{\Leftarrow}$ on (w,a) is the morphism defined by the pair $(\rho'_{(w,a)},\mathrm{id}_a)$.

It is straightforward to see that the definitions above actually give 2-natural transformations $\delta : c\bar{!}c^{\Leftarrow}\longrightarrow \mathrm{id}_{\mathbb{A}//z}$ and $\rho:\mathrm{id}_{\mathbb{A}/y}\longrightarrow c^{\Leftarrow}c\bar{!}$. We prove below that δ and ρ satisfy the triangle identities.

Let (w, a) be an object of $\mathbb{A}/y$.

The image of the morphism $\rho_{(w,a)}$ by the 2-functor $c\bar{!} : \mathbb{A}/y \to \mathbb{A}//z$ is the morphism $(\rho'_{(w,a)}, \mathrm{id}_{ca})$ between $c\bar{!}(w, a) = (w, ca)$ and $(ca \downarrow c, c\bar{!}c^{\Leftarrow}c\bar{!}(a))$ in $\mathbb{A}//z$, while the component $\delta_{c\bar{!}(w,a)} = \delta_{(w,ca)}$ is the morphism $\big((ca)^{\Rightarrow}(c), \chi^{ca\downarrow c}\big)$.

By the definition of $\rho'_{(w,a)}$, we have that $(ca)^{\Rightarrow}(c) \cdot \rho'_{(w,a)} = \mathrm{id}_w$ and $\chi^{ca\downarrow c} * \mathrm{id}_{\rho'_{(w,a)}} = \mathrm{id}_{ca}$. Therefore $\delta_{c\bar{!}(w,a)} \cdot c\bar{!}\left(\rho_{(w,a)}\right)$ is the identity on $c\bar{!}(a)$. This proves the first triangle identity.

Let (x, b) be an object of $\mathbb{A}//z$. Denoting by $(c \cdot c^{\Rightarrow}(b) \downarrow c, \chi^{c\cdot c^{\Leftarrow}(b)\downarrow c})$ the comma object of $c \cdot c^{\Leftarrow}(b)$ along c, we have that the morphism

$$c^{\Leftarrow}\left(\delta_{(x,b)}\right) : c^{\Leftarrow}c\bar{!}c^{\Leftarrow}(x, b) \to c^{\Leftarrow}(x, b)$$

in $\mathbb{A}/y$ is defined by the pair $(\delta', \mathrm{id}_{c^{\Leftarrow}c\bar{!}c^{\Leftarrow}(b)})$ in which δ' is the unique morphism in $\mathbb{A}$ making the diagrams

$$\begin{tikzcd}
x & & b\downarrow c \arrow[ll, "b^{\Rightarrow}(c)"'] \\
& c\cdot c^{\Leftarrow}(b)\downarrow c \arrow[ul, "b^{\Rightarrow}(c)\cdot(c\cdot c^{\Leftarrow}(b))^{\Rightarrow}(c)"] \arrow[ur, "\delta'"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
b\downarrow c \arrow[dr, "c^{\Leftarrow}(b)"] & \\
& y \\
c\cdot c^{\Leftarrow}(b)\downarrow c \arrow[uu, "\delta'"] \arrow[ur, "c^{\Leftarrow}c\bar{!}c^{\Leftarrow}(b)"'] &
\end{tikzcd}$$

commute, and the equation

$$\begin{tikzcd}
c\cdot c^{\Leftarrow}(b)\downarrow c \arrow[dr, "\delta'"] \arrow[drr, bend left, "f\cdot a^{\Rightarrow}(c)"] \arrow[ddr, bend right, "c^{\Leftarrow}(a)"'] \arrow[drr, phantom, "="] \arrow[ddr, phantom, "="] & & & & c\cdot c^{\Leftarrow}(b)\downarrow c \arrow[r, "(c\cdot c^{\Leftarrow}(b))^{\Rightarrow}(c)"] \arrow[dd, "c^{\Leftarrow}c\bar{!}c^{\Leftarrow}(b)"'] & b\downarrow c \arrow[d, "c^{\Leftarrow}(b)"] \arrow[dr, "b^{\Rightarrow}(c)"] & \\
& b\downarrow c \arrow[r, "b^{\Rightarrow}(c)"] \arrow[d, "c^{\Leftarrow}(b)"] & x \arrow[d, "b"] & = & & y \arrow[d, "c"] & x \arrow[dl, "b"] \\
& y \arrow[r, "c"'] & z & & y \arrow[r, "c"'] & z &
\arrow[from=2-3, to=3-2, phantom, "\overset{\chi^{b\downarrow c}}{\Longleftarrow}"]
\arrow[from=2-6, to=2-5, phantom, "\overset{\chi^{c\cdot c^{\Leftarrow}(b)\downarrow c}}{\Longleftarrow}"]
\arrow[from=2-7, to=2-6, phantom, "\overset{\chi^{b\downarrow c}}{\Longleftarrow}"]
\end{tikzcd} \tag{5.8.5}$$

holds.

Since, by the definition of ρ, the underlying morphism $\rho'_{c^{\Leftarrow}(x,b)}$ of the component of ρ on $c^{\Leftarrow}(x, b)$ is such that the equations

$$\chi^{c\cdot c^{\Leftarrow}(b)\downarrow c} * \mathrm{id}_{\rho'_{c^{\Leftarrow}(x,b)}} = \mathrm{id}_{c\cdot c^{\Leftarrow}(b)},\ \left(c \cdot c^{\Leftarrow}(b)\right)^{\Rightarrow}(c) \cdot \rho'_{c^{\Leftarrow}(b)} = \mathrm{id}_{b\downarrow c},\ c^{\Leftarrow}c\bar{!}c^{\Leftarrow}(b) \cdot \rho'_{c^{\Leftarrow}(x,b)} = c \cdot c^{\Leftarrow}(a)$$

