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DISTRIBUTIVE IDEMPOTENTS IN AN ORDER-ENRICHED CATEGORY

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Abstract. We introduce distributive maps between lattices and consider the categorical assumption that distributive idempotents split. We explore this assumption in the context of a categorical axiomatization of the category of locales. The assumption is shown to be stable under groupoids (this includes slice stability) and we further show that it implies that triquotient surjections are effective descent morphisms. This result follows even without assuming that the underlying (axiomatized) category of locales has coequalizers.

1. Introduction

Marta Bunge showed for any étale complete localic groupoid $\mathbb{G}$ that the topos of $\mathbb{G}$-equivariant sheaves, $B\mathbb{G}$, classifies principal $\mathbb{G}$-bundles, [B90]. In that paper she also says that some negative results about $B\mathbb{G}$, '... suggest that toposes are not the right kind of structures to consider when dealing with $\mathbb{G}$-bundles for a general $\mathbb{G}$'. The author in previous work ([T05], [T10], [T12], [T17]) has attempted to axiomatise a *category of spaces*, thinking about the category of locales as the canonical example. What has been important is to find axioms that are closed under the formation of the category of $\mathbb{G}$-objects; that is, if $\mathcal{C}$ is a category of spaces then so too must be $[\mathbb{G},\mathcal{C}]$ for any (or as many as possible) groupoids $\mathbb{G}$ internal to $\mathcal{C}$. This is important because there are examples (e.g. locally connected groups) where $B\mathbb{G}$ is trivial but $[\mathbb{G},\mathcal{C}]$ is not. Anything we can do axiomatically can then be lifted to this broader context, providing evidence that this context, i.e. these 'categories of spaces', are perhaps the right kind of structure that Bunge was after.

In broad terms a category of spaces $\mathcal{C}$ is an order-enriched category that has a Sierpiński object $\mathbb{S}$ that classifies closed and open subobjects and for which double exponentiation $\mathbb{S}^{\mathbb{S}^{(-)}}$ is defined. Double exponentiation determines a double power monad on the category of spaces and from this the upper and lower power monads can be constructed. However for any reasonable theory to emerge[1] we require that the upper and lower power monads are coKZ and KZ respectively. This is not true without adding it in as an additional

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[1] For example, to prove the Hofmann-Mislove theorem [T05] or construct an ideal completion with the correct properties, [T12].

assumption; indeed two additional assumptions are needed, order dual to one another. The main aim of this paper is to resolve this by introducing a single axiom that covers both.

We do this by defining the notion of a *distributive* endomorphism on a lattice A; this is a map $\delta : A \longrightarrow A$ such that $\forall a, b, c \in A$,

$$\delta(\delta a \wedge \delta(\delta b \vee \delta c)) = \delta(\delta(\delta a \wedge \delta b) \vee \delta(\delta a \wedge \delta c)).$$

If the endomorphism is idempotent and splits, then the condition is equivalent to requiring the splitting to be a distributive lattice. It is quite a simple lattice theoretic notion and we explore a number of examples below.

We will apply this notion axiomatically to $\mathcal{C}$ by saying that certain distributive idempotents split; from this it follows easily that the upper and lower power monads are coKZ and KZ respectively (as we need for a reasonable theory to emerge). The technical work of the paper is then about checking that the new axiom is closed under the formation of $\mathbb{G}$-objects. We also go a bit further and use the new axiom to verify, (i) that surjections naturally defined via the double power monad are of effective descent; and, (ii) that for reasonable groupoids in $\mathbb{G}$, $[\mathbb{G}, \mathcal{C}]$ has a well behaved connected components adjunction. This last is analogous to checking that $B\mathbb{G}$ has a unique geometric morphism back to **Set**. These two results are new axiomatically and could not be obtained using previous assumptions; crucially, whilst both results are all about constructing certain coequalizers, they are obtained without assuming that our category of spaces has coequalizers in general.

The paper is structured as follows. In the next section we provide technical prequisites. Whilst some of the initial material and examples consist of straightforward lattice and locale theory, most of the work is fairly technical. So unfortunately the technical demands on the reader are quite high, though we have attempted to marshal the material as clearly as possible. The next section then re-introduces distributive endomorphism in the context of an order enriched category, proving some basic results about them and providing plenty of examples. Next we write out the categorical axioms for a category of spaces, recall some known results that follow from them and repeat in technical detail the aims of the paper. The remainder of the paper is essentially about meeting those aims: proving axiomatic stability, showing that certain surjections are of effective descent and that certain connected components exist.

2. Prerequisites

The reader will need to be comfortable with categorical terms and familiar with the theory of locales, in particular the terms frame, suplattice, preframe and directed complete partial order (dcpo); see, e.g. Part C of [J02]. Our general context will be a cartesian order-enriched category $\mathcal{C}$. So, for example, for any objects X, Y and Z of $\mathcal{C}$, $(\_, \_) : \mathcal{C}(Z, X) \times \mathcal{C}(Z, Y) \longrightarrow \mathcal{C}(Z, X \times Y)$ is an order isomorphism (and not just a bijection). We will be interested in *order-internal* lattices in $\mathcal{C}$, by which we mean that finite joins(meets) are required to be left(right) adjoint to finite diagonals (where ‘finite’ includes the nullary case). We will sometimes use point set notation to argue about diagrams in categories. A reflexive pair of arrows is a pair of arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$ together with a common section (i.e. a map $s : B \longrightarrow A$ such that $fs = gs = Id_B$). A morphism $f : X \longrightarrow Y$ is *of effective descent* or *is an effective descent morphism* if the pullback functor $f^* : \mathcal{C}/Y \longrightarrow \mathcal{C}/X$ is monadic. Note that f^* reflects isomorphisms if f is a pullback stable regular epimorphism. So, for a pullback stable regular epimorphism, to prove that it is of effective descent you just need to check there is a pullback stable coequalizer for any pair of morphisms (over Y) that is f^*-split (this is by application of Beck's theorem).

Adjunctions that satisfy Frobenius reciprocity (Frobenius adjunctions) will also play a role in what follows. Any adjunction between two categories, with the left adjoint going from left to right, can be sliced at any object of the codomain and an adjunction is *stably* Frobenius if it is Frobenius at each such slice. Any pullback adjunction $\Sigma_f \dashv f^*$ is stably Frobenius. Given an internal groupoid $\mathbb{G} = (G_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} G_0, m : G_1 \times_{G_0} G_1 \longrightarrow G_1, s : G_0 \longrightarrow G_1, i : G_1 \longrightarrow G_1)$ there is always a functor $\mathbb{G}^* : \mathcal{C} \longrightarrow [\mathbb{G}, \mathcal{C}]$ whose codomain is the category of $\mathbb{G}$-objects; $\mathbb{G}^*$ takes X to X_{G_0} (i.e. $\pi_1 : G_0 \times X \longrightarrow G_0$) equipped with the trivial $\mathbb{G}$-action. The functor $\mathbb{G}^*$ does not always have a left adjoint (connected components cannot in general be formed) but it will do if for every $\mathbb{G}$-object $(X_p, a : G_1 \times_{G_0} X \longrightarrow X)$ the coequalizer of π_2, a exists. If the coequalizer exists and the resulting coequalizer diagram is pullback stable then the connected components adjunction $\Sigma_{\mathbb{G}} \dashv \mathbb{G}^*$ is stably Frobenius. Note that for any morphism $f : X \longrightarrow Y$ we use the notation X_f when considering f as an object of the slice category $\mathcal{C}/Y$. Also, we use the notation $\Sigma_Y \dashv Y^* : \mathcal{C}/Y \rightleftarrows \mathcal{C}$ for the pullback adjunction of $! : Y \longrightarrow 1$ and X_Y for Y^*X (i.e. for $\pi_1 : Y \times X \longrightarrow Y$).

Moving on now to locale theory, we recall the definition of weak triquotient assignment:

2.1. Definition. *Given a locale map $f : X \longrightarrow Y$ a* weak triquotient assignment on f *is a dcpo homomorphism $f_\# : \mathcal{O}X \longrightarrow \mathcal{O}Y$ satisfying* $\forall a \in \mathcal{O}X$, $\forall b \in \mathcal{O}Y$

$$\begin{aligned} f_\#(a \vee f^*b) &\leq f_\#(a) \vee b \\ f_\#(a) \wedge b &\leq f_\#(a \wedge f^*b) \end{aligned}$$

A locale map f is open (proper) if and only if f^* has a left(right) adjoint that is a weak triquotient assignment on f. In this way, weak triquotient assignements can be used to think about how both open and proper maps behave in a manner that covers both classes. A weak triquotient assignment $f_\#$ is a *triquotient assignment* if further $f_\#(0) = 0$ and $f_\#(1) = 1$; note that this condition is equivalent to $f_\# f^* = Id_{\mathcal{O}Y}$ and so any f with a triquotient assignment is necessarily an epimorphism in **Loc**. A morphism with a triquotient assigment is a *triquotient surjection.*

2.2. Remark. *If* $\psi : Z \longrightarrow X$ *is an isomorphism and* $f_\#$ *a (weak) triquotient on* $f : X \longrightarrow Y$ *then* $f_\#(\psi^{-1})^*$ *is a (weak) triquotient assignment on* $f\psi$*. This follows because* $(\psi^{-1})^*$ *is a lattice homomorphism.*

Finally we clarify how dcpo homomorphisms between frames correspond to natural transformations. It is with this correspondence that localic definitions and results can be placed in a categorical context. For any locale X we write $\mathbb{S}^X$ for the presheaf $\mathbf{Loc}(\_ \times X, \mathbb{S}) : \mathbf{Loc}^{op} \longrightarrow \mathbf{Set}$. We use this notation even if X is not exponentiable in $\mathbf{Loc}$ (however, note that $\mathbb{S}^X$ is the exponential $y\mathbb{S}^{yX}$ in $[\mathbf{Loc}^{op}, \mathbf{Set}]$ where y is the Yoneda embedding).