hold, we get that the equations

$$\begin{tikzcd}
b\downarrow c \arrow[dr, "\rho'_{c^{\Leftarrow}(x,b)}"] & & & & & & \\
& c\cdot c^{\Leftarrow}(b)\downarrow c \arrow[dr, "\delta'"] & & & & & \\
& & b\downarrow c \arrow[r, "b^{\Rightarrow}(c)"] \arrow[dd, "c^{\Leftarrow}(b)"'] & x \arrow[dd, "b"] & & b\downarrow c \arrow[r, "b^{\Rightarrow}(c)"] \arrow[dd, "c^{\Leftarrow}(b)"'] & x \arrow[dd, "b"] \\
& & & & = & & \\
& & y \arrow[r, "c"'] & z & & y \arrow[r, "c"'] & z
\arrow[from=3-3, to=5-4, phantom, "\overset{\chi^{b\downarrow c}}{\Longleftarrow}"]
\arrow[from=3-6, to=5-7, phantom, "\overset{\chi^{b\downarrow c}}{\Longleftarrow}"]
\end{tikzcd} \tag{5.8.6}$$

$$c^{\Leftarrow}(b)\cdot\delta'\cdot\rho'_{c^{\Leftarrow}(b)} = c^{\Leftarrow}(b), \qquad b^{\Rightarrow}(c)\cdot\delta'\cdot\rho'_{c^{\Leftarrow}(b)} = b^{\Rightarrow}(c)$$

hold. Since, by the universal property of the comma object of b along c, the morphism satisfying the three equations above is unique, we conclude that $\delta'\cdot\rho'_{c^{\Leftarrow}(x,b)}$ is the identity on $b\downarrow c$. This proves that

$$c^{\Leftarrow}(\delta_{(x,b)})\cdot\rho_{c^{\Leftarrow}(x,b)} = \mathrm{id}_{c^{\Leftarrow}(x,b)}$$

which proves the second triangle identity. ∎

5.9. Corollary. *If $\mathbb{A}$ has comma objects (along identities), then $\mathbb{A}/y \to \mathbb{A}//y$ has a right 2-adjoint which is defined by the comma object along the identity id_y.*

Proof. It follows from Theorem 5.8 and the fact that the inclusion $\mathbb{A}/y \to \mathbb{A}//y$ is actually given by the 2-functor $\mathrm{id}_y\bar{!} : \mathbb{A}/y \to \mathbb{A}//y$ and, hence, it is left 2-adjoint to the 2-functor

$$\mathrm{id}_y^{\Leftarrow} : \mathbb{A}//y \to \mathbb{A}/y.$$

∎

By Theorem 5.8 and the fact that, given a morphism $c : y \to z$ of a 2-category $\mathbb{A}$,

$$\begin{tikzcd}
\mathbb{A}//z & \mathbb{A}/z \arrow[l, "\mathrm{id}_z\bar{!}"] & \mathbb{A}/y \arrow[l, "c!"] \arrow[ll, bend right, "c\bar{!}"']
\end{tikzcd} \tag{5.9.1}$$

commutes, we get that:

5.10. Theorem. *Let $\mathbb{A}$ be a 2-category, and $c : y \to z$ a morphism of $\mathbb{A}$. If $\mathbb{A}$ has comma objects and pullbacks along c, we have the following commutative diagram of 2-adjunctions*

$$\begin{tikzcd}
\mathbb{A}//z \arrow[r, bend right, "\mathrm{id}_z^{\Leftarrow}"'] \arrow[r, phantom, "\perp"] \arrow[rr, bend right=60, "c^{\Leftarrow}"'] & \mathbb{A}/z \arrow[l, bend right, "\mathrm{id}_z\overline{!}"'] \arrow[r, bend right, "c^*"'] \arrow[r, phantom, "\perp"] & \mathbb{A}/y \arrow[l, bend right, "c!"'] \arrow[ll, bend right=60, "c\overline{!}"']
\end{tikzcd} \tag{5.10.1}$$

which means that the composition of the 2-adjunction $c! \dashv c^* : \mathbb{A}/z \to \mathbb{A}/y$ with $\mathrm{id}_z\overline{!} \dashv id_z^{\Leftarrow} : \mathbb{A}//z \to \mathbb{A}/z$ is, up to 2-natural isomorphism, the 2-adjunction

$$c\overline{!} \dashv c^{\Leftarrow} : \mathbb{A}//z \to \mathbb{A}/y.$$

Given a 2-category $\mathbb{A}$, it is clear that, for any object y of $\mathbb{A}$, the 2-adjunction $\mathrm{id}_y! \dashv \mathrm{id}_y^* : \mathbb{A}/y \to \mathbb{A}/y$ is 2-naturally isomorphic to the identity 2-adjunction $\mathrm{id}_{\mathbb{A}/y} \dashv \mathrm{id}_{\mathbb{A}/y}$ and, in particular, is an idempotent 2-adjunction.

In the setting of Theorem 5.8, that is to say, the comma version of the change-of-base 2-functor, the 2-adjunction

$$\mathrm{id}_y\overline{!} \dashv \mathrm{id}_y^{\Leftarrow} : \mathbb{A}//y \to \mathbb{A}/y,$$

is far from being isomorphic to the identity 2-adjunction. It is not even idempotent in most of the cases. It is, however, always lax idempotent and a Kleisli 2-adjunction. More precisely:

5.11. Theorem. *Let $\mathbb{A}$ be a 2-category, and y an object of $\mathbb{A}$. If $\mathbb{A}$ has comma objects along id_y, then the 2-adjunction*

$$\begin{tikzcd}
\mathbb{A}//y \arrow[r, bend right, "\mathrm{id}_y^{\Leftarrow}"'] \arrow[r, phantom, "\perp_{(\delta,\rho)}"] & \mathbb{A}/y \arrow[l, bend right, "\mathrm{id}_y\overline{!}"']
\end{tikzcd} \tag{5.11.1}$$

is lax idempotent. Moreover, it is a Kleisli 2-adjunction and, hence, $\mathrm{id}_y^{\Leftarrow}$ is a pre-Kock-Zöberlein 2-functor.

Proof. In order to verify that (5.11.1) is a Kleisli 2-adjunction, it is enough to see that $\mathrm{id}_y\overline{!}$ is bijective on objects. In particular, we conclude that $\mathrm{id}_y^{\Leftarrow}$ is 2-premonadic. Therefore, in order to prove that $\mathrm{id}_y^{\Leftarrow}$ is a pre-Kock-Zöberlein 2-functor, it remains only to prove that the 2-adjunction (5.11.1) is lax idempotent.

We prove below that

$$\mathrm{id}_{\mathrm{id}_y\overline{!}} * \rho \dashv \delta * \mathrm{id}_{\mathrm{id}_y\overline{!}} \tag{5.11.2}$$

is a rari adjunction and, hence, it satisfies the condition iii) of Theorem 3.15, which implies that the 2-adjunction (5.11.1) is lax idempotent.

For short, throughout this proof, we denote $\mathrm{id}_{\mathrm{id}_y\overline{!}} * \rho$ by $\overline{\rho}$, and $\delta * \mathrm{id}_{\mathrm{id}_y\overline{!}}$ by $\overline{\delta}$.

Recall that, given an object $(x, b) \in \mathbb{A}/y$, we have that $\overline{\delta}_{(x,b)}$ is defined by the pair

$$\left(\underline{\overline{\delta}}_{(x,b)},\ \begin{tikzcd} b\downarrow \mathrm{id}_y \arrow[r, "b^{\Rightarrow}(\mathrm{id}_y)"] \arrow[d, "\mathrm{id}_y^{\Leftarrow}(b)"'] \arrow[dr, phantom, "\overset{\chi^{b\downarrow\mathrm{id}_y}}{\Longleftarrow}"] & x \arrow[d, "b"] \\ y \arrow[r, "\mathrm{id}_y"'] & y \end{tikzcd}\right)$$

in which, as suggested by the notation, the 2-cell is the comma object in $\mathbb{A}$, and

$$\underline{\overline{\delta}}_{(x,b)} := b^{\Rightarrow}(\mathrm{id}_y).$$

Moreover, recall that, given an object $(x, b) \in \mathbb{A}/y$, we have that $\overline{\rho}_{(x,b)} = \left(\underline{\overline{\rho}}_{(x,b)}, \mathrm{id}_b\right)$ in which $\underline{\overline{\rho}}_{(x,b)}$ is the unique morphism of $\mathbb{A}$ such that the equations

$$\underline{\overline{\delta}}_{(x,b)} \cdot \underline{\overline{\rho}}_{(x,b)} = \mathrm{id}_x, \quad \mathrm{id}_y^{\Leftarrow}(b) \cdot \underline{\overline{\rho}}_{(x,b)} = b, \quad \text{and}$$

$$\begin{tikzcd} x \arrow[d, bend right=50, "b"'] \arrow[d, bend left=50, "b"] \arrow[d, phantom, "="] \\ y \end{tikzcd} \quad = \quad \begin{tikzcd} x \arrow[dr, "\underline{\overline{\rho}}_{(x,b)}"] \arrow[drr, bend left=30, "\mathrm{id}_x"] \arrow[ddr, bend right=30, "b"'] \arrow[drr, phantom, "=" near start] \arrow[ddr, phantom, "="] & & \\ & b\downarrow \mathrm{id}_y \arrow[r, "b^{\Rightarrow}(\mathrm{id}_y) = \underline{\overline{\delta}}_{(x,b)}"] \arrow[d, "\mathrm{id}_y^{\Leftarrow}(b)"'] \arrow[dr, phantom, "\overset{\chi^{b\downarrow\mathrm{id}_y}}{\Longleftarrow}"] & x \arrow[d, "b"] \\ & y \arrow[r, "\mathrm{id}_y"'] & y \end{tikzcd} \tag{5.11.3}$$

hold.

For each object $(x,b) \in \mathbb{A}/y$, the pair of 2-cells $\left(\chi^{b\downarrow \mathrm{id}_y}, \mathrm{id}_{\overline{\underline{\delta}}_{(x,b)}}\right)$ satisfies the equation

$$\begin{tikzcd}
b\downarrow \mathrm{id}_y \arrow[dr, "\overline{\underline{\delta}}_{(x,b)}"] \arrow[dddrr, bend right=60, "\mathrm{id}_y^{\Leftarrow}(b)"'] & & & & & b\downarrow \mathrm{id}_y \arrow[dd, "\mathrm{id}_{b\downarrow \mathrm{id}_y}"'] \arrow[ddr, bend left, "\overline{\underline{\delta}}_{(x,b)}"] & & \\
\overset{\chi^{b\downarrow \mathrm{id}_y}}{\Longleftarrow} & x \arrow[dr, "\overline{\underline{\rho}}_{(x,b)}"] \arrow[ddr, bend right, "b"'] & & & & & = & \\
& = & b\downarrow \mathrm{id}_y \arrow[r, "\overline{\underline{\delta}}_{(x,b)}"] \arrow[d, "\mathrm{id}_y^{\Leftarrow}(b)"'] \arrow[dr, phantom, "\overset{\chi^{b\downarrow \mathrm{id}_y}}{\Longleftarrow}"] & x \arrow[d, "b"] & = & b\downarrow \mathrm{id}_y \arrow[r, "\overline{\underline{\delta}}_{(x,b)}"] \arrow[d, "\mathrm{id}_y^{\Leftarrow}(b)"'] \arrow[dr, phantom, "\overset{\chi^{b\downarrow \mathrm{id}_y}}{\Longleftarrow}"] & x \arrow[d, "b"] & (5.11.4) \\
& & y \arrow[r, "\mathrm{id}_y"'] & y & & y \arrow[r, "\mathrm{id}_y"'] & y &
\end{tikzcd}$$

and, hence, by the universal property of the comma object, there is a unique 2-cell $\Gamma_{(x,b)}$ such that the equations

$$\mathrm{id}_{\mathrm{id}_y^{\Leftarrow}(b)} * \Gamma_{(x,b)} = \chi^{b\downarrow \mathrm{id}_y} \quad \text{and} \quad \mathrm{id}_{\overline{\underline{\delta}}_{(x,b)}} * \Gamma_{(x,b)} = \mathrm{id}_{\overline{\underline{\delta}}_{(x,b)}}$$

hold. The 2-cells $\Gamma_{(x,b)}$ define a modification

$$\Gamma : \overline{\rho} \cdot \overline{\delta} \Longrightarrow \mathrm{id}_{\mathrm{id}_y \overline{!} \mathrm{id}_y^{\Leftarrow} \mathrm{id}_y \overline{!}}$$

which we claim to be the counit of the adjunction (5.11.2).

The first triangle identity holds, since, by the definition of Γ above,

$$\mathrm{id}_{\overline{\underline{\delta}}_{(x,b)}} * \Gamma_{(x,b)} = \mathrm{id}_{\overline{\underline{\delta}}_{(x,b)}}$$

for every object $(x,b) \in \mathbb{A}/y$.