2.3. Theorem. *Naturally in locales* X *and* Y *there is an order isomorphism between the poset of dcpo homomorphisms* $\mathcal{O}X \longrightarrow \mathcal{O}Y$ *and natural transformations* $\mathbb{S}^X \longrightarrow \mathbb{S}^Y$*. Under this isomorphism a frame homomorphism* $f^* : \mathcal{O}X \longrightarrow \mathcal{O}Y$ *corresponds to* $\mathbb{S}^f$*, i.e. the exponential in* $[\mathbf{Loc}^{op}, \mathbf{Set}]$

Proof. [VT04]. ∎

We use $\overline{\mathbf{Loc}}^{op}$ for the full subcategory of $[\mathbf{Loc}^{op}, \mathbf{Set}]$ consisting of presheaves of the form $\mathbb{S}^X$. By the Theorem $\overline{\mathbf{Loc}}^{op}$ is equivalent to the category of frames with dcpo homomorphisms between them.

3. Distributive idempotents

We start by writing out more formally some of the material covered in the Introduction.

3.1. Proposition. *Let* $\mathcal{C}$ *be an order-enriched category with finite products.*

(i) If $\delta : A \longrightarrow A$ *is an idempotent in* $\mathcal{C}$*, with a splitting* $A \xrightarrow{q} Fix(\delta) \xrightarrow{i} A$ *(i.e.* $iq = \delta$ *and* $qi = Id_{Fix(\delta)}$*), then* $Fix(\delta)$ *inherits any order-internal lattice structure from* A*.*

(ii) If $\delta : A \longrightarrow A$ *and* $\gamma : B \longrightarrow B$ *are two idempotents with* A *and* B *order-internal lattices, then any lattice homomorphism* $f : A \longrightarrow B$ *that commutes with* δ *and* γ *(that is,* $\gamma f = f\delta$*) induces a lattice homomorphism* $Fix(f) : Fix(\delta) \longrightarrow Fix(\gamma)$*.*

(iii) If A *is an order-internal lattice and* $\delta : A \longrightarrow A$ *a split idempotent, then* $Fix(\delta)$

is an order-internal distributive lattice if and only if the diagram

$$\begin{array}{ccc}
A \times A \times A & \xrightarrow{(\delta\pi_1,\delta\pi_2,\delta\pi_1,\delta\pi_3)} & A \times A \times A \times A \\
{\scriptstyle (\pi_1, \vee_A(\delta\pi_2, \delta\pi_3))}\downarrow & & \downarrow{\scriptstyle \delta \wedge_A \times \delta\wedge_A} \\
A \times A & & A \times A \\
{\scriptstyle \delta \times \delta}\downarrow & & \downarrow{\scriptstyle \delta\vee_A} \\
A \times A & \xrightarrow{\delta\wedge_A} & A
\end{array}$$

commutes.

(iv) The diagram in (iii) commutes if and only if the same diagram, but with joins and meets interchanged, commutes.

Proof. (i) Verify that $1 \xrightarrow{0_A} A \xrightarrow{q} Fix(\delta)$, where 0_A is the bottom of A, is left adjoint to $! : Fix(\delta) \longrightarrow 1$; this defines the nullary join of $Fix(\delta)$. Binary join is $Fix(\delta) \times Fix(\delta) \xrightarrow{i \times i} A \times A \xrightarrow{\vee_A} A \xrightarrow{q} Fix(\delta)$. To check that this join is compatible with the order we calculate

$$\begin{aligned}
\vee_{Fix(\delta)}\Delta_{Fix(\delta)} &= q \vee_A (i \times i)\Delta_{Fix(\delta)} \\
&= q \vee_A \Delta_A i \\
&\sqsubseteq qi = Id_{Fix(\delta)}
\end{aligned}$$

and similarly $Id_{Fix(\delta)\times Fix(\delta)} \sqsubseteq \Delta_{Fix(\delta)}\vee_{Fix(\delta)}$ so that $\vee_{Fix(\delta)} \dashv \Delta_{Fix(\delta)}$.

The definitions and calculations are essentially the same for nullary and binary meet.

(ii) Follows easily from the explicit descriptions just given of finitary join and meet operations on the fixed objects of the idempotents. The induced map $Fix(f)$ is given by $q_\gamma f i_\delta$.

(iii) As $\delta(\delta(\_) \wedge_A \delta(\_))$ factors as $A \times A \xrightarrow{q \times q} Fix(\delta) \times Fix(\delta) \xrightarrow{\wedge_{Fix(\delta)}} Fix(\delta) \xrightarrow{i} A$ (and similarly for join) this is a routine diagram chase exploiting the fact that $q \times q \times q$ is an epimorphism (split by $i \times i \times i$).

(iv) A lattice is distributive if and only if meet distributes over join, if and only if join distributes over meet. So if the diagram in (iii) commutes, $Fix(\delta)$ is distributive, so the same reasoning used in (iii) can be applied, but this time using the fact that join distributes over meet in $Fix(\delta)$ to prove that the interchanged diagram commutes. ∎

3.2. Definition. *A morphism $\delta : A \longrightarrow A$ on an order-internal lattice is said to be* distributive *if the diagram in part (iii) of the Proposition commutes.*

If idempotents split, part (iii) of the Proposition shows that distributive idempotents always give rise to order-internal distributive lattices, and part (iv) shows that the property of being distributive is stable under reversal of the order enrichment on $\mathcal{C}$. (Whilst we have defined distributive for any endomorphism, I have not been able to think of any interesting examples which are not idempotent.)

3.3. Example. *Trivially the identity map on a lattice A is distributive if and only if A is a distributive lattice.*

3.4. Example. *For any lattice A, every element $x \in A$ gives rise to a distributive idempotent $a \mapsto x$ (i.e. the constant map to x). The fixed point distributive lattice that arises in this way is the trivial (singleton) distributive lattice. Therefore the injection of the splitting of a distributive idempotent will not necessarily preserve top or bottom.*

3.5. Example. *The projection of the splitting of any order preserving idempotent will necessarily preserve top and bottom. To see that binary meet and join is not preserved by the projection in general, consider the power set of $\{a, b\}$, with an idempotent sending $\{a\}$ to $\{b\}$ and all other subsets fixed. The image of the join (meet) of $\{a\}$ and $\{b\}$ is $\{a, b\}$ ($\{\}$), but the join (meet) of the image of $\{a\}$ and the image of $\{b\}$ is $\{b\}$.*

3.6. Example. *For a non-distributive example consider the M_3 lattice (i.e. $\{0 \le x, y, z \le 1\}$). Similarly to the power set of $\{a, b\}$, we can project onto $\{0 \le z \le 1\}$ and obtain the 3 element chain (a distributive lattice) as the set of fixed points; therefore the induced idempotent is distributive.*

3.7. Example. *If $\delta : A \longrightarrow A$ is an idempotent join semilattice homomorphism then it is clear that δ is distributive if A is distributive. This essentially covers the reasoning needed to show that the fixed points of an idempotent suplattice homomorphism on a frame form a frame. Similarly idempotent meet semilattice homomorphisms are distributive if A is distributive and this shows that the fixed points of any idempotent preframe homomorphism on a frame form a frame.*

3.8. Example. *Any morphism $\delta : A \longrightarrow A$ with $\delta a \wedge \delta b = \delta(\delta a \wedge \delta b)$ and $\delta(\delta a \vee \delta b) = \delta a \vee \delta b$ is distributive if A is distributive; this follows by an obvious substitution into the equation used to define distributive morphism. This example covers the idempotents that arise from semi-triquotient maps introduced in [M22]. I.e. this example includes the idempotents $q^* q_0$, where q^* is an injective frame homomorphism split by a dcpo homomorphism q_0. This provides examples where the idempotent is neither necessarily a meet or a join semilattice homomorphism; specifically, Example 3.5.*

3.9. Example. *We say that an idempotent $\delta : A \longrightarrow A$ is a* weak triquotient *if $\delta a \wedge \delta b \le \delta(a \wedge \delta b)$ and $\delta(a \vee \delta b) \le \delta a \vee \delta b$. If A is distributive then any weak triquotient idempotent is distributive as the defining conditions are stronger than the previous example. Moreover the injection of the splitting of a weak trquotient idempotent δ must preserve binary joins* *and meets. To see this for binary joins, recall that join in* $Fix(\delta)$ *is given by* $q(i(\_)\vee_A i(\_))$*; then we have that*

$$\begin{aligned} i(c_1 \vee c_2) &= i(q(i(c_1) \vee i(c_2))) \\ &= \delta(i(c_1) \vee i(c_2)) \\ &= \delta(i(c_1) \vee \delta(i(c_2))) \\ &\leq \delta(i(c_1)) \vee \delta(i(c_2)) \\ &= i(c_1) \vee i(c_2). \end{aligned}$$

An order dual argument shows that i *preserves binary meets. Any weak triquotient assignment* $p_\#$ *on a locale map* $p : X \longrightarrow Y$ *gives rise to two weak triquotient idempotents:* $p^* p_\#$ *and* $p_\# p^*$.

3.10. Example. *Next,* $\delta : A \longrightarrow A$ *is a* triquotient *idempotent if it is a weak triquotient idempotent,* $\delta(0) = 0$ *and* $\delta(1) = 1$*. Because the surjection of any splitting of an idempotent necessarily preserves top and bottom, the injection of the splitting of any idempotent* δ *must preserve top and bottom if* $\delta(0) = 0$ *and* $\delta(1) = 1$*. This observation, combined with the previous example, shows that the injection of the splitting of a triquotient idmpotent must be a lattice homomorphism (i.e. preserves all finitary joins and meets). Triquotient idempotents are introduced in [M22]. Any triquotient assignment* $p_\#$ *on a locale map* $p : X \longrightarrow Y$ *gives rise to a triquotient idempotent:* $p^* p_\#$*; its fixed set is* $\mathcal{O}Y$*. In the other direction notice that any triquotient idempotent* $\delta : \mathcal{O}X \longrightarrow \mathcal{O}X$ *that is also a dcpo homomorphism splits as* $p^* p_\#$ *where* $p_\#$ *is a triquotient assignment on* p.