Finally, for each object $(x,b) \in \mathbb{A}/y$, $\Gamma_{(x,b)} * \mathrm{id}_{\overline{\underline{\rho}}_{(x,b)}}$ is such that

$$\mathrm{id}_{\mathrm{id}_y^{\Leftarrow}(b)} * \Gamma_{(x,b)} * \mathrm{id}_{\overline{\underline{\rho}}_{(x,b)}} = \chi^{b\downarrow \mathrm{id}_y} * \mathrm{id}_{\overline{\underline{\rho}}_{(x,b)}} = \mathrm{id}_b$$

by (5.11.3), and, of course,

$$\mathrm{id}_{\overline{\underline{\delta}}_{(x,b)}} * \Gamma_{(x,b)} * \mathrm{id}_{\overline{\underline{\rho}}_{(x,b)}} = \mathrm{id}_{\overline{\underline{\delta}}_{(x,b)} \cdot \overline{\underline{\rho}}_{(x,b)}}.$$

Therefore, by the universal property of the comma object $b \downarrow \mathrm{id}_y$, we get that $\Gamma_{(x,b)} * \mathrm{id}_{\overline{\underline{\rho}}_{(x,b)}} = \mathrm{id}_{\mathrm{id}_{\overline{\underline{\rho}}_{(x,b)}}}$. This completes the proof that the second triangle identity holds. ■

6. Admissibility

Throughout this section,

$$\begin{tikzcd}
\mathbb{A} \arrow[r, bend right, "G"'] \arrow[r, phantom, "\perp(\varepsilon,\eta)"] & \mathbb{B} \arrow[l, bend right, "F"']
\end{tikzcd}$$

is a given 2-adjunction. By abuse of language, given any 2-functor $H : \mathbb{A} \to \mathbb{B}$, for each object x in $\mathbb{A}$, we denote by the same $\check{H}$ the 2-functors

$$\check{H} : \mathbb{A}/x \to \mathbb{B}/H(x), \quad \check{H} : \mathbb{A}/x \to \mathbb{B}//H(x), \quad \check{H} : \mathbb{A}//x \to \mathbb{B}//H(x)$$

pointwise defined by H. Moreover, given a morphism $f : w \to x$ of $\mathbb{A}$, we denote by

$$f\underline{!} : \mathbb{A}//w \to \mathbb{A}//x$$

the 2-functor defined by the *direct image* between the lax comma 2-categories, whose restriction to $\mathbb{A}/w$ is equal to $f\overline{!}$.

6.1. PROPOSITION. *If G is a locally fully faithful 2-functor then, for each object x of $\mathbb{A}$, both $\check{G} : \mathbb{A}/x \to \mathbb{B}/G(x)$ and $\check{G} : \mathbb{A}//x \to \mathbb{B}//G(x)$ are locally fully faithful.*

6.2. THEOREM. *For any object $y \in \mathbb{A}$, we have two 2-adjunctions*

$$\begin{tikzcd}
\mathbb{A}/y \arrow[r, bend right, "\check{G}"'] \arrow[r, phantom, "\perp"] & \mathbb{B}/G(y) \arrow[l, bend right, "\varepsilon_y!\circ\check{F}"'] & \textit{and} & \mathbb{A}//y \arrow[r, bend right, "\check{G}"'] \arrow[r, phantom, "\perp"] & \mathbb{B}//G(y) \arrow[l, bend right, "\varepsilon_y\underline{!}\circ\check{F}"']
\end{tikzcd} \tag{6.2.1}$$

where the counit and the unit of these 2-adjunctions are defined pointwise by the counit and unit of $F \dashv G$.

6.3. COROLLARY. *For each object $y \in \mathbb{A}$, the 2-adjunctions*

$$\begin{tikzcd}
\mathbb{A}/y \arrow[r, bend right, "\check{G}"'] \arrow[r, phantom, "\perp"] & \mathbb{B}/G(y) \arrow[l, bend right, "\varepsilon_y!\circ\check{F}"'] & \textit{and} & \mathbb{A}//y \arrow[r, bend right, "\check{G}"'] \arrow[r, phantom, "\perp"] & \mathbb{B}//G(y) \arrow[l, bend right, "\varepsilon_y\underline{!}\circ\check{F}"']
\end{tikzcd} \tag{6.3.1}$$

are lax idempotent (premonadic) if, and only if, $F \dashv G$ is lax idempotent (premonadic).

Henceforth, we further assume that $\mathbb{B}$ has comma objects and pullbacks whenever necessary. Recall that, in this case, by Section 5, for each object y of $\mathbb{B}$, we have 2-adjunctions

$$\eta_y! \dashv \eta_y^* : \mathbb{B}/GF(y) \to \mathbb{B}/y \text{ and } \eta_y\overline{!} \dashv \eta_y^{\Leftarrow} : \mathbb{B}//GF(y) \to \mathbb{B}/y$$

in which the right 2-adjoints are given respectively by the pullback and the comma object along η_y.

6.4. Definition. [Simple, admissible and 2-admissible 2-functors] The 2-functor G is called *simple/2-admissible* if $F \dashv G$ is lax idempotent/pre-Kock-Zöberlein, and, for every $y \in \mathbb{B}$,

$$\begin{tikzcd}
\mathbb{A}//y \arrow[rr, bend right, "\check{G}"'] \arrow[rr, phantom, "\perp"] & & \mathbb{B}//G(y) \arrow[ll, bend right, "\varepsilon_y\bar{!}\circ\check{F}"']
\end{tikzcd} \tag{6.4.1}$$

is simple/2-admissible w.r.t. $\eta_y! \dashv \eta_y^{\Leftarrow}$ (see Definitions 4.7 and 4.12).

We say that G is *admissible w.r.t. the basic fibration* if G is fully faithful, and, for every $y \in \mathbb{B}$,

$$\begin{tikzcd}
\mathbb{A}/y \arrow[rr, bend right, "\check{G}"'] \arrow[rr, phantom, "\perp"] & & \mathbb{B}/G(y) \arrow[ll, bend right, "\varepsilon_y!\circ\check{F}"']
\end{tikzcd} \tag{6.4.2}$$

is admissible w.r.t. $\eta_y! \dashv \eta_y^*$.

6.5. Remark. The notion of admissibility w.r.t. the basic fibration is just the direct strict 2-dimensional generalization of the classical notion of admissibility (also called semi-left-exact reflective functor) [6, 3], while the notion of simplicity coincides with that introduced in [10].

In order to establish the direct consequences of the results of Section 4 for the case of 2-admissibility and simplicity, we set some notation below. For each y of $\mathbb{B}$, we consider the 2-adjunctions

$$\begin{tikzcd}
\mathbb{A}//F(y) \arrow[rr, bend right, "\check{G}"'] \arrow[rr, phantom, "\perp(\varepsilon,\eta)"] & & \mathbb{B}//GF(y) \arrow[ll, bend right, "\varepsilon_{F(y)}\underline{!}\circ\check{F}"'] \arrow[loop above, "\mathcal{T}"] \arrow[rr, bend right, "\eta_y^{\Leftarrow}"'] \arrow[rr, phantom, "\perp(\delta,\rho)"] & & \mathbb{B}/y \arrow[ll, bend right, "\eta_y\bar{!}"']
\end{tikzcd} \tag{6.5.1}$$

in which, by abuse of language, we denote respectively by ε and η the counit and unit defined pointwise, and $\mathcal{T} = (T, \mu, \eta)$ the 2-monad induced by $\varepsilon_{F(y)}! \circ \check{F} \dashv \check{G}$.

In this case, the composition of 2-adjunctions above is given by

$$\begin{tikzcd}
\mathbb{A}//F(y) \arrow[rrrr, bend right, "\eta_y^{\Leftarrow}\circ\check{G}"'] \arrow[rrrr, phantom, "\perp\left(\varepsilon\cdot\left(\mathrm{id}_{\check{F}}*\delta*\mathrm{id}_{\check{G}}\right),\ \alpha\right)"] & & & & \mathbb{B}/y \arrow[llll, bend right, "\check{F}"'] \arrow[loop right, "\mathcal{R}"]
\end{tikzcd} \tag{6.5.2}$$

where $\alpha = \left(\mathrm{id}_{\eta_y^{\Leftarrow}} * \eta * \mathrm{id}_{\eta_y\bar{!}}\right) \cdot \rho$, and we denote by $\mathcal{R} = (R, v, \alpha)$ the 2-monad induced by $\check{F} \dashv \eta_y^{\Leftarrow} \circ \check{G}$.

6.6. Remark. [α] Given an object $(x,b) \in \mathbb{B}/y$,

$$\alpha_{(x,b)} : (x,b) \to \eta_y^{\Leftarrow}\check{G}\check{F}\,(x,b)$$

is defined by the unique morphism $\alpha_b : w \to GF(b) \downarrow \eta_y$ in $\mathbb{B}$ such that the equations

$$\begin{tikzcd}
x \arrow[dr, "\alpha_{(x,b)}"] & & & & & \\
& GF(b)\downarrow\eta_y \arrow[r, "(GF(b))^{\Rightarrow}(\eta_y)"] \arrow[d, "\eta_y^{\Leftarrow}(GF(b))"'] & GF(x) \arrow[d, "GF(b)"] \arrow[dl, phantom, "\overset{\chi^{ca\downarrow c}}{\Longleftarrow}"] & = & x \arrow[r, "\eta_x"] \arrow[d, "b"'] & GF(x) \arrow[d, "GF(b)"] \arrow[dl, phantom, "="] \\
& y \arrow[r, "\eta_y"'] & GF(y) & & y \arrow[r, "\eta_y"'] & GF(y)
\end{tikzcd} \tag{6.6.1}$$

$$(GF(b))^{\Rightarrow}\,(\eta_y)\ \cdot\ \alpha_b = \eta_w \qquad \text{and} \qquad \eta_y^{\Leftarrow}\,(GF(b))\ \cdot\ \alpha_b = b$$

hold.

6.7. Remark. The composition of $\varepsilon_{F(y)}! \circ \check{F}$ with $\eta_y!$ is given by $\check{F}$. More precisely, the diagrams

$$\begin{tikzcd}
\mathbb{A}//F(y) & \mathbb{B}//GF(y) \arrow[l, "\varepsilon_{F(y)}\underline{!}\circ\check{F}"] & \mathbb{B}/y \arrow[l, "\eta_y\bar{!}"] \arrow[ll, bend right, "\check{F}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A}/F(y) & \mathbb{B}/GF(y) \arrow[l, "\varepsilon_{F(y)}!\circ\check{F}"] & \mathbb{B}/y \arrow[l, "\eta_y!"] \arrow[ll, bend right, "\check{F}"']
\end{tikzcd}$$

commute.

As direct consequences of the main results of Section 4, we get the following corollaries.

6.8. Corollary. [Simplicity [10]] *Let G be pre-Kock-Zöberlein. The 2-adjunction*

$$(F \dashv G, \varepsilon, \eta) : \mathbb{A} \to \mathbb{B}$$

is simple if, and only if, for each $y \in \mathbb{B}$,

$$\mathrm{id}_T * \alpha \dashv \mu \cdot (\mathrm{id}_T * \delta * \mathrm{id}_T)$$

in which $(\mathrm{id}_T * \alpha)$ is pointwise defined by $(\mathrm{id}_T * \alpha)_b := T(\alpha_{(x,b)})$, and $\mu \cdot (\mathrm{id}_T * \delta * \mathrm{id}_T)$ is pointwise defined by

$$(\mu \cdot (\mathrm{id}_T * \delta * \mathrm{id}_T))_b \quad := \quad \begin{tikzcd}
T\,(T(b)\downarrow\eta_y) \arrow[r, "T\left(\delta_{T(b)}\right)"] \arrow[d, "T\left(\eta_y^{\Leftarrow}(T(b))\right)"'] & TT(x) \arrow[r, "\mu_x"] \arrow[d, "TT(b)"] \arrow[dl, phantom, "\overset{T\left(\chi^{T(b)\downarrow\eta_y}\right)}{\Longleftarrow}"] & T(x) \arrow[d, "T(b)"] \arrow[dl, phantom, "="] \\
T(y) \arrow[r, "T(\eta_y)"'] & TT(y) \arrow[r, "\mu_y"'] & T(y)
\end{tikzcd}$$