3.11. Example. *Not all dcpo splittings of frame injections are triquotient assigments, so this provides examples that satisfy Example 3.8 but are not triquotient; Example 3.5 can again be used. Certainly weak triquotient idempotents can arise from weak triquotients* $p_\#$ *on locale maps* p *that are not necessarily surjections; any constant map is a weak triquotient for* p*, and this will not be a surjection (unless the codomain, and therefore domain, of* p *is trivial). This provides examples of weak triquotient idempotents that are not triquotient idempotents. The example using the constant map gives a trivial fixed points frame, but the images of any open or proper map provide a plentiful supply of non-trivial examples; e.g. any open or closed sublocale.*

3.12. Example. *If* $\delta : \mathcal{O}X \longrightarrow \mathcal{O}X$ *is a distributive idempotent dcpo homomorphism then* $Fix(\delta)$ *is a frame. From Proposition 3.1 (iii)* $Fix(\delta)$ *is a distributive lattice and it is a dcpo as* δ *is a dcpo homomorphism. So it is complete and we just need to check that* *meet distributes over directed join to conclude that it is a frame. This is straightforward:*

$$\begin{aligned} a \wedge_{Fix(\delta)} \bigvee^{\uparrow}_{Fix(\delta)} S &= q(ia \wedge_{\mathcal{O}X} i \bigvee^{\uparrow}_{Fix(\delta)} S) \\ &= q(ia \wedge_{\mathcal{O}X} \bigvee^{\uparrow}_{\mathcal{O}X} \{is | s \in S\}) \\ &= q(\bigvee^{\uparrow}_{\mathcal{O}X} \{ia \wedge_{\mathcal{O}X} is | s \in S\}) \\ &= \bigvee^{\uparrow}_{Fix(\delta)} \{q(ia \wedge_{\mathcal{O}X} is) | s \in S\}) \\ &= \bigvee^{\uparrow}_{Fix(\delta)} \{a \wedge_{Fix(\delta)} s) | s \in S\}). \end{aligned}$$

Notice that this also implies that distributive idempotents split in $\overline{\mathbf{Loc}}^{op}$ (use Theorem 2.3).

4. Categorical context

We now clarify our categorical context:

Axiom 1. $\mathcal{C}$ is an order-enriched cartesian category with finite coproducts.

Axiom 2. For any morphism $f : X \longrightarrow Y$ the functor $f^* : \mathcal{C}/Y \longrightarrow \mathcal{C}/X$ preserves finite coproducts.

Axiom 3. (Sierpiński object) $\mathcal{C}$ has an order-internal distributive lattice $\mathbb{S}$ such that for any object X the pullback $i^* : \mathcal{C}(X, \mathbb{S}) \longrightarrow Sub(X)$ is an injection for both $i = 0_{\mathbb{S}}$ and $i = 1_{\mathbb{S}}$.

Axiom 4. Any natural transformation $\alpha : \mathbb{S}^X \longrightarrow \mathbb{S}^Y$ which is also a lattice homomorphism is of the form $\mathbb{S}^f$ for some unique $f : Y \longrightarrow X$.

Axiom 5. (Double coverage axiom) For any equalizer diagram $E \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in $\mathcal{C}$ the diagram

$$\mathbb{S}^X \times \mathbb{S}^X \times \mathbb{S}^Y \underset{\sqcap(Id\times\sqcup)(Id\times Id\times\mathbb{S}^g)}{\overset{\sqcap(Id\times\sqcup)(Id\times Id\times\mathbb{S}^f)}{\rightrightarrows}} \mathbb{S}^X \xrightarrow{\mathbb{S}^e} \mathbb{S}^E$$

is a coequalizer in $\overline{\mathcal{C}}^{op}$.

Axiom 6. (Double power axiom) For any object X of $\mathcal{C}$ the exponential $y\mathbb{S}^{\mathbb{S}^X}$ in $[\mathcal{C}^{op}, \mathbf{Set}]$ exists and is representable.

Axiom 7. Distributive idempotents split in $\overline{\mathcal{C}}^{op}$.

Axioms 1-6 are true when $\mathcal{C} = \mathbf{Loc}$; see [T10] for all but Axiom 4, which is clear from Theorem 2.3. Further:

(a) They are stable under the reversal of order enrichment. This is trivial for all of them but Axiom 5, and clear for Axiom 5 by exploiting the distributivity assumption on $\mathbb{S}$.

(b) They are slice stable (Theorem 3.3 of [T10] and Proposition 3.1.3 of [T12] for Axiom 4). The Sierpiński object needed for Axiom 3 in $\mathcal{C}/X$ is $\mathbb{S}_X$.

(c) They are stable under the formation of the category of $\mathbb{G}$-objects; see the last section of [T17].

4.1. Definition. *A category* $\mathcal{C}$ *is a* category of spaces *if it satisfies Axioms 1-7.*

Example 3.12 shows how Axiom 7 is satisfied when $\mathcal{C} = \mathbf{Loc}$. Therefore:

4.2. Proposition. **Loc** *is a category of spaces.*

Given Example 3.7, Axiom 7 implies the Axiom 6 used in [T05] to provide a categorical account of the Hofmann-Mislove theorem. It is this connection that supports the general claim made in the Introduction that a reasonable 'locale-like' theory can emerge with Axiom 7.

5. Categorical change of base and pullback stability

We now need to include some comments about how change of base works axiomatically and how maps with triquotient assignments, now defined axiomatically, pullback. The technical results that we recall here only exploit Axioms 1-6 and will be key for the rest of the paper. They are all known aspsects of locale theory, so this Section is nothing more than writing out some known locale theory results, but now for categories of spaces.

5.1. Axiomatic change of base. To start, recall Lemma 3.1 of [T10] which shows how change of base extends to natural transformations; that is, for any morphism $f : X \longrightarrow Y$ of $\mathcal{C}$, the pullback adjunction $\Sigma_f \dashv f^*$ extends to $f^\# \dashv f_* : \overline{\mathcal{C}/X}^{op} \rightleftarrows \overline{\mathcal{C}/Y}^{op}$. Recall from the proof of Lemma 3.1 in [T10] that the units and counits of $f^\# \dashv f_*$ are determined by the counits and units of $\Sigma_f \dashv f^*$ respectively via $\bar{\eta} = \mathbb{S}_Y^{\epsilon}$ and $\bar{\epsilon} = \mathbb{S}_X^{\eta}$. In fact $\bar{\epsilon}$ is always a regular epimorphism:

5.2. Lemma. *Let* $\mathcal{C}$ *be a category satisfying Axioms 1-6.*

(i) If $E \xrightarrow{e} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ *is an equalizer in* $\mathcal{C}$ *and the pair* f, g *are coreflexive then*

$$\mathbb{S}^Y \overset{\mathbb{S}^f}{\underset{\mathbb{S}^g}{\rightrightarrows}} \mathbb{S}^X \xrightarrow{\mathbb{S}^e} \mathbb{S}^E$$

is a reflexive coequalizer in $\overline{\mathcal{C}}^{op}$.

(ii) Let Z_p *be an object of* $\mathcal{C}/X$.

$$\mathbb{S}_X^{X^*\Sigma_X X^*\Sigma_X Z_p} \overset{\mathbb{S}_X^{\eta_{X^*\Sigma_X Z_p}}}{\underset{\mathbb{S}_X^{X^*\Sigma_X \eta_{Z_p}}}{\rightrightarrows}} \mathbb{S}_X^{X^*\Sigma_X Z_p} \xrightarrow{\mathbb{S}_X^{\eta_{Z_p}}} \mathbb{S}_X^E$$

is a reflexive coequalizer in $\overline{\mathcal{C}/X}^{op}$.

Proof. (i) Let $s : Y \longrightarrow X$ be a common section of f and g. The pair of arrows displayed in Axiom 5 then factors through the pair $\mathbb{S}^f$, $\mathbb{S}^g$ via $\mathbb{S}^X \times \mathbb{S}^X \times \mathbb{S}^Y \xrightarrow{\mathbb{S}^s \times \mathbb{S}^s \times \mathbb{S}^{Id}} \mathbb{S}^Y \times \mathbb{S}^Y \times \mathbb{S}^Y \xrightarrow{\sqcap(Id \times \sqcup)} \mathbb{S}^Y$. To see this note that $\mathbb{S}^f$ is a lattice homomorphism for any f (so Axiom 4 is really an 'if and only if'). Therefore (i) follows from Axiom 5.