Proof. The result follows from Corollary 6.3 and Theorem 4.8. ∎

6.9. Corollary. *Assume that $F \dashv G$ is lax idempotent. We have that $F \dashv G$ is simple provided that, for each $y \in \mathbb{B}$, $\eta_y^{\Leftarrow} T\delta\check{G}$ or $F\delta T$ is invertible.*

Proof. It follows from Corollary 6.3 and Corollary 4.10. ■

6.10. Corollary. [2-admissibility] *Assume that G is pre-Kock-Zöberlein. The 2-adjunction $(F \dashv G, \varepsilon, \eta) : \mathbb{A} \to \mathbb{B}$ is 2-admissible if and only if it is simple and, for every object $y \in \mathbb{B}$ and every object $a : w \to F(y)$ of $\mathbb{A}//F(y)$, the morphism defined by*

$$\begin{tikzcd}
F\left(G(a) \downarrow \eta_y\right) \arrow[r, "F\left(\delta_{G(a)}\right)"] \arrow[d, "F\left(\eta_y^{\Leftarrow}(G(a))\right)"'] \arrow[dr, phantom, "\overset{F\left(\chi^{G(a)\downarrow\eta_y}\right)}{\Longleftarrow}"] & FG(w) \arrow[r, "\varepsilon_w"] \arrow[d, "FG(a)"] \arrow[dr, phantom, "="] & w \arrow[d, "a"] \\
F(y) \arrow[r, "F(\eta_y)"'] & FGF(y) \arrow[r, "\varepsilon_{F(y)}"'] & F(y)
\end{tikzcd}$$

in $\mathbb{A}//F(y)$ is a regular epimorphism, i.e. the morphism defined by

$$\left(\varepsilon_w \cdot F\left(\delta_{G(a)}\right), \mathrm{id}_{\varepsilon_{F(y)}} * F\left(\chi^{G(a)\downarrow\eta_y}\right)\right) : \varepsilon_{F(y)}!\,\check{F}\,\eta_y^{\Leftarrow}\,\check{G}(a) \to a$$

is a regular epimorphism in $\mathbb{A}//F(y)$.

Proof. The result follows from Corollary 6.3 and Theorem 4.13. ■

6.11. Corollary. *If G is pre-Kock-Zöberlein then $F \dashv G$ is 2-admissible, provided that, for each $y \in \mathbb{B}$, $\check{F}\delta\check{G}$ is invertible.*

Proof. It follows from Corollary 6.3 and Corollary 4.15. ■

It should be noted that by Lemma 3.20 we can conclude that the notion of simplicity w.r.t. the basic fibration (admissibility w.r.t. the basic fibration) coincides with the notion of simplicity (2-admissibility) if $\mathbb{A}$ and $\mathbb{B}$ are locally discrete. This shows that the notion of simplicity and 2-admissibility can be seen as generalizations of the classical notions of simplicity and admissibility/semi-left exact reflective functors [6, 3] when categories are seen as locally discrete 2-categories. Furthermore, Theorem 6.13 shows that classical admissibility implies 2-admissibility in the presence of comma objects.

6.12. Proposition. *Assume that $F \dashv G$ is pre-Kock-Zöberlein, and $\mathbb{A}$ has comma objects. The 2-adjunction $F \dashv G$ is simple (2-admissible) if, and only if, for each object $y \in \mathbb{B}$, the 2-adjunction*

$$\begin{tikzcd}
\mathbb{A}//F(y) \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathrm{id}_{F(y)}^{\Leftarrow}"'] & \mathbb{A}/F(y) \arrow[l, bend right, "\mathrm{id}_{F(y)}\bar{!}"']
\end{tikzcd} \qquad (6.12.1)$$

is simple (2-admissible) w.r.t. the composite of the 2-adjunctions

$$\begin{tikzcd}
\mathbb{A}/F(y) \arrow[r, bend right, "\check{G}"'] \arrow[rr, bend right=40, "\eta_y^*\circ\check{G}"'] \arrow[r, phantom, "\perp{\scriptstyle(\varepsilon,\eta)}"] & \mathbb{B}/GF(y) \arrow[l, bend right, "\varepsilon_{F(y)}!\circ\check{F}"'] \arrow[r, bend right, "\eta_y^*"'] \arrow[r, phantom, "\perp"] & \mathbb{B}/y \arrow[l, bend right, "\eta_y!"'] \arrow[ll, bend right=40, "\check{F}"']
\end{tikzcd}$$

(6.12.2)

PROOF. By definition, $F \dashv G$ is simple (2-admissible) if, and only if, for each object $y \in \mathbb{B}$, the composition of the 2-adjunctions of (6.5.1) is lax idempotent (pre-Kock-Zöberlein). Since G is right 2-adjoint, it preserves comma objects and, hence, we get that

$$\begin{tikzcd}
\mathbb{A}//F(y) \arrow[d, bend left, "\mathrm{id}_{F(y)}^{\Leftarrow}"] & & \mathbb{A}//F(y) \arrow[d, bend left, "\check{G}"] & & \mathbb{A}//F(y) \arrow[d, bend left, "\check{G}"] \\
\mathbb{A}/F(y) \arrow[u, bend left, "\mathrm{id}_{F(y)}\bar{!}"] \arrow[u, phantom, "\dashv"] \arrow[d, bend left, "\check{G}"] & \cong & \mathbb{B}//GF(y) \arrow[u, bend left, "\varepsilon_{F(y)}\underline{!}\circ\check{F}"] \arrow[u, phantom, "\underset{\dashv}{(\varepsilon,\eta)}"] \arrow[d, bend left, "\mathrm{id}_{GF(y)}^{\Leftarrow}"] & \cong & \mathbb{B}//GF(y) \arrow[u, bend left, "\varepsilon_{F(y)}\underline{!}\circ\check{F}"] \arrow[u, phantom, "\underset{\dashv}{(\varepsilon,\eta)}"] \arrow[d, bend left, "\eta_y^{\Leftarrow}"] \\
\mathbb{B}/GF(y) \arrow[u, bend left, "\varepsilon_{F(y)}!\circ\check{F}"] \arrow[u, phantom, "\underset{\dashv}{(\varepsilon,\eta)}"] \arrow[d, bend left, "\eta_y^*"] & & \mathbb{B}/GF(y) \arrow[u, bend left, "\mathrm{id}_{GF(y)}\bar{!}"] \arrow[u, phantom, "\dashv"] \arrow[d, bend left, "\eta_y^*"] & & \mathbb{B}/y \arrow[u, bend left, "\eta_y\bar{!}"] \arrow[u, phantom, "\underset{\dashv}{(\delta,\rho)}"] \\
\mathbb{B}/y \arrow[u, bend left, "\eta_y!"] \arrow[u, phantom, "\dashv"] & & \mathbb{B}/y \arrow[u, bend left, "\eta_y!"] \arrow[u, phantom, "\dashv"] & &
\end{tikzcd}$$

(6.12.3)

in which the second 2-natural isomorphism follows from Theorem 5.10. By the definitions of simplicity and 2-admissibility (see Definitions 4.12 and 4.7), the proof is complete. ■

6.13. THEOREM. *Provided that* $\mathbb{A}$ *has comma objects, if* $(F \dashv G) : \mathbb{A} \to \mathbb{B}$ *is admissible w.r.t. the basic fibration, then it is* 2*-admissible.*

PROOF. By Theorem 5.11, the 2-functor $\mathrm{id}_{F(y)}^{\Leftarrow}$ (the right 2-adjoint of (6.12.1)) is a pre-Kock-Zöberlein 2-functor for every $y \in \mathbb{B}$.

If $F \dashv G$ is admissible w.r.t. the basic fibration, we get that, for every $y \in \mathbb{B}$, $\eta_y^* \circ \check{G}$ is full reflective. Therefore $\eta_y^* \circ \check{G} \circ \mathrm{id}_{F(y)}^{\Leftarrow}$ is a pre-Kock-Zöberlein 2-functor by Corollary 4.15. By Proposition 6.12, this means that $F \dashv G$ is 2-admissible. ■

7. Examples

The references [10, 7] provide several examples of simple 2-adjunctions/monads. In this section, we give examples of 2-admissible 2-adjunctions which, in particular, are also examples of simple 2-adjunctions.

Our first example of 2-admissible 2-adjunction is the identity. The result below follows directly from Theorem 5.11.

7.1. Lemma. *Let* $\mathbb{A}$ *be any* 2*-category with comma objects. The* 2*-adjunction* $\mathrm{id}_{\mathbb{A}} \dashv \mathrm{id}_{\mathbb{A}}$ *is* 2*-admissible.*

Of course, the identity is also an example of admissible 2-functor w.r.t. the basic fibration. Moreover, by Theorem 6.13, examples of admissible 2-functors w.r.t. the basic fibrations give us a wide class of examples of 2-admissible 2-functors.

7.2. Theorem. *Let* ord *be the* 2*-category of preordered sets, and* cat *the* 2*-category of small categories. The inclusion* 2*-functor* $\mathsf{ord} \to \mathsf{cat}$ *has a left* 2*-adjoint and it is admissible w.r.t. the basic fibration (and, hence, also* 2*-admissible).*

Proof. It is known that the underlying adjunction is admissible (w.r.t. the basic fibration) [39]. Since cat is a complete 2-category, we get that the 2-adjunction is admissible w.r.t. the basic fibration. ■

Free cocompletions of 2-categories also give us a good source for examples of admissibility w.r.t. the basic fibration. In particular, the most basic cocompletion is the free addition of the initial object.

7.3. Theorem. *Let* $\mathbb{A}$ *be a* 2*-category with pullbacks and an initial object* 0*. We denote by* $\overline{\mathbb{A}}$ *the free addition of an initial object. If* $\mathbb{A}(-,0) : \mathbb{A}^{\mathrm{op}} \to \mathsf{Cat}$ *is constantly equal to the empty category, the canonical* 2*-functor*

$$G : \mathbb{A} \to \overline{\mathbb{A}}$$

is admissible w.r.t. the basic fibration (and, hence, if $\mathbb{A}$ *has comma objects, it is* 2*-admissible as well).*

Proof. In fact $\mathbb{A} \to \overline{\mathbb{A}}$ has a left 2-adjoint if and only if $\mathbb{A}$ has initial object. Moreover, provided that $\mathbb{A}$ has initial object, we denote by η the unit of this 2-adjunction and by $\overline{0}$ the initial object freely added.

We have that η_x is invertible whenever $x \neq \overline{0}$. Therefore, in this case,

$$\eta_x^* \circ \check{G} : \mathbb{A}/x \to \overline{\mathbb{A}}/x$$

is fully faithful.

Moreover, $\eta_{\overline{0}}^{*} \circ \check{G} : \mathbb{A}/0 \to \overline{\mathbb{A}}/\overline{0}$ is clearly an isomorphism, since $\mathbb{A}/0$ and $\overline{\mathbb{A}}/\overline{0}$ are both empty.

This completes the proof that G is admissible w.r.t. the basic fibration and, hence, 2-admissible provided that it has comma objects. ■

Another example is the free cocompletion of a 2-category under (finite) coproducts.

7.4. DEFINITION. Let $\mathbb{A}$ be a 2-category. We define the 2-category $\mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$ as follows. The objects of $\mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$ are finite families of objects of $\mathbb{A}$, which can be seen as (possibly empty) lists of objects

$$(x_1, \ldots, x_n).$$

In this case, a morphism $(x_1, \ldots, x_n) \to (y_1, \ldots, y_m)$ is a list $t = (t_0, \ldots, t_n)$ in which

$$t_0 : \{1, \ldots, n\} \to \{1, \ldots, m\}$$

is a function, and, for $j > 0$,

$$t_j : x_j \to y_{t_0(j)}$$

is a morphism of $\mathbb{A}$. The composition and, hence, the identities are defined pointwise. Finally, given morphisms

$$t = (t_0, \ldots, t_n), t' = (t'_0, \ldots, t'_n) : (x_1, \ldots, x_n) \to (y_1, \ldots, y_m)$$

of $\mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$, there is no 2-cell $t \Rightarrow t'$, provided that $t_0 \neq t'_0$. Otherwise, a 2-cell $\tau : t \Rightarrow t'$ is a finite family of 2-cells

$$\left(\tau_j : t_j \Rightarrow t'_j : x_j \to y_{t_0(j)}\right)_{j\in\{1,\ldots,n\}}$$

of $\mathbb{A}$. The horizontal and vertical compositions are again defined pointwise.