(ii) In $\mathcal{C}/X$ there is a coreflexive equalizer diagram

$$Z_p \xrightarrow{\eta_{Z_p}} X^*\Sigma_X Z_p \underset{X^*\Sigma_X \eta_{Z_p}}{\overset{\eta_{X^*\Sigma_X Z_p}}{\rightrightarrows}} X^*\Sigma_X X^*\Sigma_X Z_p;$$

the common section is $X^*\epsilon_{\Sigma_X Z_p}$. To see this a little bit more explicitly notice that η_{Z_p} is $Z \xrightarrow{(p, Id_Z)} X \times Z$; so $\eta_{X^*\Sigma_X Z_p}$ is $\Delta_X \times Id_Z : X \times Z \longrightarrow X \times X \times Z$ and $X^*\Sigma_X \eta_{Z_p}$ is $Id_X \times (p, Id_Z) : X \times Z \longrightarrow X \times X \times Z$ (and $X^*\epsilon_{\Sigma_X Z_p}$ is $\pi_{13} : X \times X \times X \longrightarrow X \times Z$). So (ii) follows from (i) carried out in $\mathcal{C}/X$. ∎

Beck-Chevalley also works at the level of categories:

5.3. Lemma. *If*

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{\pi_2} & Z \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & Y \end{array}$$

is a pullback diagram in $\mathcal{C}$ *then* $(\pi_1)_* \pi_2^\# = f^\# p_*$*; that is,*

$$\begin{array}{ccc} \overline{\mathcal{C}/X \times_Y Z}^{op} & \xleftarrow{(\pi_2)^\#} & \overline{\mathcal{C}/Z}^{op} \\ {\scriptstyle (\pi_1)_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \overline{\mathcal{C}/X}^{op} & \xleftarrow{f^\#} & \overline{\mathcal{C}/Y}^{op} \end{array}$$

commutes.

Proof. This follows from the definition of the extended change of base adjunction given in Lemma 3.1 of [T10]. It is easiest to change base and see that all is required is a check of the case $Y = 1$. ∎

5.4. Pullback stability of triquotients. It should now be clear how to define (weak) triquotient assignments on maps in $\mathcal{C}$ given their definition relative to the category of locales and the fact that dcpo homomorphisms between frames can be represented as natural transformations (Theorem 2.3). Being a triquotient assignment and the property of being distributive are preserved by change of base in both directions:

5.5. Lemma. *For any morphism $f : X \longrightarrow Y$ in a category $\mathcal{C}$ satisfying Axioms 1-6, both of the functors $f^{\#}$ and f_* preserve distributive idempotents and triquotient assignments.*

Proof. Recall Lemma 3.2 of [T10], which shows that the extended adjunction $f^{\#} \dashv f_*$ preserves meets and joins in both directions. From this the lemma is clear as the definition of a distributive morphism and triquotient assignment is only in terms of meets and joins. ∎

The following final proposition for this Section recalls how triquotients assignments interact with change of base and from this the usual 'Beck-Chevalley for pullback squares' result follows axiomatically:

5.6. Proposition. *Let $\mathcal{C}$ be a category satisfying Axioms 1-6.*

(a) For any morphism $p : Z \longrightarrow Y$ of $\mathcal{C}$ there is an order isomorphism between triquotient assignments $p_{\#} : \mathbb{S}^Z \longrightarrow \mathbb{S}^Y$ on p and triquotient assignments $\mathbb{S}_Y^{Z_p} \longrightarrow \mathbb{S}_Y$ on $!^{Z_p} : Z_p \longrightarrow 1$. We use the notation $\alpha^{p\#} : \mathbb{S}_Y^{Z_p} \longrightarrow \mathbb{S}_Y$ for the unique natural transformation corresponding to $p_{\#}$ under this order isomorphism. Then: (i) $\alpha^{p\#}\mathbb{S}_Y^{\eta_{Z_p}} = \widetilde{p_{\#}}$ (where $\widetilde{(\_)}$ denotes adjoint transpose accros $Y^{\#} \dashv Y_*$); and, (ii) $p_{\#} = Y_*(\alpha^{p\#})$.

(b) If $p_{\#}$ is a (weak) triquotient assignment on $p : Z \longrightarrow Y$ then for any $f : X \longrightarrow Y$ there is a unique (weak) triquotient assignment $(\pi_1)_{\#}$ on $\pi_1 : X \times_Y Z \longrightarrow X$ such that $(\pi_1)_{\#}\mathbb{S}^{\pi_2} = \mathbb{S}^f p_{\#}$. Explicitly, $(\pi_1)_{\#} = X_* f^{\#}(\alpha^{p\#})$.

Proof. (a) See the main result (Theorem 5.5) of [T10]. It shows that natural transformations $\mathbb{S}_X^{Y_f} \longrightarrow \mathbb{S}_X$ are in order isomorphism with weak triquotient assignments on f with the relationship (i) holding; (ii) follows from (i) by naturalness of the adjunction.

Now any natural transformation $\mathbb{S}_X^{Y_f} \longrightarrow \mathbb{S}_X$ is a weak triquotient assignment on $!^{Y_f}$ (see Lemma 4.3 of [T10]). So (a) follows by checking that property of preserving 0 and 1 is unchanged under the order order isomorphism. This is clear from the construction of the isomorphism (or see Corollary 24 of [T04] for more detail).

(b) See Proposition 6.1 [T10]. ∎

6. Restatement of technical aims

The remaining aim of the paper is to show that the categorical assumption that distributive idempotents split in $\overline{\mathcal{C}}^{op}$

(i) is slice stable,

(ii) is groupoid stable; i.e. true of $[\mathbb{G}, \mathcal{C}]$ if true of $\mathcal{C}$, for any internal groupoid $\mathbb{G}$,

(iii) implies triquotient surjections are effective descent morphisms; and,

(iv) implies that $[\mathbb{G}, \mathcal{C}]$ has a stably Frobenius connected components adjunction, provided $!^{\mathbb{G}} : \mathbb{G} \longrightarrow 1$ is a triquotient surjection relative to $[\mathbb{G}, \mathcal{C}]$.

6.1. Remark. *The importance of (iii) and (iv) is that these can be shown* without assuming $\mathcal{C}$ has coequalizers. *Indeed we can't assume all coequalizers exist if we want (ii) to be true for an axiomatic approach to locale theory. This is because coequalizers would need to be pullback stable to lift to $[\mathbb{G}, \mathcal{C}]$, and coequalizers in* **Loc** *are not pullback stable.*

6.2. Remark. *The condition '$!^{\mathbb{G}}$ is a triquotient surjection' in (iv) is quite natural: it captures the notion of bounded geometric morphism in topos theory and is a condition under which an axiomatic theory of bounded geometric morphism works, see [T18]. Further, as we will show below, the condition is true for all open (and proper) localic groupoids; i.e. the usual cases of interest are covered.*

6.3. Remark. *It is trivial that Axiom 7 is stable under reversal of the order enrichment (see Proposition 3.1 (iv)). Therefore the property of being a category of spaces is stable under reversal of the order enrichment. The importance of reversing the order enrichement is that discrete objects are mapped to compact Hausdorff objects and vice versa. Any axiomatic result about discrete objects therefore has a compact Hausdorff dual.*

In short, the categorical assumption 'distributive idempotents split in $\overline{\mathcal{C}}^{op}$', does the job of allowing us to develop locale theory via categorical axioms and neatly replaces the requirement for two order-dual axioms deployed in [T05] to prove various aspects of locale theory axiomatically.

7. Slice stability

7.1. Proposition. *Let X be an object of a category of spaces $\mathcal{C}$. Then $\mathcal{C}/X$ is a category of spaces.*

Proof. We have already commented that Axioms 1-6 are slice stable so we only need to prove that if Axioms 1-7 are true in $\mathcal{C}$ then Axiom 7 is true in $\mathcal{C}/X$.

Say we are given a distributive idempotent $\delta : \mathbb{S}_X^{Z_p} \longrightarrow \mathbb{S}_X^{Z_p}$ where Z_p is an object of $\mathcal{C}/X$. By Lemma 5.2 (ii) there is a reflexive coequalizer diagram

$$\mathbb{S}_X^{X^*\Sigma_X X^*\Sigma_X Z_p} \underset{\mathbb{S}_X^{X^*\Sigma_X \eta_{Z_p}}}{\overset{\mathbb{S}_X^{\eta_{X^*\Sigma_X Z_p}}}{\rightrightarrows}} \mathbb{S}_X^{X^X \Sigma_X Z_p} \xrightarrow{\mathbb{S}_X^{\eta_{Z_p}}} \mathbb{S}_X^E$$

with $\mathbb{S}_X^{X^*\epsilon_{\Sigma_X Z_p}}$ the common section of the reflexive pair. As $X^\# \dashv X_*$ extends $\Sigma_X \dashv X^*$ (contravariantly) this reflexive pair can be rewritten in terms of the unit and counit of the extended adjunction:

$$X^\# X_* X^\# X_*(\mathbb{S}_X^{Z_p}) \begin{array}{c} \xrightarrow{\bar{\epsilon}_{X^\# X_* \mathbb{S}_X^{Z_p}}} \\ \xrightarrow{X^\# X_* \bar{\epsilon}_{\mathbb{S}_X^{Z_p}}} \\ \xleftarrow[X^\# \bar{\eta}_{X_* \mathbb{S}_X^{Z_p}}]{} \end{array} X^\# X_*(\mathbb{S}_X^{Z_p}) \quad (*)$$

Then by naturality of $X_\# \dashv X_*$ we see that each of the morphisms of (*) commutes with the idempotents $X^\# X_*(\delta)$ and $X^\# X_* X^\# X_*(\delta)$. Now if δ is a distributive idempotent then so is $X_*(\delta)$ (Lemma 5.5) which splits (via $\mathbb{S}^{Z_\delta}$ say) by assumption that Axiom 7 is true in $\mathcal{C}$. Splittings are preserved by all functors, so both $X^\# X_*(\delta)$ and $X^\# X_* X^\# X_*(\delta)$ split and so by part (ii) of Proposition 3.1 we know that the three lattice homomorphisms of (*) give rise to three lattice homomorphisms between the splittings. Axiom 4 in $\mathcal{C}/X$ then tells us that these three lattice homomorphisms can be written $\mathbb{S}_X^{a_1}$, $\mathbb{S}_X^{a_2}$ and $\mathbb{S}_X^{s}$ where s splits a_1 and a_2; let $E \xrightarrow{e} X^*Z_\delta$ be the equalizer of a_1, a_2. It follows by Lemma 5.2 (i) that in $\overline{\mathcal{C}/X}^{op}$ there is a coequalizer:

$$X^{\#}X_*X^{\#}Fix(X_*(\delta)) \underset{\mathbb{S}_X^{a_2}}{\overset{\mathbb{S}_X^{a_1}}{\rightrightarrows}} X^{\#}Fix(X_*(\delta)) \xrightarrow{\mathbb{S}_X^{e}} \mathbb{S}_X^{E}.$$