There is an obvious full faithful 2-functor $I_{\mathbb{A}} : \mathbb{A} \to \mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$ which takes each object x to the family (x). As observed above, the 2-category $\mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$ is the *free cocompletion* of $\mathbb{A}$ under finite coproducts. In particular, we have:

7.5. PROPOSITION. *The fully faithful 2-functor*

$$I_{\mathbb{A}} : \mathbb{A} \to \mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$$

has a left 2-adjoint if and only if $\mathbb{A}$ has finite coproducts. In this case, the left 2-adjoint is given by the coproduct. More precisely, a 2-cell

$$(\tau_1, \ldots, \tau_n) : (t_0, \ldots, t_n) \Longrightarrow (t'_0, \ldots, t'_n) : (x_1, \ldots, x_n) \to (y_1, \ldots, y_m)$$

in $\mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$ is taken to the unique 2-cell

$$\begin{tikzcd}[column sep=huge] \coprod\limits_{j=1}^{n} x_j \arrow[r, bend left=50, ""{name=U, below}] \arrow[r, bend right=50, ""{name=D, above}] \arrow[Rightarrow, from=U, to=D] & \coprod\limits_{j=1}^{m} y_j \end{tikzcd} \tag{7.5.1}$$

induced by the 2-cells

$$\left(\begin{tikzcd} x_i \arrow[r, bend left=50, "t_i"] \arrow[r, bend right=50, "t'_i"'] \arrow[r, phantom, "\Downarrow"] & y_{t_0(i)} \arrow[r] & \coprod_{j=1}^{m} y_j \end{tikzcd}\right)_{i\in\{1,\dots,n\}} \tag{7.5.2}$$

in which the second arrows are the components of the universal cocone that gives the coproduct.

7.6. Remark. If we replace *finite families* with *arbitrary families* in Definition 7.4, we get the concept of $\mathsf{Fam}\,(\mathbb{A})$ which corresponds to the free cocompletion of $\mathbb{A}$ under coproducts.

We say that a 2-category $\mathbb{A}$ *has finite limits* if it has finite products, pullbacks and comma objects. The well-known notion of extensive category has an obvious (strict) 2-dimensional analogue. In order to simplify the hypothesis on completion of the 2-category $\mathbb{A}$, we are going to consider lextensive 2-categories.

7.7. Definition. [Lextensive 2-category] A 2-category $\mathbb{A}$ is lextensive if it has finite limits and coproducts, and, for every finite family of objects $(y_1, \dots, y_n)$, the 2-functor

$$\begin{array}{rcl} \prod_{j=1}^{n} \mathbb{A}/y_j & \to & \mathbb{A}/\coprod_{j=1}^{n} y_j \\ (a_j : w_j \to y_j)_{j\in\{1,\dots n\}} & \mapsto & \coprod_{j=1}^{n} a_j \end{array}$$

defined pointwise by the coproduct is a (Cat-)equivalence.

7.8. Theorem. *Let $\mathbb{A}$ be a lextensive 2-category. We consider the 2-adjunction*

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend right=40, "I"'] \arrow[r, phantom, "\perp_{(\varepsilon,\eta)}"] & \mathsf{Fam}_{\mathsf{fin}}\,(\mathbb{A}) \arrow[l, bend right=40] \end{tikzcd}$$

in which the right 2-adjoint is the canonical inclusion. For each finite family $Y = (y_j)_{j\in\{1,\dots,n\}}$ of objects in $\mathbb{A}$, there is a (canonical) 2-natural isomorphism

$$\begin{tikzcd} \prod_{j=1}^{n} \mathbb{A}/y_j \arrow[r, "\prod_{j=1}^{n} I_{\mathbb{A}/y_j}"] \arrow[d, "\simeq"'] \arrow[rd, phantom, "\cong"] & \prod_{j=1}^{n} \mathsf{Fam}_{\mathsf{fin}}\,(\mathbb{A}/y_j) \arrow[d, "\simeq"] \\ \mathbb{A}/\coprod_{j=1}^{n} y_j \arrow[r, "\eta_Y^{*}\circ \check{I}_{\mathbb{A}}"'] & \mathsf{Fam}_{\mathsf{fin}}\,(\mathbb{A})\,/\,(y_j)_{j\in\{1,\dots,n\}} \end{tikzcd} \tag{7.8.1}$$

Proof. The equivalence 2-functor

$$\prod_{j=1}^{n} \mathsf{Fam}_{\mathsf{fin}}\left(\mathbb{A}/y_j\right) \to \mathsf{Fam}_{\mathsf{fin}}\left(\mathbb{A}\right) / \left(y_j\right)_{j\in\{1,\ldots,n\}}$$

is such that each object

$$A = \left(\left(a_{(1,1)}, \ldots, a_{(1,m_1)}\right), \ldots, \left(a_{(n,1)}, \ldots, a_{(n,m_n)}\right)\right)$$

is taken to

$$t^A = \left(t^A_l\right)_{l\in\{0,(1,1),\ldots,(1,m_1),\ldots,(n,m_n)\}}$$

in which $t^A_0(j,k) := j$ and $t^A_{(j,k)} := a_{(j,k)}$. The action on morphisms and 2-cells is then pointwise defined. ■

7.9. Corollary. *Let $\mathbb{A}$ be a lextensive 2-category. The 2-functor $I_{\mathbb{A}} : \mathbb{A} \to \mathsf{Fam}_{\mathsf{fin}}(\mathbb{A})$ is admissible w.r.t. the basic fibration and, hence, 2-admissible.*

Proof. In fact, since products of fully faithful 2-functors are fully faithful, we get that $\eta^*_Y I_{\mathbb{A}}$ is fully faithful by the 2-natural isomorphism (7.8.1). ■

7.10. Remark. Definition 7.7 has an obvious infinite analogue, the definition of *infinitary lextensive 2-category.* For an infinitary lextensive 2-category $\mathbb{A}$, we have an analogous result w.r.t. $\mathsf{Fam}(\mathbb{A})$. More precisely,

$$I_{\mathbb{A}} : \mathbb{A} \to \mathsf{Fam}(\mathbb{A})$$

is admissible w.r.t. the basic fibration (and, hence, 2-admissible) whenever $\mathbb{A}$ is infinitary extensive.

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