It then follows that $\mathbb{S}_X^E$ is the required splitting of δ as both $\mathbb{S}_X^e$ and $\mathbb{S}_X^{\eta_{Z_p}}$ are coequalizers. That they are both coequalizers allows the splitting morphisms of δ to be defined, as the reflexive pairs of both $\mathbb{S}_X^e$ and $\mathbb{S}_X^{\eta_{Z_p}}$ commute with the splitting morphisms of $X^{\#}X_*(\delta)$ and $X^{\#}X_*X^{\#}X_*(\delta)$. In other words, written out as a diagram we have:

$$\begin{array}{ccccc}
\mathbb{S}_X^{(X\times X\times Z)_{\pi_1}} & \rightrightarrows & \mathbb{S}_X^{(X\times Z)_{\pi_1}} & \xrightarrow{\mathbb{S}_X^{\eta_{Z_p}}} & \mathbb{S}_X^{Z_p} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{S}_X^{(X\times X\times Z_\delta)_{\pi_1}} & \rightrightarrows & \mathbb{S}_X^{(X\times Z_\delta)_{\pi_1}} & \xrightarrow{\mathbb{S}_X^{e}} & \mathbb{S}_X^{E} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{S}_X^{(X\times X\times Z)_{\pi_1}} & \rightrightarrows & \mathbb{S}_X^{(X\times Z)_{\pi_1}} & \xrightarrow{\mathbb{S}_X^{\eta_{Z_p}}} & \mathbb{S}_X^{Z_p}
\end{array}$$

where the rows are all coequalizers in $\overline{\mathcal{C}/X}^{op}$. The middle and left hand side vertical arrows are splittings of $X^{\#}X_*(\delta)$ and $X^{\#}X_*X^{\#}X_*(\delta)$ respectively. As they commute with the pairs of arrows that define the coequalizers, the right hand vertical arrows can be added and form a splitting for δ as required. ∎

8. Groupoid stability

8.1. Proposition. *If* $\mathbb{G} = (d_0, d_1 : G_1 \rightrightarrows G_0, ...)$ *is a groupoid internal to a category of spaces* $\mathcal{C}$ *then* $[\mathbb{G}, \mathcal{C}]$ *is a category of spaces.*

Proof. We have commented already how this is covered for Axioms 1-6 in [T17]. The key insight, covered in [T17], is that for any two $\mathbb{G}$-objects $(X_f, a), (Y_g, b)$, natural transformations $\Delta : \mathbb{S}_{\mathbb{G}}^{(X_f,a)} \longrightarrow \mathbb{S}_{\mathbb{G}}^{(Y_g,b)}$ are in natural order isomorphism with those natural transformations $\delta : \mathbb{S}_{G_0}^{X_f} \longrightarrow \mathbb{S}_{G_0}^{Y_g}$ such that

$$\begin{array}{ccc} \mathbb{S}_{G_0}^{X_f} & \xrightarrow{\delta} & \mathbb{S}_{G_0}^{Y_g} \\ \downarrow{\scriptstyle \mathbb{S}_{G_0}^{a}} & & \downarrow{\scriptstyle \mathbb{S}_{G_0}^{b}} \\ \mathbb{S}_{G_0}^{G_1 \times_{G_0} X} & \xrightarrow{(d_1)_* d_0^{\#} \delta} & \mathbb{S}_{G_0}^{G_1 \times_{G_0} Y} \end{array}$$

commutes. From the construction of the order isomorphism it is clear that if $\Delta : \mathbb{S}_{\mathbb{G}}^{(Z_p,a)} \longrightarrow \mathbb{S}_{\mathbb{G}}^{(Z_p,a)}$ is a distributive idempotent then so is the corresponding $\delta : \mathbb{S}_{G_0}^{Z_p} \longrightarrow \mathbb{S}_{G_0}^{Z_p}$. But then we can form the following diagram

$$\begin{array}{ccccc} \mathbb{S}_{G_0}^{Z_p} & \xrightarrow{q_\delta} & \mathbb{S}_{G_0}^{Z_\delta} & \xrightarrow{i_\delta} & \mathbb{G}_{G_0}^{Z_p} \\ \downarrow{\scriptstyle \mathbb{S}_{G_0}^{a}} & & \downarrow{\scriptstyle \mathbb{S}_{G_0}^{\bar{a}}} & & \downarrow{\scriptstyle \mathbb{S}_{G_0}^{a}} \\ \mathbb{S}_{G_0}^{G_1 \times_{G_0} Z_p} & \xrightarrow{(d_1)_* d_0^{\#} q_\delta} & \mathbb{S}_{G_0}^{G_1 \times_{G_0} Z_\delta} & \xrightarrow{(d_1)_* d_0^{\#} i_\delta} & \mathbb{G}_{G_0}^{G_1 \times_{G_0} Z_p} \end{array}$$

by application of the previous proposition, where indeed the middle down arrow is determined by a morphism $\bar{a} : G_1 \times_{G_0} Z_\delta \longrightarrow Z_\delta$ of $\mathcal{C}/G_0$ by application of part (ii) of Proposition 3.1 and Axiom 4. It can then be checked that $(Z_\delta, \bar{a})$ is a $\mathbb{G}$-object using the fact that (Z_p, a) is a $\mathbb{G}$-object. ∎

9. Triquotient surjections are of effective descent

In this section we complete aim (iii) from above, assuming a category of spaces $\mathcal{C}$. We start by providing a particular criteria for when a fork in $\mathcal{C}$ is a coequalizer.

9.1. Lemma. *Say in $\overline{\mathcal{C}}^{op}$ we have a split equalizer diagram*

$$\mathbb{S}^Q \underset{q_\#}{\overset{\mathbb{S}^q}{\rightleftarrows}} \mathbb{S}^Y \begin{array}{c} \xrightarrow{\mathbb{S}^f} \\ \xrightarrow{\mathbb{S}^g} \\ \xleftarrow[\alpha]{} \end{array} \mathbb{S}^X$$

with $\alpha \mathbb{S}^g = Id_{\mathbb{S}^Y}$ and $q_\#$ a triquotient assignment on q. Then $X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{q} Q$ is a coequalizer diagram in $\mathcal{C}$.

Proof. Certainly $qf = qg$ as $\mathbb{S}^{qf} = \mathbb{S}^f \mathbb{S}^q = \mathbb{S}^g \mathbb{S}^q = \mathbb{S}^{qg}$ (apply the uniqueness part of Axiom 4). So to prove that q is the coequalizer of f and g we need to show that for any l with $lf = lg$ we have that $q_\#\mathbb{S}^l$ is a lattice homomorphism, since then by Axiom 4 $q_\#\mathbb{S}^l = \mathbb{S}^h$ for a unique morphism h of $\mathcal{C}$. But,

$$\begin{aligned} q_\#\mathbb{S}^l(b_1 \vee b_2) &= q_\#(\mathbb{S}^l b_1 \vee \mathbb{S}^l b_2) \\ &= q_\#(\mathbb{S}^l b_1 \vee \alpha\mathbb{S}^g\mathbb{S}^l b_2) \\ &= q_\#(\mathbb{S}^l b_1 \vee \alpha\mathbb{S}^f\mathbb{S}^l b_2) \\ &= q_\#(\mathbb{S}^l b_1 \vee \mathbb{S}^q q_\#\mathbb{S}^l b_2) \\ &\leq q_\#\mathbb{S}^l b_1 \vee q_\#\mathbb{S}^l b_2 \end{aligned}$$

with the last line using the fact that $q_\#$ is a triquotient idempotent. A symmetric argument shows that $q_\#\mathbb{S}^l$ preserves binary meets. Certainly top and bottom are preserved as $q_\#$ preserves top and bottom. Therefore $q_\#\mathbb{S}^l$ is a lattice homomorphism as required. ■

The next Proposition is really the key technical insight of the paper as it shows how splitting distributive idempotents allows us to form the pullback stable coequalizers that we need.

9.2. Proposition. *(a) If $f, g : X \rightrightarrows Y$ is a pair of morphisms in $\mathcal{C}$ and $g_\#$ a triquotient assignment on g such that $\mathbb{S}^f g_\#\mathbb{S}^f = \mathbb{S}^g g_\#\mathbb{S}^f$, then the pair f, g has a coequalizer, $q : Y \longrightarrow Q$.*

(b) Any triquotient surjection $p : Z \longrightarrow Y$ is a pullback stable coequalizer.

(c) Given f, g and $g_\#$ as in (a), then the coequalizer diagram $X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{q} Q$ pullback stable.

Proof. (a) Let $\delta = g_\#\mathbb{S}^f$, then by exploiting $\mathbb{S}^f g_\#\mathbb{S}^f = \mathbb{S}^g g_\#\mathbb{S}^f$ and the fact that $g_\#$ is a triquotient assignment on g, we see that $\delta : \mathbb{S}^Y \longrightarrow \mathbb{S}^Y$ is a triquotient idempotent which therefore is split (recall Example 3.10). Say the splitting is $\mathbb{S}^Y \xrightarrow{\gamma} \mathbb{S}^Q \xhookrightarrow{\epsilon} \mathbb{S}^Y$. Further recall from Example 3.10 that we know (i) ϵ is a lattice homomorphism and so must therefore be of the form $\mathbb{S}^q$ for some unique morphism $q : Y \longrightarrow Q$ of $\mathcal{C}$ (Axiom 5); and, (ii) $\gamma = q_\#$ a triquotient assignment on q. But then

$$\mathbb{S}^Q \underset{q_\#}{\overset{\mathbb{S}^q}{\rightleftarrows}} \mathbb{S}^Y \underset{g_\#}{\overset{\mathbb{S}^f,\ \mathbb{S}^g}{\rightrightarrows\!\!\leftarrow}} \mathbb{S}^X$$

is a split equalizer diagram and we can apply the previous Lemma (Lemma 9.1).

(b) The pullback stability aspect is clear as the property of being a triquotient surjection is pullback stable (see (b) of Proposition 5.6). So we just have to check that p is a coequalizer. Let $(\pi_1)_\#$ be the unique triquotient assignment on $\pi_1 : Z \times_Y Z \longrightarrow Z$ such that $(\pi_1)_\#\mathbb{S}^{\pi_2} = \mathbb{S}^p p_\#$ where $p_\#$ is a triquotient assignment witnessing that p is a triquotient surjection. Then

$$\begin{aligned}\mathbb{S}^{\pi_2}(\pi_1)_\#\mathbb{S}^{\pi_2} &= \mathbb{S}^{\pi_2}\mathbb{S}^{p}p_\# \\ &= \mathbb{S}^{\pi_1}\mathbb{S}^{p}p_\# \\ &= \mathbb{S}^{\pi_1}(\pi_1)_\#\mathbb{S}^{\pi_2}\end{aligned}$$

and so by (a) the kernel pair $\pi_1, \pi_2 : Z \times_Y Z \rightrightarrows Z$ of p has a coequalizer; it is given by the splitting of the idempotent $(\pi_1)_\#\mathbb{S}^{\pi_2}$, which we know to be $\mathbb{S}^p p_\#$.

(c) Following the notation of Proposition 5.6 (a) consider the diagram

$$\mathbb{S}_Q \underset{\alpha^{q\#}}{\overset{\mathbb{S}_Q^{!_{Y_q}}}{\rightleftarrows}} \mathbb{S}_Q^{Y_q} \; \begin{matrix} \xrightarrow{\mathbb{S}_Q^{f}} \\ \underset{q_*(\alpha^{g\#})}{\overset{\mathbb{S}_Q^{g}}{\rightleftarrows}} \end{matrix} \; \mathbb{S}_Q^{X_{qg}}$$

in $\overline{\mathcal{C}/Q}^{op}$. We know that $\alpha^{q\#}$ is a triquotient assignment on $!^{Y_q}$ (Proposition 5.6 (a)) so to complete the proof we in fact just need to check that the diagram is a split equalizer. Since then for any $k : K \longrightarrow Q$ we only need to note that the diagram (and the property that $\alpha^{q\#}$ is a triquotient assigment) is preserved by $K_* k^\#$ and so gives rise to a coequalizer by Lemma 9.1.

That the diagram is a split equalizer follows by recalling that the counit of $Q^\# \dashv Q_*$ at $\mathbb{S}_Q^{Y_q}$ is $\mathbb{S}_Q^{\eta_{Y_q}} : \mathbb{S}_Q^{Y_Q} \xrightarrow{\mathbb{S}_Q^{(q,Id_Y)}} \mathbb{S}_Q^{Y_q}$, where this last is an epimorphism. For example $\mathbb{S}_Q^{\eta_{Y_q}} Q^\#(g_\#) = q_*(\alpha^{g\#})\mathbb{S}_Q^{\eta_{X_{qg}}}$ is true because under the adjunction $Q^\# \dashv Q_*$ it is equivalent to checking $g^\# = Q_*(q_*(\alpha^{g\#}))$ which is true as $Q_* q_* = Y_*$ (recall again part (a) of Proposition 5.6 to see that $q^\# = Y_*(\alpha^{g\#})$ by construction of $\alpha^{g\#}$). To see that $\mathbb{S}_Q^{!_{Y_q}}\alpha^{q\#} = q_*(\alpha^{g\#})\mathbb{S}_Q^{f}$ check that $\mathbb{S}_Q^{!_{Y_q}}\alpha^{q\#}\mathbb{S}_Q^{\eta_{Y_q}} = q_*(\alpha^{g\#})\mathbb{S}_Q^{f}\mathbb{S}_Q^{\eta_{Y_q}}$, which amounts to checking that

$$\mathbb{S}_Q^{Y_Q} \xrightarrow{\widetilde{q_\#}} \mathbb{S}_Q \xrightarrow{\mathbb{S}_Q^{!_{Y_q}}} \mathbb{S}_Q^{Y_q} \quad \text{(I)}$$

is equal to

$$\mathbb{S}_Q^{Y_Q} \xrightarrow{\mathbb{S}_Q^{f\times Id_Q}} \mathbb{S}_Q^{X_Q} \xrightarrow{Q^\#(g_\#)} \mathbb{S}_Q^{Y_Q} \xrightarrow{\mathbb{S}_Q^{\eta_{Y_q}}} \mathbb{S}_Q^{Y_q} \quad \text{(II)}$$

as $\mathbb{S}_Q^{f}\mathbb{S}_Q^{\eta_{Y_q}} = \mathbb{S}_Q^{\eta_{X_{qg}}}\mathbb{S}_Q^{f\times Id_Q}$. The adjoint transpose of (I) across $Q^\# \dashv Q_*$ is $\mathbb{S}^Y \xrightarrow{q_\#} \mathbb{S}^Q \xrightarrow{\mathbb{S}^q} \mathbb{S}^Y$. Because $\mathbb{S}_Q^{f\times Id_Q} = Q^\#(\mathbb{S}^f)$ the adjoint transpose of (II) is $\mathbb{S}^Y \xrightarrow{\mathbb{S}^f} \mathbb{S}^X \xrightarrow{g_\#} \mathbb{S}^Y$ which is equal to the adjoint transpose of (I) by definition of q. ∎

The following is originally due to Plewe [P97] for locales; what is new here is that we are able to avoid an assumption that the ambient category $\mathcal{C}$ has coequalizers.

9.3. THEOREM. *Any triquotient surjection $p : Z \longrightarrow Y$ is of effective descent.*

Proof. First it is clear by change of base that we can assume $Y = 1$ and we know that p^* reflects isomorphisms by Proposition 9.2 (b). So we just have to create a pullback stable coequalizer diagram for any pair $A \underset{a_2}{\overset{a_1}{\rightrightarrows}} B$ that is p^*-split.

By considering the three pullback squares

$$\begin{array}{ccccc} A \times Z & \underset{a_2 \times Id_Z}{\overset{a_1 \times Id_Z}{\rightrightarrows}} & B \times Z & \xrightarrow{\pi_2} & Z \\ \downarrow \pi_1^A & & \downarrow \pi_1^B & & \downarrow \\ A & \underset{a_2}{\overset{a_1}{\rightrightarrows}} & B & \longrightarrow & 1 \end{array}$$

we see by the uniqueness of weak triquotient assignments on pullback squares that there is a unique triquotient assignment $(\pi_1^A)_\#$ on π_1^A such that $(\pi_1^A)_\# \mathbb{S}^{a_1 \times Id_Z} = \mathbb{S}^{a_1}(\pi_1^B)_\#$ and $(\pi_1^A)_\# \mathbb{S}^{a_2 \times Id_Z} = \mathbb{S}^{a_2}(\pi_1^B)_\#$ where $(\pi_1^B)_\#$ is the unique triquotient assignment on π_1^B that satisfies Beck-Chevalley for the right hand pullback square (i.e. $(\pi_1^B)_\# \mathbb{S}^{\pi_2} = \mathbb{S}^{!^B} p_\#$ where $p_\#$ is the triquotient assignment that exists by assumption that p is a triquotient surjection). The map $(\pi_1^A)_\#$ works for both as it is also the unique triquotient assignment satisfying Beck-Chevalley for the composite pullback (which is the same for each left hand square).

Now as a_1, a_2 is p^*-split we have a split coequalizer $A \times Z \underset{a_2 \times Id_Z}{\overset{a_1 \times Id_Z}{\rightrightarrows}} B \times Z \xrightarrow{q} Q$. So we have a split equalizer diagram:

$$\mathbb{S}^Q \underset{\mathbb{S}^r}{\overset{\mathbb{S}^q}{\rightleftarrows}} \mathbb{S}^{B \times Z} \begin{array}{c} \xrightarrow{\mathbb{S}^{a_1 \times Id_Z}} \\ \xrightarrow{\mathbb{S}^{a_2 \times Id_Z}} \\ \xleftarrow[\mathbb{S}^j]{} \end{array} \mathbb{S}^{A \times Z}$$

where $\mathbb{S}^j$ is the splitting of $\mathbb{S}^{a_1 \times Id_Z}$.

Let $\gamma = (\pi_1^B)_\# \mathbb{S}^j \mathbb{S}^{\pi_1^A}$. Then,

$$\begin{aligned} \gamma(d \vee \mathbb{S}^{a_1} c) &= (\pi_1^B)_\# \mathbb{S}^j \mathbb{S}^{\pi_1^A}(d \vee \mathbb{S}^{a_1} c) \\ &= (\pi_1^B)_\# (\mathbb{S}^j \mathbb{S}^{\pi_1^A} d \vee \mathbb{S}^j \mathbb{S}^{\pi_1^A} \mathbb{S}^{a_1} c) \\ &= (\pi_1^B)_\# (\mathbb{S}^j \mathbb{S}^{\pi_1^A} d \vee \mathbb{S}^j \mathbb{S}^{a_1 \times Id_Z} \mathbb{S}^{\pi_1^B} c) \\ &= (\pi_1^B)_\# (\mathbb{S}^j \mathbb{S}^{\pi_1^A} d \vee \mathbb{S}^{\pi_1^B} c) \\ &\leq (\pi_1^B)_\# (\mathbb{S}^j \mathbb{S}^{\pi_1^A} d) \vee c \\ &= \gamma(d) \vee c \end{aligned}$$

and, by an order dual argument, $\gamma(d) \wedge c \leq \gamma(d \wedge \mathbb{S}^{a_1} c)$. Therefore γ is a weak triquotient assignment on a_1; but it is actually, further, a triquotient assignment since $(\pi_1^B)_\#$ preserves top and bottom (as it is a triquotient assignment). Finally

$$\begin{aligned}
\mathbb{S}^{a_2}\gamma\mathbb{S}^{a_2} &= \mathbb{S}^{a_2}(\pi_1^B)_\#\mathbb{S}^j\mathbb{S}^{\pi_1^A}\mathbb{S}^{a_2} \\
&= \mathbb{S}^{a_2}(\pi_1^B)_\#\mathbb{S}^j\mathbb{S}^{a_2\times Id_Z}\mathbb{S}^{\pi_1^B} \\
&= \mathbb{S}^{a_2}(\pi_1^B)_\#\mathbb{S}^q\mathbb{S}^r\mathbb{S}^{\pi_1^B} \\
&= (\pi_1^A)_\#\mathbb{S}^{a_2\times Id_Z}\mathbb{S}^q\mathbb{S}^r\mathbb{S}^{\pi_1^B} \\
&= (\pi_1^A)_\#\mathbb{S}^{a_1\times Id_Z}\mathbb{S}^q\mathbb{S}^r\mathbb{S}^{\pi_1^B} \\
&= \mathbb{S}^{a_1}(\pi_1^B)_\#\mathbb{S}^q\mathbb{S}^r\mathbb{S}^{\pi_1^B} \\
&= \mathbb{S}^{a_1}\gamma\mathbb{S}^{a_2}
\end{aligned}$$

where the last step is a reversal of the first three steps. It follows that a_1, a_2 have a pullback stable coequalizer by Proposition 9.2. ∎

10. Connected component adjunctions

Given an internal groupoid $\mathbb{G}$, $\mathbb{G}$ itself is an object of $[\mathbb{G},\mathcal{C}]$; the underlying object is $(G_1)_{d_1}$ and the structure map $\Sigma_{d_1}d_0^*(G_1)_{d_1} \longrightarrow (G_1)_{d_1}$ is the groupoid multiplication $m(f,g) = fg$. For example, if $\mathbb{G}$ is a group G then this object is G with group multiplication as the action. Or, if $\mathbb{G}$ is $X \rightrightarrows X$ so that $[\mathbb{G},\mathcal{C}]$ is the slice $\mathcal{C}/X$, then this object is the terminal object X_{Id_X}.

In this section we prove that provided the unique map $\mathbb{G} \longrightarrow 1$ is a triquotient surjection in $[\mathbb{G},\mathcal{C}]$, then $[\mathbb{G},\mathcal{C}]$ has a stably Frobenius connected components adjunction. This is important information as without an assumption that $\mathcal{C}$ has coequalizers we can't assume that a connected components functor exists.

Let us first motivate the condition that $\mathbb{G} \longrightarrow 1$ is a triquotient surjection:

10.1. Proposition. *If $\mathbb{G}$ is open, $\mathbb{G} \longrightarrow 1$ is a triquotient surjection.*

Recall that $\mathbb{G}$ is *open* if $d_0 : G_1 \longrightarrow G_0$ (equivalently d_1) is an open map. Because d_0 has a section (the unit map $s : G_0 \longrightarrow G_1$), if $\mathbb{G}$ is open, we in fact know that d_0 is an open surjection. (In detail: as $\exists_{d_0} \dashv \mathbb{S}^{d_0}$, we have that $\mathbb{S}^{d_0}\exists_{d_0}\mathbb{S}^{d_0} = \mathbb{S}^{d_0}$; but $\mathbb{S}^{d_0}$ is a monic split by $\mathbb{S}^s$, so $\exists_{d_0}\mathbb{S}^{d_0} = Id$.)

Note that by order duality the proposition also allows the same conclusion for proper groupoids, though we do not explore that aspect in this paper.

Proof. A morphism $f : X \longrightarrow Y$ is an open surjection if and only if $\mathbb{S}_Y^{!^{X_f}} : \mathbb{S}_Y \longrightarrow \mathbb{S}_Y^{X_f}$ has a left adjoint $\exists_{X_f} : \mathbb{S}_Y^{X_f} \longrightarrow \mathbb{S}_Y$ that preserves top. This can be seen by exploiting the main result (Proposition 5.4) of [T10]; more details are covered in Section 7 of [T04]. So if $\mathbb{G}$ is open we know that there exists $\exists_{(G_1)_{d_1}} : \mathbb{S}_{G_0}^{(G_1)_{d_1}} \longrightarrow \mathbb{S}_{G_0}$, left adjoint to $\mathbb{S}_{G_0}^{!^{(G_1)_{d_1}}}$, that preserves top. By naturality of the counit of the extended adjunction $(d_1)_* \dashv d_1^\#$ the diagram

$$\begin{array}{ccc} \mathbb{S}_{G_0}^{(G_1)_{d_1}} & \xrightarrow{\exists_{(G_1)_{d_1}}} & \mathbb{S}_{G_0} \\ {\scriptstyle \mathbb{S}_{G_0}^{\pi_2}}\downarrow & & \downarrow{\scriptstyle \mathbb{S}_{G_0}^{d_1}} \\ \mathbb{S}_{G_0}^{\Sigma_{d_1} d_1^*(G_1)_{d_1}} & \xrightarrow{(d_1)_* d_1^{\#}(\exists_{(G_1)_{d_1}})} & \mathbb{S}_{G_0}^{\Sigma_{d_1} d_1^* 1} \end{array} \qquad (*)$$

commutes. But we have a groupoid, so there is an isomorphism $\psi : d_0^*((G_1)_{d_1}) \longrightarrow d_1^*((G_1)_{d_1})$ given by $(f, g) \mapsto (f, fg)$ (the inverse is $(k, h) \mapsto (k, k^{-1}h)$), for which $\mathbb{S}_{G_0}^{\psi^{-1}} \mathbb{S}_{G_0}^{m} = \mathbb{S}_{G_0}^{\pi_2}$. Now, to conclude that $!^{\mathbb{G}}$ is a triquotient surjection we just have to check that

$$\begin{array}{ccc} \mathbb{S}_{G_0}^{(G_1)_{d_1}} & \xrightarrow{\exists_{(G_1)_{d_1}}} & \mathbb{S}_{G_0} \\ {\scriptstyle \mathbb{S}_{G_0}^{m}}\downarrow & & \downarrow{\scriptstyle \mathbb{S}_{G_0}^{d_1}} \\ \mathbb{S}_{G_0}^{\Sigma_{d_1} d_0^*(G_1)_{d_1}} & \xrightarrow{(d_1)_* d_0^{\#}(\exists_{(G_1)_{d_1}})} & \mathbb{S}_{G_0}^{\Sigma_{d_1} d_0^* 1} \end{array}$$

commutes because then $\exists_{(G_1)_{d_1}}$ corresponds to a morphism of $\overline{[\mathbb{G}, \mathcal{C}]}^{op}$ which preserves top and is left adjoint to $\mathbb{S}_{\mathbb{G}}^{!^{\mathbb{G}}}$. But

$$\mathbb{S}_{G_0}^{\Sigma_{d_1} d_0^* 1} \xrightarrow{\mathbb{S}_{G_0}^{\Sigma_{d_1} d_0^*(!^{(G_1)_{d_1}})}} \mathbb{S}_{G_0}^{\Sigma_{d_1} d_0^*(G_1)_{d_1}} \xrightarrow{\mathbb{S}_{G_0}^{\psi^{-1}}} \mathbb{S}_{G_0}^{\Sigma_{d_1} d_1^*(G_1)_{d_1}}$$

is equal to $\mathbb{S}_{G_0}^{\pi_1} = \mathbb{S}_{G_0}^{\Sigma_{d_1} d_1^*(!^{(G_1)_{d_1}})}$. To see this note that $\Sigma_{d_1} d_0^*(!^{(G_1)_{d_1}})$ is just $\pi_1 : G_1 \times_{G_0} G_1 \longrightarrow G_1$ (as $!^{(G_1)_{d_1}} : G_1 \longrightarrow G_0$ is just $d_1 : G_1 \longrightarrow G_0$) and $\psi^{-1}(k, h) = (k, k^{-1}h)$. They therefore both have the same left adjoint and since the extended change of base functors preserve adjoints (and of course $\mathbb{S}_{G_0}^{\psi} \dashv \mathbb{S}_{G_0}^{\psi^{-1}}$) we have that $[(d_1)_* d_0^{\#}(\exists_{(G_1)_{d_1}})]\mathbb{S}_{G_0}^{\psi} = (d_1)_* d_1^{\#}(\exists_{(G_1)_{d_1}})$. Therefore

$$\begin{aligned} [(d_1)_* d_0^{\#}(\exists_{(G_1)_{d_1}})]\mathbb{S}_{G_0}^{m} &= [(d_1)_* d_0^{\#}(\exists_{(G_1)_{d_1}})]\mathbb{S}_{G_0}^{\psi}\mathbb{S}_{G_0}^{\psi^{-1}}\mathbb{S}_{G_0}^{m} \\ &= [(d_1)_* d_1^{\#}(\exists_{(G_1)_{d_1}})]\mathbb{S}_{G_0}^{\psi^{-1}}\mathbb{S}_{G_0}^{m} \\ &= [(d_1)_* d_1^{\#}(\exists_{(G_1)_{d_1}})]\mathbb{S}_{G_0}^{\pi_2} \\ &= \mathbb{S}_{G_0}^{d_1}\exists_{(G_1)_{d_1}} \qquad \text{(by (*))} \end{aligned}$$

as required. ∎

10.2. Proposition. *If* $\mathbb{G} \longrightarrow 1$ *is a triquotient surjection in* $[\mathbb{G}, \mathcal{C}]$*, then* $[\mathbb{G}, \mathcal{C}]$ *has a stably Frobenius connected components adjunction.*

Proof. There is a triquotient assignment on $\mathbb{G} \longrightarrow 1$ and so, via the description of natural transformations relative to $[\mathbb{G}, \mathcal{C}]$, recalled in the proof of Proposition 8.1, we know that there is a triquotient assignment $!^{\mathbb{G}}_{\#} : \mathbb{S}^{(G_1)_{d_1}}_{G_0} \longrightarrow \mathbb{S}_{G_0}$ such that

$$\begin{array}{ccc} \mathbb{S}^{(G_1)_{d_1}}_{G_0} & \xrightarrow{\;!^{\mathbb{G}}_{\#}\;} & \mathbb{S}_{G_0} \\ \Big\downarrow \mathbb{S}^m_{G_0} & & \Big\downarrow \mathbb{S}^{d_1}_{G_0} \\ \mathbb{S}^{\Sigma_{d_1} d_0^*(G_1)_{d_1}}_{G_0} & \xrightarrow{(d_1)_* d_0^{\#}(!^{\mathbb{G}}_{\#})} & \mathbb{S}^{(G_1)_{d_1}}_{G_0} \end{array}$$

commutes. However this is equivalent to

$$\begin{array}{ccc} \mathbb{S}^{(G_1)_{d_0}}_{G_0} & \xrightarrow{\;!^{\mathbb{G}^{op}}_{\#}\;} & \mathbb{S}_{G_0} \\ \Big\downarrow \mathbb{S}^m_{G_0} & & \Big\downarrow \mathbb{S}^{d_0}_{G_0} \quad (+) \\ \mathbb{S}^{\Sigma_{d_0} d_1^*(G_1)_{d_0}}_{G_0} & \xrightarrow{(d_0)_* d_1^{\#}(!^{\mathbb{G}^{op}}_{\#})} & \mathbb{S}^{(G_1)_{d_0}}_{G_0} \end{array}$$

commuting where $!^{\mathbb{G}^{op}}_{\#}$ is the map : $\mathbb{S}^{(G_1)_{d_0}}_{G_0} \xrightarrow{\mathbb{S}^i_{G_0}} \mathbb{S}^{(G_1)_{d_1}}_{G_0} \xrightarrow{!^{\mathbb{G}}_{\#}} \mathbb{S}_{G_0}$ (i is the groupoid inverse). To see this consider $\psi = i$ in Remark 2.2, applied in $\mathcal{C}/G_0$.

Define $(d_0)_{\#} : \mathbb{S}^{G_1} \longrightarrow \mathbb{S}^{G_0}$ to be $(G_0)_*(!^{\mathbb{G}^{op}}_{\#})$; it is a triquotient assignment on $d_0 : G_1 \longrightarrow G_0$ (see (a) of Proposition 5.6; so $!^{\mathbb{G}^{op}}_{\#} = \alpha^{(d_0)_{\#}}$ using the notation of that proposition).

Now let (X_p, a) be a $\mathbb{G}$-object. Apply $X_* p^{\#}$ to (+). Then I claim that we obtain a commuting square

$$\begin{array}{ccc} \mathbb{S}^{G_1 \times_{G_0} X} & \xrightarrow{(\pi_2)_{\#}} & \mathbb{S}^X \\ \Big\downarrow \mathbb{S}^{m \times Id_X} & & \Big\downarrow \mathbb{S}^{\pi_2} \\ \mathbb{S}^{G_1 \times_{G_0} G_1 \times_{G_0} X} & \xrightarrow{(\pi_{23})_{\#}} & \mathbb{S}^{G_1 \times_{G_0} X} \end{array}$$

where $(\pi_2)_{\#}$ is the unique triquotient assignment for Beck-Chevalley of the right hand pullback square of

$$\begin{array}{ccccc} G_1 \times_{G_0} G_1 \times_{G_0} X & \xrightarrow{Id_{G_1} \times a} & G_1 \times_{G_0} X & \xrightarrow{\pi_1} & G_1 \\ \downarrow{\scriptstyle \pi_{23}} & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle d_0} \quad (*) \\ G_1 \times_{G_0} X & \xrightarrow{a} & X & \xrightarrow{p} & G_0 \end{array}$$

and $(\pi_{23})_\#$ is the unique triquotient for Beck-Chevalley of the left hand pullback square. That is we have $(\pi_2)_\# \mathbb{S}^{\pi_1} = \mathbb{S}^p (d_0)_\#$ and $(\pi_{23})_\# \mathbb{S}^{Id \times a} = \mathbb{S}^a (\pi_2)_\#$.

To prove this claim firstly recall that $(\pi_2)_\# = X_* p^\# (\alpha^{(d_0)_\#})$ from (b) of Proposition 5.6 so $(\pi_2)_\# = X_* p^\# (!^{\mathbb{G}^{op}}_\#)$. To complete proof of the claim we calculate:

$$\begin{aligned} X_* p^\# (d_0)_* d_1^\# (!^{\mathbb{G}^{op}}_\#) &= X_* (\pi_2)_* \pi_1^\# d_1^\# (!^{\mathbb{G}^{op}}_\#) \text{ by Lemma 5.3} \\ &= X_* (\pi_2)_* a^\# p^\# (!^{\mathbb{G}^{op}}_\#) \text{ as } pa = d_1 \pi_1 \text{ by def. of } \mathbb{G}\text{-object} \\ &= (G_1 \times_{G_0} X)_* (pa)^\# (\alpha^{(d_0)_\#}) \\ &= (\pi_{23})_\# \end{aligned}$$

where the last line is by uniqueness of triquotients satisfying Beck-Chevalley for the outer rectangle in (*).

But then

$$\begin{aligned} \mathbb{S}^a (\pi_2)_\# \mathbb{S}^a &= (\pi_{23})_\# \mathbb{S}^{Id_{G_1} \times a} \mathbb{S}^a \\ &= (\pi_{23})_\# \mathbb{S}^{m \times Id_X} \mathbb{S}^a \text{ (def. of } \mathbb{G}\text{-object)} \\ &= \mathbb{S}^{\pi_2} (\pi_2)_\# \mathbb{S}^a \end{aligned}$$

and so we can apply Proposition 9.2 to complete the proof. (It has been commented in Section 2 that to construct a stably Frobenius connected components adjunction we just need to construct a coequalizer for the pair $\pi_2, a : G_1 \times_{G_0} X \rightrightarrows X$ and check that the resulting coequalizer diagram is pullback stable.) ■

10.3. Remark. *The above analysis provides an answer to an intuitive question. As is well known and effectively recalled above, both proper and open localic groupoids have connected component adjunctions. As triquotient surjection appears to be the natural common generalisation of both proper surjection and open surjection, we might be tempted to define a 'triquotient' localic groupoid to be one for which d_0 is a triquotient surjection. A hope might then be that we can prove that triquotient groupoids have connected component adjunctions, covering the proper and open cases in one go. The trouble is that d_0 is a split surjection and therefore is always a triquotient surjection, so this doesn't work (we don't* *have a connected component adjunction for every localic groupoid). The intuitively correct generalisation here seems to be to consider triquotient surjections relative to the category of $\mathbb{G}$-objects rather than the base category. Indeed this appears to be key to understanding how to isolate the correct notion of bounded morphism between categories of spaces (see [T18]).*

11. Dedication

I knew Marta mostly through her work, though we did correspond on our joint interest in understanding what the correct notion of 'upper power topos' should be. It is fair to say that we had different ideas on this topic, but I was chuffed that there was at least someone thinking about the problem. I spent time understanding her work on fundamental groupoids/Galois theory for toposes; I would have loved to have finally been able to say 'got it'. My guess is that when we finally get a good account it will reference back to her work. I also found occassions where if only I had taken the time to study her work more carefully I would have understood that quite a bit of my thinking was already known, particularly around stack completions. I am saddened that she is gone, but am certain that she will live on through her work.

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References

[B90] Bunge, M. *An application of descent to a classification theorem for toposes*. Mathematical Proceedings of the Cambridge Philosophical Society, 107(1), 59–79. (1990).

[J02] Johnstone, P.T. *Sketches of an elephant: A topos theory compendium*. Vols 1, 2, Oxford Logic Guides **43**, **44**, Oxford Science Publications, 2002.

[M22] Manuell, G. *Presenting quotient locales*, arXiV2207.05116. July (2022).

[P97] Plewe, T. *Localic triquotient maps are effective descent maps* Mathematical Proceedings of the Cambridge Philosophical Society 122(01):17 - 43 (1997)

[T04] Townsend, C.F. *An axiomatic account of weak triquotient assignments in locale theory*; early draft. www.christophertownsend.org.

[T05] Townsend, C.F. *A categorical account of the Hofmann-Mislove theorem* Math. Proc. Camb. Philos. Soc. 139, No.3, 441-455 (2005).

[T10] Townsend, C.F. *An axiomatic account of weak triquotient assignments in locale theory*, Journal of Pure and Applied Algebra, Volume 214, Issue 6, 2010, Pages 729-739.

[T12] Townsend, C.F. *Aspects of slice stability in Locale Theory* Georgian Mathematical Journal. Vol. 19, Issue **2**, (2012) 317-374.

[T17] Townsend, C.F. *Stability of Properties of Locales Under Groups*. Appl Categor Struct Vol. 25 Issue **3** (2017), 363-380.

[T18] Townsend, C.F. *Double power monad preserving adjunctions are Frobenius.*. Theory and Applications of Categories, Vol. 33, No. 17, 2018, pp. 476-491

[VT04] Vickers, S.J. and Townsend, C.F. *A Universal Characterization of the Double Power Locale*. Theo.Comp. Sci. 316 (2004), 297-321.

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